$(\inf, \sup)$-Hesitant Fuzzy Interior ideals in Semigroups

Pannawit Khamrot, Aiyared Iampan, Thiti Gaketem

***Abstract*—In this paper, we give the concept of an $(\inf, \sup)$-hesitant fuzzy interior ideals, show that it is a general concept of interval-valued fuzzy interior ideals, and investigate its related properties. Characterizations of $(\inf, \sup)$-hesitant fuzzy interior ideals are established in terms of sets, fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, and hesitant fuzzy sets. Moreover, we discuss fuzzy interior ideals, and Pythagorean fuzzy interior ideals of semigroups in terms of $(\inf, \sup)$-hesitant fuzzy interior ideals and interval-valued fuzzy interior ideals.**

***Index Terms*—$(\inf, \sup)$-hesitant fuzzy interior ideals, interval-valued fuzzy interior ideals, Hesitant fuzzy interior ideals, Pythagorean fuzzy interior ideals.**

I. Introduction

After the concept of fuzzy sets was proposed by Zadeh [1], has been widely and successfully applied in many branches: medical science, theoretical physics, computer science, finite state machine, automata, artificial intelligence, expert, robotics, control engineering and theory of groups, semigroups, ternary semigroup, BCK/BCI-algebras, KU-algebra, UP-algebras, etc. Many general, extended and related concepts of fuzzy sets have been introduced and studied such as interval-valued fuzzy sets [2], intuitionistic fuzzy sets [3], Pythagorean fuzzy sets [4], bipolar fuzzy sets [5], [6], [7] , hesitant fuzzy sets [8] and so forth. In 2016, Muhiuddin et al. [9] introduced inf-hesitant fuzzy subalgebras and ideals in BCK/BCI-algebras and investigated their properties. Later, in 2015, Jun et al. [10] studied the concepts and properties of a hesitant fuzzy subgroupoid (left ideal, right ideal, and ideal) of a groupoid, a hesitant fuzzy subgroup (normal subgroup and quotient subgroup) of a group, and a hesitant fuzzy subring (left ideal, right ideal, and ideal) of a ring. In 2021, U. Jittburus and P. Julatha [11] used of inf-hesitant fuzzy set studied properties in semigroup, Γ-semigroup and logical algebra systems. Recently, U. Jittburus et al. [12] studied inf-hesitant fuzzy ideals and inf-hesitant fuzzy translations in semigroups. In the same year Chunsee et al. [13] studied properties of $(\inf, \sup)$-hesitant fuzzy bi-ideals in semigroups. Moreover, Phummee et al. [14] discussed the concept of a fuzzy interior ideal in semigroup.

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P. Khamrot is a lecturer at the Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University Technology Lanna Phitsanulok, Phitsanulok, Thailand. (e-mail: pk_g@rmutl.ac.th).

A. Iampan is a lecturer at the Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. (e-mail: aiyared.ia@up.ac.th).

T. Gaketem is a lecturer at the Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. (corresponding author to provide email: thiti.ga@up.ac.th).

In this paper, we was extended the concept of SUP-hesitant fuzzy interior ideal to concept an $(\inf, \sup)$-hesitant fuzzy interior ideals, show that it is a general concept of interval-valued fuzzy interior ideals, and investigate its related properties. Characterizations of $(\inf, \sup)$-hesitant fuzzy interior ideals are established in terms of sets, fuzzy sets, negative fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, bipolar fuzzy sets and hesitant fuzzy sets. Moreover, characterizations of fuzzy interior ideals, and Pythagorean fuzzy interior ideals of semigroups are discussed in terms of $(\inf, \sup)$-hesitant fuzzy interior ideals and interval-valued fuzzy interior ideals.

II. Preliminaries

From now on throughout this work, $\ddot{\mathfrak{S}}$ is represented as a semigroup and $\ddot{\mathfrak{K}}$ a nonempty set unless otherwise specified. An **subsemigroup** $\ddot{\mathfrak{K}}$ of $\ddot{\mathfrak{S}}$ if $\ddot{\mathfrak{K}}^2 \subseteq \ddot{\mathfrak{K}}$. By an **ideal** of $\ddot{\mathfrak{S}}$ we mean a nonempty subset $\ddot{\mathfrak{K}}$ of $\ddot{\mathfrak{S}}$ such that $\ddot{\mathfrak{S}}\ddot{\mathfrak{K}} \subseteq \ddot{\mathfrak{K}}$ and $\ddot{\mathfrak{K}}\ddot{\mathfrak{S}} \subseteq \ddot{\mathfrak{K}}$. A **interior ideal** $\ddot{\mathfrak{K}}$ of $\ddot{\mathfrak{S}}$ if $\ddot{\mathfrak{K}}$ is a subsemigroup of $\ddot{\mathfrak{S}}$ and $\ddot{\mathfrak{S}}\ddot{\mathfrak{K}}\ddot{\mathfrak{S}} \subseteq \ddot{\mathfrak{K}}$. A semigroup $\ddot{\mathfrak{S}}$ is called a *regular* if for each $\ddot{u} \in \ddot{\mathfrak{S}}$, there exists $\ddot{x} \in \ddot{\mathfrak{S}}$ such that $\ddot{u} = \ddot{u}\ddot{x}\ddot{u}$. A semigroup $\ddot{\mathfrak{S}}$ is called a *left* (right) regular if for each $\ddot{u} \in \ddot{\mathfrak{S}}$, there exists $\ddot{a} \in \ddot{\mathfrak{S}}$ such that $\ddot{u} = \ddot{a}\ddot{u}^2$ (resp. $\ddot{u} = \ddot{u}^2\ddot{a}$). A semigroup $\ddot{\mathfrak{S}}$ called an *intra-regular* if for each $\ddot{u} \in \ddot{\mathfrak{S}}$, there exist $\ddot{a}, \ddot{b} \in \ddot{\mathfrak{S}}$ such that $\ddot{u} = \ddot{a}\ddot{u}^2 b$. A semigroup $\ddot{\mathfrak{S}}$ is called a *semisimple* if every ideal of $\ddot{\mathfrak{S}}$ is an idempotent. It is evident that $\ddot{\mathfrak{S}}$ is a semisimple if and only if $\ddot{u} \in (\ddot{\mathfrak{S}}\ddot{u}\ddot{\mathfrak{S}})(\ddot{\mathfrak{S}}\ddot{u}\ddot{\mathfrak{S}})$ for every $\ddot{u} \in \ddot{\mathfrak{S}}$, that is there exist $\ddot{w}, \ddot{y}, \ddot{z} \in \ddot{\mathfrak{S}}$ such that $\ddot{u} = \ddot{w}\ddot{u}\ddot{y}\ddot{u}\ddot{z}$.[15].

A **fuzzy set** (FS) [1] in $\ddot{\mathfrak{Y}}$ is an arbitrary function from $\ddot{\mathfrak{Y}}$ into $[0, 1]$. A FS δ on $\ddot{\mathfrak{S}}$ is called a **fuzzy subsemigroup** (Fss) on $\ddot{\mathfrak{S}}$ if it satisfies the following conditions: $\min\{\delta(\ddot{u}), \delta(\ddot{v})\} \leq \delta(\ddot{u}\ddot{v})$ for all $\ddot{u}, \ddot{v} \in \ddot{\mathfrak{S}}$. A FS δ on $\ddot{\mathfrak{S}}$ is called a **fuzzy left ideal** (Fli) on $\ddot{\mathfrak{S}}$ if it satisfies the following conditions: $\delta(\ddot{v}) \leq \delta(\ddot{u}\ddot{v})$ for all $\ddot{u}, \ddot{v} \in \ddot{\mathfrak{S}}$. A FS δ on $\ddot{\mathfrak{S}}$ is called a **fuzzy right ideal** (Fri) on $\ddot{\mathfrak{S}}$ if it satisfies the following conditions: $\delta(\ddot{u}) \leq \delta(\ddot{u}\ddot{v})$ for all $\ddot{u}, \ddot{v} \in \ddot{\mathfrak{S}}$. A **fuzzy ideal** (Fi) δ on $\ddot{\mathfrak{S}}$, if it is both Fli and a Fri of $\ddot{\mathfrak{S}}$. A **fuzzy interior ideal** (Fii) δ on $\ddot{\mathfrak{S}}$ if δ is a Fss of $\ddot{\mathfrak{S}}$ and $\delta(\ddot{v}) \leq \delta(\ddot{u}\ddot{v}\ddot{w})$ for all $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$.

An **intuitionistic fuzzy set** (IFS) [3] on $\ddot{\mathfrak{Y}}$ is an object having the form $\ddot{\mathfrak{P}} = \left\{(\ddot{u}, \delta(\ddot{u}), \eta(\ddot{u})) \,\middle|\, \ddot{u} \in \ddot{\mathfrak{Y}}\right\}$ when the functions $\delta : \ddot{\mathfrak{Y}} \to [0, 1]$ denote the degree of membership and $\eta : \ddot{\mathfrak{Y}} \to [0, 1]$ denote the degree of nonmembership, and $0 \leq \delta(\ddot{u}) + \eta(\ddot{u}) \leq 1$ for all $\ddot{u} \in \ddot{\mathfrak{Y}}$. We denote (δ, η) for the PFS $\{(\ddot{u}, \delta(\ddot{u}), \eta(\ddot{u})) \mid \ddot{u} \in \ddot{\mathfrak{Y}}\}$. An IFS (δ, η) on $\ddot{\mathfrak{S}}$ is called a **intuitionistic fuzzy subsemigroup** (IFss) of $\ddot{\mathfrak{S}}$ if $\min\{\delta(\ddot{u}), \delta(\ddot{v})\} \leq \delta(\ddot{u}\ddot{v})$ and $\max\{\eta(\ddot{u}), \eta(\ddot{v})\} \geq \eta(\ddot{u}\ddot{v})$ for all $\ddot{u}, \ddot{v} \in \ddot{\mathfrak{S}}$. An IFS (δ, η) on $\ddot{\mathfrak{S}}$ is called an **intuitionistic fuzzy left ideal** (IFli) of $\ddot{\mathfrak{S}}$ if $\delta(\ddot{v}) \leq \delta(\ddot{u}\ddot{v})$ and $\eta(\ddot{v}) \geq$

$\eta(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. An IFS (δ,η) on $\ddot{\mathfrak{S}}$ is called an **intuitionistic fuzzy right ideal** (iFri) of $\ddot{\mathfrak{S}}$ if $\delta(\ddot{u})\leq\delta(\ddot{u}\ddot{v})$ and $\eta(\ddot{u})\geq\eta(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. An IFS (δ,η) on $\ddot{\mathfrak{S}}$ is called an **intuitionistic fuzzy ideal** (IFi) of $\ddot{\mathfrak{S}}$, if it is both an IFli and an IFri of $\ddot{\mathfrak{S}}$. A PFS (δ,η) on $\ddot{\mathfrak{S}}$ is called an **intuitionistic fuzzy interior ideal** (IFii) of $\ddot{\mathfrak{S}}$ if (δ,η) is an IFss of $\ddot{\mathfrak{S}}$ and $\delta(\ddot{v})\leq\delta(\ddot{u}\ddot{v}\ddot{w})$ and $\eta(\ddot{v})\geq\eta(\ddot{u}\ddot{v}\ddot{w})$ for all $\ddot{u},\ddot{v},\ddot{w}\in\ddot{\mathfrak{S}}$.

A **Pythagorean fuzzy set** (PFS) [16], [4] on $\ddot{\mathfrak{Y}}$ is an object having the form $\ddot{\mathfrak{P}}=\left\{(\ddot{u},\delta(\ddot{u}),\eta(\ddot{u}))\,\middle|\,\ddot{u}\in\ddot{\mathfrak{Y}}\right\}$ when the functions $\delta:\ddot{\mathfrak{Y}}\to[0,1]$ denote the degree of membership and $\eta:\ddot{\mathfrak{Y}}\to[0,1]$ denote the degree of nonmembership, and $0\leq(\delta(\ddot{u}))^2+(\eta(\ddot{u}))^2\leq 1$ for all $\ddot{u}\in\ddot{\mathfrak{Y}}$. We denote (δ,η) for the PFS $\{(\ddot{u},\delta(\ddot{u}),\eta(\ddot{u}))\mid\ddot{u}\in\ddot{\mathfrak{Y}}\}$. Then $(\frac{\delta}{1+i},\frac{\eta}{1+i})$, $(\frac{i+\delta}{1+2j},\frac{i+\eta}{1+2j})$ and $(\frac{\delta}{1+i},\frac{\delta}{1+i})$ are PFSs in $\ddot{\mathfrak{Y}}$ for each FSs δ and η in $\ddot{\mathfrak{Y}}$ and positive integers i and j such that $i\leq j$. Thus, the concept of PFSs is an extension of the concept from FSs. A PFS (δ,η) on $\ddot{\mathfrak{S}}$ is called a **Pythagorean fuzzy subsemigroup** (PFss) of $\ddot{\mathfrak{S}}$ if $\min\{\delta(\ddot{u}),\delta(\ddot{v})\}\leq\delta(\ddot{u}\ddot{v})$ and $\max\{\eta(\ddot{u}),\eta(\ddot{v})\}\geq\eta(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. A PFS (δ,η) on $\ddot{\mathfrak{S}}$ is called a **Pythagorean fuzzy left ideal** (PFli) of $\ddot{\mathfrak{S}}$ if $\delta(\ddot{v})\leq\delta(\ddot{u}\ddot{v})$ and $\eta(\ddot{v})\geq\eta(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. A PFS (δ,η) on $\ddot{\mathfrak{S}}$ is called a **Pythagorean fuzzy right ideal** (PFri) of $\ddot{\mathfrak{S}}$ if $\delta(\ddot{u})\leq\delta(\ddot{u}\ddot{v})$ and $\eta(\ddot{u})\geq\eta(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. A PFS$(\delta,\eta)$ on $\ddot{\mathfrak{S}}$ is called a **Pythagorean fuzzy ideal** (PFi) of $\ddot{\mathfrak{S}}$, if it is both a PFli and a PFri of $\ddot{\mathfrak{S}}$. A PFS (δ,η) on $\ddot{\mathfrak{S}}$ is called a **Pythagorean fuzzy interior ideal** (PFii) of $\ddot{\mathfrak{S}}$ if (δ,η) is a PFss of $\ddot{\mathfrak{S}}$ and $\delta(\ddot{v})\leq\delta(\ddot{u}\ddot{v}\ddot{w})$ and $\eta(\ddot{v})\geq\eta(\ddot{u}\ddot{v}\ddot{w})$ for all $\ddot{u},\ddot{v},\ddot{w}\in\ddot{\mathfrak{S}}$.

By an interval number $\breve{t}$ we mean an interval $[t^-,t^+]$, where $0\leq t^-\leq t^+\leq 1$. We denote $\mathcal{C}([0,1])$ for the set of all interval numbers. For each elements $\breve{s}=[s^-,s^+]$, $\breve{t}=[t^-,t^+]\in\mathcal{C}([0,1])$, define the operations $\precsim$, $\succcurlyeq$, $=$, $\prec$, $\succ$, rmin and rmax as follows:

(1) $\breve{s}\precsim\breve{t}\Leftrightarrow s^-\leq t^-$ and $s^+\leq t^+$,
(2) $\breve{s}\succcurlyeq\breve{t}\Leftrightarrow s^-\geq t^-$ and $s^+\geq t^+$,
(3) $\breve{s}=\breve{t}\Leftrightarrow s^-=t^-$ and $s^+=t^+$,
(4) $\breve{s}\prec\breve{t}\Leftrightarrow\breve{s}\precsim\breve{t}$ and $\breve{s}\neq\breve{t}$,
(5) $\breve{s}\succ\breve{t}\Leftrightarrow\breve{s}\succcurlyeq\breve{t}$ and $\breve{s}\neq\breve{t}$,
(6) $\mathrm{rmin}\{\breve{s},\breve{t}\}=[\min\{s^-,t^-\},\min\{s^+,t^+\}]$,
(7) $\mathrm{rmax}\{\breve{s},\breve{t}\}=[\max\{s^-,t^-\},\max\{s^+,t^+\}]$.

An **interval-valued fuzzy set** (IvFS) [2] on $\ddot{\mathfrak{Y}}$ is defined to be a function $\breve{\pi}:\ddot{\mathfrak{Y}}\to\mathcal{C}([0,1])$, where $\breve{\pi}(\ddot{u})=[\pi^-(\ddot{u}),\pi^+(\ddot{u})]$ for all $\ddot{u}\in\ddot{\mathfrak{Y}}$, π^- and π^+ are FSs in $\ddot{\mathfrak{Y}}$ such that $\pi^-\leq\pi^+$. Thus, the concept of IvFSs is an extension of the concept of FSs. An IvFS $\breve{\pi}$ on $\ddot{\mathfrak{S}}$ is called an **interval-valued fuzzy subsemigroup** (IvFss) [17] of $\ddot{\mathfrak{S}}$ if $\mathrm{rmin}\{\breve{\pi}(\ddot{u}),\breve{\pi}(\ddot{v})\}\precsim\breve{\pi}(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. An IvFS $\breve{\pi}$ on $\ddot{\mathfrak{S}}$ is called an **interval-valued fuzzy left ideal** (IvFli) [17] of $\ddot{\mathfrak{S}}$ if $\breve{\pi}(\ddot{v})\precsim\breve{\pi}(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. An IvFS $\breve{\pi}$ on $\ddot{\mathfrak{S}}$ is called an **interval-valued fuzzy right ideal** (IvFri) [17] of $\ddot{\mathfrak{S}}$ if $\breve{\pi}(\ddot{u})\precsim\breve{\pi}(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. An IvFS $\breve{\pi}$ on $\ddot{\mathfrak{S}}$ is called an **interval-valued fuzzy ideal** (IvFI) [17] of $\ddot{\mathfrak{S}}$, if it is both qn IvFli and an IvFri. An IvFS $\breve{\pi}$ on $\ddot{\mathfrak{S}}$ is called an **interval-valued fuzzy bi-ideal** (IvFii) [17] of $\ddot{\mathfrak{S}}$ if $\breve{\pi}$ is an IvFss of $\ddot{\mathfrak{S}}$ and $\breve{\pi}(\ddot{v})\precsim\breve{\pi}(\ddot{u}\ddot{v}\ddot{w})$ for all $\ddot{u},\ddot{v},\ddot{w}\in\ddot{\mathfrak{S}}$.

A **hesitant fuzzy set** (HFS) [8], [18] on $\ddot{\mathfrak{Y}}$ is defined to be a function $\widetilde{\kappa}:\ddot{\mathfrak{Y}}\to\wp([0,1])$ when $\wp([0,1])$ is the set of all subsets of $[0,1]$. Then $\mathcal{C}([0,1])\subseteq\wp([0,1])$ and we see that every IvFS on $\ddot{\mathfrak{Y}}$ is a HFS on $\ddot{\mathfrak{Y}}$. Thus, the concept of HFSs is both a generalization of the concept of IvFSs, and an extension of the concept of FSs. A HFS $\widetilde{\kappa}$ is a **hesitant fuzzy subsemigroup** (HFss) [8] of $\ddot{\mathfrak{S}}$ if $\widetilde{\kappa}(\ddot{u})\cap\widetilde{\kappa}(\ddot{v})\subseteq\widetilde{\kappa}(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. A HFS $\widetilde{\kappa}$ is a **hesitant fuzzy left ideal** (HFli) [8] of $\ddot{\mathfrak{S}}$ if $\widetilde{\kappa}(\ddot{v})\subseteq\widetilde{\kappa}(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. A HFS $\widetilde{\kappa}$ is a **hesitant fuzzy right ideal** (HFri) [8] of $\ddot{\mathfrak{S}}$ if $\widetilde{\kappa}(\ddot{v})\subseteq\widetilde{\kappa}(\ddot{u}\ddot{v})$ for all $\ddot{u},\ddot{v}\in\ddot{\mathfrak{S}}$. A HFS $\widetilde{\kappa}$ is a **hesitant fuzzy ideal** (HFi) [8] of $\ddot{\mathfrak{S}}$ if it is both an HFli and an HFri of $\ddot{\mathfrak{S}}$. A HFS $\widetilde{\kappa}$ is a **hesitant fuzzy interior ideal** (HFii) [8] of $\ddot{\mathfrak{S}}$ if $\widetilde{\kappa}$ is a HFss of $\ddot{\mathfrak{S}}$ and $\widetilde{\kappa}(\ddot{v})\subseteq\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w})$ for all $\ddot{u},\ddot{v},\ddot{w}\in\ddot{\mathfrak{S}}$.

III. $(\inf,\sup)$-Hesitant Fuzzy interior ideals

In this section, we introduce the concept of $(\inf,\sup)$-hesitant fuzzy bi-ideals of semigroups, investigate its properties and give its examples. Later, we show that the concept is a general concept of interval-valued fuzzy bi-ideals of subsemigroups. Finally, we investigate characterizations of the concept of $(\inf,\sup)$-hesitant fuzzy bi-ideals of semigroups in terms of sets, FSs, negative fuzzy sets, PFSs, IvFSs and HFSs.

For each element $\Theta\in\wp([0,1])$ and HFS $\widetilde{\kappa}$ of $\ddot{\mathfrak{S}}$, define elements $\mathrm{SUP}\,\Theta$ and $\mathrm{INF}\,\Theta$ [13], [12] of $[0,1]$ and a subset $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$ of $\ddot{\mathfrak{Y}}$ as follows:

$$\mathrm{SUP}\,\Theta=\begin{cases}\sup\Theta & \text{if } \Theta\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathrm{INF}\,\Theta=\begin{cases}\inf\Theta & \text{if } \Theta\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]=\{\ddot{u}\in\ddot{\mathfrak{S}}\mid\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u})\geq\mathrm{SUP}\,\Theta\},$$

$$[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]=\{\ddot{u}\in\ddot{\mathfrak{S}}\mid\mathrm{INF}\,\widetilde{\kappa}(\ddot{u})\geq\mathrm{INF}\,\Theta\}.$$

For $\widetilde{\kappa}\in\mathrm{HFS}(\ddot{\mathfrak{K}})$ and $\Theta\in\wp([0,1])$ define the elemets $\mathrm{SUP}\,\Theta$ and $\mathrm{INF}\,\Theta$ of $[0,1]$ the subsets $[\widetilde{\kappa};\Theta]_{\mathrm{SUP}}$ and $[\widetilde{\kappa};\Theta]_{\mathrm{INF}}$ of $\ddot{\mathfrak{K}}$ of the FS $\mathcal{F}^{\widetilde{\kappa}}$ of $\ddot{\mathfrak{K}}$ and the HFS $\mathcal{H}_{\mathrm{SUP}\,(\widetilde{\kappa};\Theta)}$ and $\mathcal{H}_{\mathrm{INF}\,(\widetilde{\kappa};\Theta)}$ on $\ddot{\mathfrak{K}}$ by

(1) $[\widetilde{\kappa};\Theta]_{\mathrm{SUP}}=\{\ddot{u}\in\ddot{\mathfrak{K}}\mid\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u})\geq\mathrm{SUP}\,\Theta\}$ and $[\widetilde{\kappa};\Theta]_{\mathrm{INF}}=\{\ddot{u}\in\ddot{\mathfrak{K}}\mid\mathrm{INF}\,\widetilde{\kappa}(\ddot{u})\geq\mathrm{INF}\,\Theta\}$.
(2) $\mathcal{F}^{\widetilde{\kappa}}(\ddot{u})=(\mathrm{INF}\,,\mathrm{SUP}\,)\widetilde{\kappa}(\ddot{u})$ for all $\ddot{u}\in\ddot{\mathfrak{K}}$.
(3) $\mathcal{H}_{\mathrm{SUP}\,(\widetilde{\kappa};\Theta)}(\ddot{u})=\{\ddot{t}\in\Theta\mid\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u})\geq\ddot{t}\}$ and $\mathcal{H}_{\mathrm{INF}\,(\widetilde{\kappa};\Theta)}(\ddot{u})=\{\ddot{t}\in\Theta\mid\mathrm{INF}\,\widetilde{\kappa}(\ddot{u})\geq\ddot{t}\}$ for all $\ddot{u}\in\ddot{\mathfrak{K}}$.

Defind $\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};[0,1])$ by $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}$ and $\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};[0,1])$ by $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}$. Then the $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}$ and $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}$ are elements in $\mathrm{IVFS}(\ddot{\mathfrak{K}})$.

Definition 3.1. *[14] A HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{S}}$ is said to be an* $(\inf,\sup)$***-hesitant fuzzy ideal*** *(($\inf$, $\sup$)-HFi) of $\ddot{\mathfrak{S}}$ if the set $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$ is an interior ideal of $\ddot{\mathfrak{S}}$ for all $\Theta\in\wp([0,1])$ when $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]\neq\emptyset$.*

Definition 3.2. *A HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{S}}$ is said to be an* $(\inf,\sup)$***-hesitant fuzzy interior ideal*** *(($\inf$, $\sup$)-HFii) of $\ddot{\mathfrak{S}}$ if the set $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$ is an interior ideal of $\ddot{\mathfrak{S}}$ for all $\Theta\in\wp([0,1])$ when $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]\neq\emptyset$.*

For any HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{Y}}$, define the FSs $\mathcal{F}^{\widetilde{\kappa}}$ and $\mathcal{F}_{\widetilde{\kappa}}$ in $\ddot{\mathfrak{Y}}$ by $\mathcal{F}^{\widetilde{\kappa}}(\ddot{u})=\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u})$, and $\mathcal{F}_{\widetilde{\kappa}}(\ddot{u})=\mathrm{INF}\,\widetilde{\kappa}(\ddot{u})$ for all $\ddot{u}\in\ddot{\mathfrak{Y}}$. A HFS $\widetilde{\varepsilon}$ on $\ddot{\mathfrak{Y}}$ is called a **supremum complement** of $\widetilde{\kappa}$ on $\ddot{\mathfrak{Y}}$ if $\mathrm{SUP}\,\widetilde{\varepsilon}(\ddot{u})=(1-\mathcal{F}^{\widetilde{\kappa}})(\ddot{u})$ for all $\ddot{u}\in\ddot{\mathfrak{Y}}$ and called an **infimum complement** [13] of $\widetilde{\kappa}$ on $\ddot{\mathfrak{Y}}$ if $\mathrm{INF}\,\widetilde{\varepsilon}(\ddot{u})=(1-\mathcal{F}_{\widetilde{\kappa}})(\ddot{u})$ for all $\ddot{u}\in\ddot{\mathfrak{Y}}$. The set of all supremum complements of $\widetilde{\kappa}$ is denoted by $\mathrm{SC}(\widetilde{\kappa})$ and the set of all infimum complements of $\widetilde{\kappa}$ is denoted by $\mathrm{IC}(\widetilde{\kappa})$. Define the HFSs $\widetilde{\kappa}^{\pm}$ and $\widetilde{\kappa}^{\mp}$ on $\ddot{\mathfrak{Y}}$ by $\widetilde{\kappa}^{\pm}(\ddot{u}) = \{(1-\mathcal{F}_{\widetilde{\kappa}})(\ddot{u})\}$ and $\widetilde{\kappa}^{\mp}(\ddot{u}) = \{(1-\mathcal{F}^{\widetilde{\kappa}})(\ddot{u})\}$ for all $\ddot{u} \in \ddot{\mathfrak{Y}}$. Then $\widetilde{\kappa}^{\pm} \in \mathrm{IC}(\widetilde{\kappa})$ and $\widetilde{\kappa}^{\mp} \in \mathrm{SC}(\widetilde{\kappa})$. Moreover, $\mathcal{F}_{\widetilde{\kappa}^{\pm}} = \mathcal{F}_{\widetilde{\varepsilon}} = 1 - \mathrm{F}_{\widetilde{\kappa}}$ for each $\varepsilon \in \mathrm{IC}(\widetilde{\kappa})$ and $\mathcal{F}^{\widetilde{\kappa}^{\mp}} = \mathcal{F}^{\widetilde{\tau}} = 1 - \mathrm{F}^{\widetilde{\kappa}}$ for each $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\kappa})$. Next, we investigate characterizations of $(\inf,\sup)$-HFbi of semigroups in terms of $\ddot{\mathfrak{S}}$.

Lemma 3.3. *Every HFii of $\ddot{\mathfrak{S}}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$.*

Proof: Suppose that $\widetilde{\kappa}$ is a HFii of $\ddot{\mathfrak{S}}$ and $\Theta \in \mathcal{P}([0,1])$ such that $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta] \neq \emptyset$. Let $\ddot{u},\ddot{w} \in \ddot{\mathfrak{S}}$ and $v \in [\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$. Then $\widetilde{\kappa}(\ddot{v}) \subseteq \widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w})$. Thus,

$$\mathrm{INF}\,\Theta \leq \mathrm{INF}\,\widetilde{\kappa}(\ddot{v}) \leq \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}).$$

Similarly, we can show that $\mathrm{SUP}\,\Theta \leq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w})$. Hecne, $\ddot{u}\ddot{v}\ddot{w} \in [\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$ so $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$ is an interior ideal of $\ddot{\mathfrak{S}}$. Therefore, $\widetilde{\kappa}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$. ∎

Lemma 3.4. *Every IvFii of $\ddot{\mathfrak{S}}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$.*

Proof: Suppose that $\breve{\pi}$ is an IvFii of $\ddot{\mathfrak{S}}$ and $\Theta \in \mathcal{P}([0,1])$ such that $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta] \neq \emptyset$. Let $\ddot{u},\ddot{w} \in \ddot{\mathfrak{S}}$ and $v \in [\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$. Then $\breve{\pi}(\ddot{v}) \precsim \breve{\pi}(\ddot{u}\ddot{v}\ddot{w})$. Thus,

$$\begin{aligned}\mathrm{INF}\,\Theta\& \leq \mathrm{INF}\\ \breve{\pi}(\ddot{u}\ddot{v}\ddot{w}) &= \breve{\pi}^{-}(\ddot{u}\ddot{v}\ddot{w})\\ &\leq \breve{\pi}^{-}(\ddot{v})\\ &= \min\{\mathrm{INF}\,\breve{\pi}(\ddot{u}\ddot{v}\ddot{w}), \mathrm{INF}\,\breve{\pi}(\ddot{u}\ddot{v}\ddot{w})\}.\end{aligned}$$

Similarly, we can show that $\mathrm{SUP}\Theta \geq \max\{\mathrm{SUP}\breve{\pi}(\ddot{u}\ddot{v}\ddot{w}), \mathrm{SUP}\breve{\pi}(\ddot{u}\ddot{v}\ddot{w})\}$. Thus, $\ddot{u}\ddot{v}\ddot{w} \in [\breve{\varsigma},\breve{\pi},\Theta]$. Hence, $[\ddot{\mathfrak{S}},\widetilde{\kappa},\Theta]$ is an interior ideal of $\ddot{\mathfrak{S}}$. Therefore, $\breve{\varsigma}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$. ∎

Example 3.5. *Consider a semigroup $(S,\cdot)$ defined by the following table:*

$\cdot$	$\ddot{w}$	$\ddot{x}$	$\ddot{y}$	$\ddot{z}$
$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$
$\ddot{x}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$
$\ddot{y}$	$\ddot{w}$	$\ddot{w}$	$\ddot{w}$	$\ddot{x}$
$\ddot{z}$	$\ddot{w}$	$\ddot{w}$	$\ddot{x}$	$\ddot{y}$

(1) *Define a HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{S}}$ by $\widetilde{\kappa}(\ddot{w}) = (0,1)\,\widetilde{\kappa}(\ddot{x}) = [0,1]\,\widetilde{\kappa}(\ddot{y}) = \breve{0}$ and $\widetilde{\kappa}(\ddot{z}) = \emptyset$. Then $\widetilde{\kappa}$ is an (inf-sup)-HFii of $\ddot{\mathfrak{S}}$ but it is not a HFii of $\ddot{\mathfrak{S}}$, since $\widetilde{\kappa}(\ddot{x}) = [0,1] \nsubseteq (0,1) = \widetilde{\kappa}(\ddot{x}) = \widetilde{\kappa}(\ddot{y}\ddot{x}\ddot{z})$.*

(2) *Define an IvFS $\breve{\pi}$ on $\ddot{\mathfrak{S}}$ by $\breve{\pi}(\ddot{w}) = [0,1]\,\breve{\pi}(\ddot{x}) = \breve{1}\,\breve{\pi}(\ddot{y}) = \breve{\pi}(\ddot{z}) = \breve{0}$. Then $\breve{\pi}$ is an $(\inf-\sup)$-IVFii of $\ddot{\mathfrak{S}}$ but it is not a IvFii of $\ddot{\mathfrak{S}}$, since $\breve{\pi}(\ddot{x}^3) = \breve{\pi}(\ddot{w}) = [0,1] \precsim \breve{1} = \breve{\pi}(\ddot{x})$.*

For $\widetilde{\kappa}$ be a HFS on $\ddot{\mathfrak{K}}$ and $\Theta \in \mathcal{P}([0,1])$, defined $\mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta) : \ddot{\mathfrak{K}} \to \mathcal{P}([0,1])$ by $\mathcal{V}(\widetilde{\kappa},\Theta)(\ddot{u}) = \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) \geq \Theta$ and $\mathcal{V}(\widetilde{\kappa},\Theta)(\ddot{u}) = \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}) \geq \Theta$ for all $\ddot{u} \in \ddot{\mathfrak{K}}$.

Theorem 3.6. *A HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{S}}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)$ is a HFii of $\ddot{\mathfrak{S}}$ for all $\Theta \in \mathcal{P}([0,1])$.*

Proof: Let $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$ and $\Theta \in \mathcal{P}([0,1])$. If $\mathcal{V}_{\mathrm{INF,SUP}}(\widetilde{\kappa},\Theta) = \emptyset$, then $\mathcal{V}_{\mathrm{INF,SUP}}(\widetilde{\kappa},\Theta)(\ddot{v}) \subseteq \mathcal{V}_{\mathrm{INF,SUP}}(\widetilde{\kappa},\Theta)(\ddot{u}\ddot{v}\ddot{w})$. Let $\ddot{r} \in \mathcal{V}_{\mathrm{INF,SUP}}(\widetilde{\kappa},\Theta)(\ddot{v})$ Since $\ddot{v} \in [\widetilde{\kappa},\widetilde{\kappa}(\ddot{v})]$ and $\widetilde{\kappa}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$ we have $\ddot{u}\ddot{v}\ddot{w} \in [\widetilde{\kappa},\widetilde{\kappa}(\ddot{v})]$.Thus,

$$\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\widetilde{\kappa}(\ddot{v}) \geq \ddot{r} \in \Theta$$

and

$$\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\widetilde{\kappa}(\ddot{v}) \geq \ddot{r} \in \Theta.$$

It implies that $\ddot{r} \in \mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)(\ddot{u}\ddot{v}\ddot{w})$. Hence, $\mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)(\ddot{v}) \subseteq \mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)(\ddot{u}\ddot{v}\ddot{w})$. Therefore, $\mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)(\ddot{u}\ddot{v}\ddot{w})$ is a HFii of $\ddot{\mathfrak{S}}$.

For the converse, let $\ddot{u},\ddot{w} \in \ddot{\mathfrak{S}}, \ddot{v} \in [\widetilde{\kappa},\Theta]$ and $\Theta \in \mathcal{P}([0,1])$. Then $\mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)(\ddot{v}) = \Theta$ and $\mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)$ is a HFii of $\ddot{\mathfrak{S}}$. Thus, $\Theta \subseteq \mathcal{V}_{\mathrm{INF},\mathrm{SUP}}(\widetilde{\kappa},\Theta)(\ddot{u}\ddot{v}\ddot{w})$. Hence, $\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\,\Theta$ and $\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\,\Theta$ so $\ddot{u}\ddot{v}\ddot{w} \in [\widetilde{\kappa},\widetilde{\kappa}(\ddot{v})]$. Therefore, $[\widetilde{\kappa},\widetilde{\kappa}(\ddot{v})]$ is an interior ideal of $\ddot{\mathfrak{S}}$. We conclude that $\widetilde{\kappa}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$. ∎

Lemma 3.7. *If $\breve{\pi} \in HFS^{*}(\ddot{\mathfrak{S}})$ is a HFii of $\ddot{\mathfrak{S}}$, then $\breve{\varsigma}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$ for all $\breve{\varsigma} \in \mathrm{IC}(\breve{\pi})$.*

Proof: Suppose that $\breve{\pi} \in HFS^{*}(\ddot{\mathfrak{S}})$ is a HFii of $\ddot{\mathfrak{S}}$ and $\breve{\varsigma} \in \mathrm{IC}(\breve{\pi})$. Let $\Theta \in \wp([0,1]), \ddot{u},\ddot{w} \in \ddot{\mathfrak{S}}, \ddot{v} \in [\breve{\varsigma},\widetilde{\kappa},\Theta]$. Then $\breve{\pi}(\ddot{v}) \subseteq \breve{\pi}(\ddot{u}\ddot{v}\ddot{w})$. Thus, $\mathrm{INF}\,\breve{\pi}(\ddot{v}) \leq \mathrm{INF}\,\breve{\pi}(\ddot{u}\ddot{v}\ddot{w})$ and $\mathrm{SUP}\,\breve{\pi}(\ddot{v}) \geq \mathrm{SUP}\,\breve{\pi}(\ddot{u}\ddot{v}\ddot{w})$. It follows that

$$\begin{aligned}\mathrm{INF}\,\Theta &\leq \mathrm{INF}\,\breve{\varsigma}(\ddot{v})\\ &= 1 - \mathrm{INF}\,\breve{\varsigma}(\ddot{v})\\ &\leq 1 - \mathrm{INF}\,\breve{\pi}(\ddot{u}\ddot{v}\ddot{w})\\ &= \mathrm{INF}\,\breve{\varsigma}(\ddot{u}\ddot{v}\ddot{w}).\end{aligned}$$

Hence, $\mathrm{INF}\,\Theta \leq \mathrm{INF}\,\breve{\varsigma}(\ddot{u}\ddot{v}\ddot{w})$. Similarly, we can show that $\mathrm{SUP}\,\Theta \geq \mathrm{SUP}\,\breve{\varsigma}(\ddot{u}\ddot{v}\ddot{w})$. Thus, $\ddot{u}\ddot{v}\ddot{w} \in [\breve{\varsigma},\breve{\pi},\Theta]$. Hence, $[\breve{\delta},\breve{\varsigma},\Theta]$ is an interior ideal of $\ddot{\mathfrak{S}}$. Therefore, $\breve{\varsigma}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$. ∎

The following theorems we can proved by relationship between of ideal and interior ideal in semigroup.

Theorem 3.8. *In regular, left (right) regular, intra-regular and semisimple semigroup, the $(\inf,\sup)$-HFi and $(\inf,\sup)$-HFii coincide.*

Theorem 3.9. *In regular, left (right) regular, intra-regular and semisimple semigroup, the IvFi and $(\inf,\sup)$-hesitant fuzzy interior ideal coincide.*

Theorem 3.10. *In regular, left (right) regular, intra-regular and semisimple semigroup, the HFi and $(\inf,\sup)$-hesitant fuzzy interior ideal coincide.*

Lemma 3.11. *Let $\breve{\pi} \in HFS(\ddot{\mathfrak{S}})$. Then the following are equivalent:*

(1) *$\widetilde{\kappa}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$.*
(2) *$\mathcal{F}_{\widetilde{\kappa}}$ is a Fii of $\ddot{\mathfrak{S}}$.*
(3) *$\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\,\widetilde{\kappa}(\ddot{v})$ and $\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v})$ for all $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$.*
(4) *$\mathrm{INF}\,\breve{\varsigma}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\,\breve{\varsigma}(\ddot{v})$, and $\mathrm{SUP}\,\breve{\varsigma}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\,\breve{\varsigma}(\ddot{v})$ for all $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$ and $\breve{\varsigma} \in IC(\widetilde{\kappa})$.*
(5) *$\mathrm{INF}\,\widetilde{\kappa}^{*}(\ddot{u}\ddot{v}\ddot{w}) \leq \mathrm{INF}\,\widetilde{\kappa}(\ddot{v})$ and $\mathrm{SUP}\,\widetilde{\kappa}^{*}(\ddot{u}\ddot{v}\ddot{w}) \leq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v})$ for all $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$.*

Proof: (1) $\Rightarrow$ (3) Let $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$. Then $\ddot{u}$ is an element of $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{u})]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{u})]_{\mathrm{SUP}}$, $\ddot{v}$ is an element of $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{v})]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{v})]_{\mathrm{SUP}}$, $\ddot{w}$ is an element of $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{w})]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{w})]_{\mathrm{SUP}}$. Thus, $\ddot{u}\ddot{v}\ddot{w}$ is an element of $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{u})]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{u})]_{\mathrm{SUP}}$, $\ddot{u}\ddot{v}\ddot{w}$ is an element of $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{v})]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{v})]_{\mathrm{SUP}}$, $\ddot{u}\ddot{v}\ddot{w}$ is an element of $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{w})]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\widetilde{\kappa}(\ddot{v})]_{\mathrm{SUP}}$. Hence, $\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\,\widetilde{\kappa}(\ddot{v})$ and $\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v})$.

(3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) They are obious.

(2) $\Rightarrow$ (1) Let $\Theta \in \wp([0,1])$, $\ddot{v} \in \ddot{\mathfrak{S}}$ $\ddot{u},\ddot{w} \in [\widetilde{\kappa};\Theta]_{\mathrm{INF}}$ and $\widetilde{\kappa},\ddot{w} \in [\widetilde{\kappa};\Theta]_{\mathrm{SUP}}$. Then

$$\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) = \mathcal{F}_{\widetilde{\kappa}}(\ddot{u}\ddot{v}\ddot{w}) \geq \widetilde{\kappa}(\ddot{v}) = \mathrm{INF}\,\widetilde{\kappa}(\ddot{v}) \geq \mathrm{INF}\,\Theta$$

and

$$\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) = \mathcal{F}_{\widetilde{\kappa}}(\ddot{u}\ddot{v}\ddot{w}) \geq \widetilde{\kappa}(\ddot{v}) = \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v}) \geq \mathrm{SUP}\,\Theta.$$

Thus, $\ddot{u}\ddot{v}\ddot{w}$ are elements $[\widetilde{\kappa};\Theta]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\Theta]_{\mathrm{SUP}}$. Hence, $[\widetilde{\kappa};\Theta]_{\mathrm{INF}}$ and $[\widetilde{\kappa};\Theta]_{\mathrm{SUP}}$ are interior ideal of $\ddot{\mathfrak{S}}$. Therefore, $\widetilde{\kappa}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$.

(3) $\Rightarrow$ (4) Let $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$ and $\breve{\zeta} \in IC(\widetilde{\kappa})$. Then $\mathrm{INF}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\,\breve{\zeta}(\ddot{v})$ and $\mathrm{SUP}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\,\breve{\zeta}(\ddot{v})$. Thus,

$$\mathrm{INF}\,\breve{\zeta}(\ddot{v}) = 1 - \mathrm{INF}\,\breve{\zeta}(\ddot{v}) \leq 1 - \mathrm{INF}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w}) = \mathrm{INF}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w})$$

and

$$\mathrm{SUP}\,\breve{\zeta}(\ddot{v}) = 1 - \mathrm{SUP}\,\breve{\zeta}(\ddot{v}) \leq 1 - \mathrm{SUP}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w}) = \mathrm{SUP}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w}).$$

Hence, $\mathrm{INF}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\,\breve{\zeta}(\ddot{v})$ and $\mathrm{SUP}\,\breve{\zeta}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\,\breve{\zeta}(\ddot{v})$.

(5) $\Rightarrow$ (3) Let $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$. Then $\mathrm{INF}\,\widetilde{\kappa}^*(\ddot{u}\ddot{v}\ddot{w}) \leq \mathrm{INF}\,\widetilde{\kappa}(\ddot{v})$, and $\mathrm{SUP}\,\widetilde{\kappa}^*(\ddot{u}\ddot{v}\ddot{w}) \leq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v})$. Thus,

$$\begin{aligned}\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) &= 1 - (1 - \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w})) \\ &= 1 - \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \\ &\geq 1 - \mathrm{INF}\,\widetilde{\kappa}(\ddot{v}), \mathrm{INF}\,\widetilde{\kappa}(\ddot{w}) \\ &= 1 - (1 - \mathrm{INF}\,\widetilde{\kappa}(\ddot{v})) \\ &= \mathrm{INF}\,\widetilde{\kappa}(\ddot{v}).\end{aligned}$$

Hence, $\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{INF}\,\widetilde{\kappa}(\ddot{v})$. Similarly, we can show that $\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) \geq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v})$. ■

The following result is an immediate consequence of Lemma 3.4 and 3.11.

Theorem 3.12. *If an IVFS $\breve{\pi}$ of $\ddot{\mathfrak{S}}$ is an IVFii of $\ddot{\mathfrak{S}}$, then $\mathcal{F}_{\widetilde{\kappa}}$ is an interior ideal of $\ddot{\mathfrak{S}}$.*

The following result is an immediate consequence of Lemma 3.7 and 3.11

Theorem 3.13. *If $\widetilde{\kappa} \in \mathrm{HFS}^*(\ddot{\mathfrak{S}})$ is an HFii of $\ddot{\mathfrak{S}}$, then $\mathcal{F}_{\breve{\zeta}}$ is an interior ideal of $\ddot{\mathfrak{S}}$ for all $\breve{\zeta} \in IC(\widetilde{\kappa})$.*

For PFS (δ,η) in $\ddot{\mathfrak{S}}$ and every element $\breve{\zeta} \in \mathcal{P}([0,1])$, define the HFS $\mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}$ and the IvFS $\mathcal{I}_{(\delta,\eta)}$ on $\ddot{\mathfrak{S}}$ by for all $\ddot{u} \in \ddot{\mathfrak{S}}$.

$$\mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}(\ddot{u}) = \left\{\ddot{t} \in \breve{\zeta} \mid \frac{(\delta(\ddot{u}))^2}{2} \leq \ddot{t} \leq \frac{1+(\eta(\ddot{u}))^2}{2}\right\}$$

and

$$\mathcal{I}_{(\delta,\eta)}(\ddot{u}) = \left[\frac{1+(\delta(\ddot{u}))^2}{2}, \frac{1+(\eta(\ddot{u}))^2}{2}\right].$$

The following theorems we study the characterizations of a PFbi (δ,η) in a semigroup via HFbi $\mathcal{H}^{\breve{\delta}}_{(\delta,\eta)}$ and the IvFS $I_{(\delta,\eta)}$.

Theorem 3.14. *Let (δ,η) be a PFS in $\ddot{\mathfrak{S}}$. Then (δ,η) is a PFii of $\ddot{\mathfrak{S}}$ if and only if $\mathcal{H}^{\breve{\zeta}}_{((\delta,\eta)}$ is a HFii of $\ddot{\mathfrak{S}}$ for all $\breve{\zeta} \in \mathcal{P}([0,1])$.*

Proof: Suppose that (δ,η) is a PFii of $\ddot{\mathfrak{S}}$. Let $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$ and $\breve{\zeta} \in \mathcal{P}([0,1])$ and $\omega \in \mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}(\ddot{v})$. Then $\omega \in \breve{\zeta}$ and $\frac{(\delta(\ddot{v}))^2}{2} \geq \omega \geq \frac{1+(\eta(\ddot{v}))^2}{2}$. By assumption, $\delta(\ddot{v}) \leq \delta(\ddot{u}\ddot{v}\ddot{w})$ and $\eta(\ddot{v}) \geq \eta(\ddot{u}\ddot{v}\ddot{w})$.
Since $\delta(\ddot{u}\ddot{v}\ddot{w}), \delta(\ddot{v}), \eta(\ddot{u}\ddot{v}\ddot{w}), \eta(\ddot{v}) \in [0,1]$ we have $(\delta(\ddot{v}))^2 \leq (\delta(\ddot{u}\ddot{v}\ddot{w}))^2$ and $(\eta(\ddot{v}))^2 \geq (\eta(\ddot{u}\ddot{v}\ddot{w}))^2$. Thus,

$$\frac{(\delta(\ddot{u}\ddot{v}\ddot{w}))^2}{2} \geq \frac{(\delta(\ddot{v}))^2}{2} \geq \omega \geq \frac{1+(\eta(\ddot{v}))^2}{2} \geq \frac{1+(\eta(\ddot{u}\ddot{v}\ddot{w}))^2}{2}.$$

Hence, $\omega \in \mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}(\ddot{u}\ddot{v}\ddot{w})$. Therefore, $\mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}(\ddot{v}) \subseteq \mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}(\ddot{u}\ddot{v}\ddot{w})$. We conclude that $\mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}$ is a HFii of $\ddot{\mathfrak{S}}$.

For the converse, suppoe that $\mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}$ is a HFii of $\ddot{\mathfrak{S}}$ and $\delta(\ddot{v}) > \delta(\ddot{u}\ddot{v}\ddot{w})$. Then there exist $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$ such that $\delta(\ddot{v}) > \delta(\ddot{u}\ddot{v}\ddot{w})$. Since $\delta(\ddot{u}\ddot{v}\ddot{w}), \delta(\ddot{v}) \in [0,1]$ we have $(\delta(\ddot{v}))^2 > (\delta(\ddot{u}\ddot{v}\ddot{w}))^2$.
Choose $\tau := \frac{1}{4}\left((\delta(\ddot{u}\ddot{v}\ddot{w}))^2 + (\delta(\ddot{v}))^2\right)$. Then $\frac{\delta(\ddot{u}\ddot{v}\ddot{w}))^2}{2} < \tau < \frac{(\delta(\ddot{v}))^2}{2}$. Thus,

$$\frac{1+(\delta(\ddot{u}\ddot{v}\ddot{w}))^2}{2} = \frac{1}{2} + \frac{1+(\delta(\ddot{u}\ddot{v}\ddot{w}))^2}{2} \leq \frac{1}{2} + \tau$$

and

$$\frac{(\delta(\ddot{v}))^2}{2} \leq \frac{1}{2} < \frac{1}{2} + \tau < \frac{1+(\delta(\ddot{v}))^2}{2}$$

It implies that $\frac{1}{2} + \tau \notin \mathcal{H}^{[0,1]}_{(\delta,\eta)}(\ddot{u}\ddot{v}\ddot{w})$ and $\frac{1}{2} + \ddot{u} \in \mathcal{H}^{[0,1]}_{(\delta,\eta)}(\ddot{v})$. By assumption, $\mathcal{H}^{[0,1]}_{(\delta,\eta)}$ is a HFii of $\ddot{\mathfrak{S}}$. Thus, $\frac{1}{2} + \ddot{u} \in \mathcal{H}^{[0,1]}_{(\delta,\eta)}(\ddot{u}\ddot{v}\ddot{w})$. It is a contradiction. Hence, $\delta(\ddot{v}) \leq \delta(\ddot{u}\ddot{v}\ddot{w})$. Similarly, we can show that $\eta(\ddot{v}) \geq \eta(\ddot{u}\ddot{v}\ddot{w})$. Therefore, (δ,η) is a PFii of $\ddot{\mathfrak{S}}$. ■

Theorem 3.15. *Let (δ,η) be a PFS in $\ddot{\mathfrak{S}}$. Then (δ,η) is a PFii of $\ddot{\mathfrak{S}}$ if and only if $\mathcal{I}_{(\delta,\eta)}$ is an IvFii of $\ddot{\mathfrak{S}}$.*

Proof: Suppose that (δ,η) is a PFii of $\ddot{\mathfrak{S}}$. Let $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$. Then $\delta(\ddot{v}) \leq \delta(\ddot{u}\ddot{v}\ddot{w})$ and $\eta(\ddot{v}) \geq \eta(\ddot{u}\ddot{v}\ddot{w})$. Since $\delta(\ddot{u}\ddot{v}\ddot{w}), \delta(\ddot{v}), \eta(\ddot{u}\ddot{v}\ddot{w}), \eta(\ddot{v}) \in [0,1]$ we have $(\delta(\ddot{v}))^2, \leq (\delta(\ddot{u}\ddot{v}\ddot{w}))^2$ and $(\eta(\ddot{v}))^2 \geq (\eta(\ddot{u}\ddot{v}\ddot{w}))^2$. Thus, $\frac{1-(\delta(\ddot{v}))^2}{2} \leq \frac{1-(\delta(\ddot{u}\ddot{v}\ddot{w}))^2}{2}$ and $\frac{1-(\eta(\ddot{v}))^2}{2} \leq \frac{1-(\eta(\ddot{u}\ddot{v}\ddot{w}))^2}{2}$. Hence, $\mathcal{I}_{(\delta,\eta)}(\ddot{v}) \precsim \mathcal{I}_{(\delta,\eta)}(\ddot{u}\ddot{v}\ddot{w})$. Therefore, $\mathcal{I}_{(\delta,\eta)}$ is an IvFii of $\ddot{\mathfrak{S}}$.

For the converse, suppoe that $\mathcal{I}_{(\delta,\eta)}$ is an IvFii of $\ddot{\mathfrak{S}}$ and $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$. Then $\mathcal{I}_{(\delta,\eta)}(\ddot{v}) \precsim \mathcal{I}_{(\delta,\eta)}(\ddot{u}\ddot{v}\ddot{w})$. Thus, $\frac{1-\max\{(\delta(\ddot{v}))^2}{2} = \frac{1-(\delta(\ddot{v}))^2}{2} \leq \frac{1-(\delta(\ddot{u}\ddot{v}\ddot{w}))^2}{2}$ and $\frac{1-\max\{(\eta(\ddot{v}))^2}{2} = \frac{1-(\eta(\ddot{v}))^2}{2} \leq \frac{1-(\eta(\ddot{u}\ddot{v}\ddot{w}))^2}{2}$. Hecne $(\delta(\ddot{u}\ddot{v}\ddot{w}))^2 \geq (\delta(\ddot{v}))^2$ and $(\eta(\ddot{u}\ddot{v}\ddot{w}))^2 \leq (\eta(\ddot{v}))^2$. Since $\delta(\ddot{u}\ddot{v}\ddot{w}), \delta(\ddot{v}), \eta(\ddot{u}\ddot{v}\ddot{w}), \eta(\ddot{v}) \in [0,1]$ we have $\delta(\ddot{u}\ddot{v}\ddot{w}) \geq \delta(\ddot{v})$ and $\eta(\ddot{u}\ddot{v}\ddot{w}) \leq \eta(\ddot{v})$. Thus, (δ,η) is a PFii of $\ddot{\mathfrak{S}}$. ■

Corollary 3.16. *Let (δ,η) is a PFii of $\ddot{\mathfrak{S}}$. Then the following statements hold.*

(1) *$\mathcal{H}^{\breve{\zeta}}_{(\delta,\eta)}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$ for all $\breve{\zeta} \in \mathcal{P}([0,1])$.*
(2) *$\mathcal{I}_{(\delta,\eta)}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$*

Proof: It follows from Lemma 3.4 and 3.7, Theorem 3.14 and 3.15. ■

For HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{S}}$ and $\ddot{t} \in [0,1]$ define

$\mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) = \{\ddot{u} \in \ddot{\mathfrak{S}} \mid \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}) \geq t\}$, $\mathfrak{U}_{\mathrm{SUP}}(\widetilde{\kappa};\ddot{t}) = \{\ddot{u} \in \ddot{\mathfrak{S}} \mid \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) \geq t\}$ and $\mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) = \{\ddot{u} \in \ddot{\mathfrak{S}} \mid \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) \leq t\}$, $\mathfrak{L}_{\mathrm{SUP}}(\widetilde{\kappa};\ddot{t}) = \{\ddot{u} \in \ddot{\mathfrak{S}} \mid \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) \leq t\}$.

We call the $\mathfrak{U}_{\mathrm{INF}}$ is a INF-*upper t-level subset*, call the $\mathfrak{U}_{\mathrm{SUP}}$ is a SUP -*upper $\ddot{t}$-level subset*, call the $\mathfrak{L}_{\mathrm{INF}}$ is a INF-*lower $\ddot{t}$-level subset*and call the $\mathfrak{L}_{\mathrm{SUP}}$ is a SUP -*lower $\ddot{t}$-level subset* [19] of $\widetilde{\kappa}$.

Theorem 3.17. *Let h is a HFS on $\ddot{\mathfrak{S}}$. Then the following statements holds.*

(1) *$\widetilde{\kappa}$ is an inf-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t})$ is either empty of an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{t} \in [0,1]$.*

(2) *$\widetilde{\kappa}^*$ is an inf-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t})$ is either empty of an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{t} \in [0,1]$.*

Proof:

(1) Suppose that $\widetilde{\kappa}$ is an inf-HFii of $\ddot{\mathfrak{S}}$ and $\ddot{t} \in [0,1]$ such that $\mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) \neq \emptyset$. Choose $\Theta = \{\ddot{t}\}$. Then $\ddot{\mathfrak{S}}[\widetilde{\kappa},\Theta] = \mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) \neq \emptyset$. By assumption, we have $\mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) = \ddot{\mathfrak{S}}[\widetilde{\kappa},\Theta]$ is an interior ideal of $\ddot{\mathfrak{S}}$.
For the converse, suppose that $\mathfrak{U}_{\mathcal{INF}}(\widetilde{\kappa};\ddot{t})$ is either empty of an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{t} \in [0,1]$ and $\Theta \in \mathcal{P}[0,1]$ such that $\ddot{\mathfrak{S}}[\widetilde{\kappa},\Theta] \neq \emptyset$. Choose $\ddot{t} = \mathrm{INF}\,\Theta$. Then $\mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) = \ddot{\mathfrak{S}}[\widetilde{\kappa},\Theta] \neq \emptyset$. By assumption, we have $\ddot{\mathfrak{S}}[\ddot{h},\Theta] = \mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t})$ is an interior ideal of $\ddot{\mathfrak{S}}$. Thus, $\widetilde{\kappa}$ is a Θ-inf-HFii of $\ddot{\mathfrak{S}}$. Hence, $\widetilde{\kappa}$ is an inf-HFii of $\ddot{\mathfrak{S}}$.

(2) Suppose that $\widetilde{\kappa}^*$ is an inf-HFii of $\ddot{\mathfrak{S}}$ and $\ddot{t} \in [0,1]$ such that $\mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) \neq \emptyset$. Choose $\Upsilon = \{1-\ddot{t}\}$. Then $\ddot{\mathfrak{S}}[\widetilde{\kappa}^*,\Upsilon] = \mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) \neq \emptyset$. By assumption, we have $\mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) = \ddot{\mathfrak{S}}[\widetilde{\kappa}^*,\Upsilon]$ is an interior ideal of $\ddot{\mathfrak{S}}$.
For the converse, suppose that $\mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t})$ is either empty of an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{t} \in [0,1]$ and $\Upsilon \in \mathcal{P}[0,1]$ such that $\ddot{\mathfrak{S}}[\ddot{h}^*,\Upsilon] \neq \emptyset$. Choose $\ddot{t} = 1 - \mathrm{INF}\,\Upsilon$. Then $\mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa};\ddot{t}) = \ddot{\mathfrak{S}}[\ddot{h}^*,\Upsilon] \neq \emptyset$. By assumption, we have $\ddot{\mathfrak{S}}[\widetilde{\kappa}^*,\Upsilon] = \mathfrak{L}_{\mathcal{INF}}(\widetilde{\kappa};\ddot{t})$ is a bi-ideal of $\ddot{\mathfrak{S}}$. Thus, $\widetilde{\kappa}^*$ is a Υ-inf-HFii of $\ddot{\mathfrak{S}}$. Hence, $\widetilde{\kappa}^*$ is an inf-HFii of $\ddot{\mathfrak{S}}$.

■

Corollary 3.18. *Let $\breve{\pi}$ be an IvFii of $\ddot{\mathfrak{S}}$. Then, for all $\ddot{t} \in [0,1]$, a non-empty subset $\mathfrak{U}_{\mathrm{INF}}(\ddot{h});\ddot{t})$ of $\ddot{\mathfrak{S}}$ is an interior ideal of $\ddot{\mathfrak{S}}$.*

Proof: It follows from Lemma 3.4 and Theorem 3.17. (1). ■

Corollary 3.19. *Let $\widetilde{\kappa}$ be an HFS$^*(\ddot{\mathfrak{S}})$ is an HFii of $\ddot{\mathfrak{S}}$. Then, for all $\ddot{t} \in [0,1]$, a non-empty subset $\mathfrak{L}_{\mathrm{INF}}(\ddot{h});\ddot{t})$ of $\ddot{\mathfrak{S}}$ is an interior ideal of $\ddot{\mathfrak{S}}$.*

Proof: It follows from Lemma 3.7 and Theorem 3.17. (2) ■

Theorem 3.20. *Let $\widetilde{\kappa}$ is a HFS on $\ddot{\mathfrak{S}}$. Then the following statements holds.*

(1) *$\widetilde{\kappa}$ is a sup-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{U}_{\mathrm{SUP}}(\widetilde{\kappa};\ddot{t})$ is either empty of an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{t} \in [0,1]$.*

(2) *$\widetilde{\kappa}^*$ is a sup-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{L}_{\mathrm{SUP}}(\widetilde{\kappa};\ddot{t})$ is either empty of an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{t} \in [0,1]$.*

Proof: It follows from Theorem 3.17. ■

Corollary 3.21. *Let $\breve{\pi}$ be an IvFii of $\ddot{\mathfrak{S}}$. Then, for all $\ddot{t} \in [0,1]$, a non-empty subset $\mathfrak{L}_{\mathcal{SUP}}(\widetilde{\kappa};\ddot{t})$ of $\ddot{\mathfrak{S}}$ is an interior ideal of $\ddot{\mathfrak{S}}$.*

Proof: It follows from Lemma 3.4 and Theorem 3.17. (1) ■

Corollary 3.22. *Let $\widetilde{\kappa}$ be an HFS$^*(\ddot{\mathfrak{S}})$ is a HFbi of $\ddot{\mathfrak{S}}$. Then, for all $\ddot{t} \in [0,1]$, a non-empty subset $\mathfrak{L}_{\mathcal{SUP}}(\widetilde{\kappa};\ddot{t})$ of $\ddot{\mathfrak{S}}$ is an interior ideal of $\ddot{\mathfrak{S}}$.*

Proof: It follows from Lemma 3.7 and Theorem 3.17. (2). ■

The following theorem is conditions for a HFS of a semigroup $\ddot{\mathfrak{S}}$ is an inf-HFbi via IFs.

Theorem 3.23. *Let $\widetilde{\kappa}$ be an element in HFS$(\ddot{\mathfrak{S}})$. Then following are equivalent:*

(1) *$\widetilde{\kappa}$ is an inf-HFii of $\ddot{\mathfrak{S}}$.*
(2) *$(\mathcal{F}_{\widetilde{\kappa}}, \mathcal{F}_{\breve{\varsigma}})$ is an IFii of $\ddot{\mathfrak{S}}$ for all $\breve{\varsigma} \in \mathrm{IC}(\breve{\pi})$.*
(3) *$(\mathcal{F}_{\widetilde{\kappa}}, \mathcal{F}_{\widetilde{\kappa}^*})$ is an IFii of $\ddot{\mathfrak{S}}$.*

Proof: It follws from Lemma 3.11. ■

The following result is an immediate consequence of Lemma 3.7 and 3.23.

Corollary 3.24. *Let $\breve{\pi}$ be an IvFii of IvFs $\ddot{\mathfrak{S}}$. Then $(\mathcal{F}_{\widetilde{\kappa}}, \mathcal{F}_{\breve{\delta}})$ is an IFii of $\ddot{\mathfrak{S}}$ for all $\breve{\delta} \in \mathrm{IC}(\breve{\pi})$.*

The following result is an immediate consequence of Lemma 3.11 and 3.23.

Corollary 3.25. *Let $\widetilde{\kappa}$ be elenemt in HFS$^*(\ddot{\mathfrak{S}})$ is an HFii of $\ddot{\mathfrak{S}}$. Then $(\mathcal{F}_{\breve{\delta}}, \mathcal{F}_{\widetilde{\kappa}})$ is an IFii of $\ddot{\mathfrak{S}}$ for all $\breve{\delta} \in \mathrm{IC}(\breve{\pi})$.*

For $\widetilde{\kappa} \in \mathrm{HFS}(\ddot{\mathfrak{S}})$ and $\nabla \in \mathcal{P}([0,1])$, defind the HFS $\mathcal{H}_{\mathrm{INF}}$ and $\mathcal{H}_{\mathrm{SUP}}$ on $\ddot{\mathfrak{S}}$ by

$$\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)(\ddot{u}) = \{\ddot{t} \in \nabla \mid \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}) \geq t\} \text{ for all } \ddot{u} \in \ddot{\mathfrak{S}}$$

and

$$\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)(\ddot{u}) = \{\ddot{t} \in \nabla \mid \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) \geq t\} \text{ for all } \ddot{u} \in \ddot{\mathfrak{S}}$$

and we denote $\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};[0,1])$ by $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}$, $\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};[0,1])$ by $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}$. Then the following statemetnd hold:

(1) $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}(\ddot{u})$ and $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}(\ddot{u})$ are subsets ∇ for all $\ddot{u} \in \ddot{\mathfrak{S}}$.
(2) $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}(\ddot{u})$ and $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}(\ddot{u})$ are elenemts IvFS$(\ddot{\mathfrak{S}})$
(3) $0 = \min \mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}(\ddot{u}) \leq \mathrm{INF}\,\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)(\ddot{u}) \leq \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}) = \max \mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}(\ddot{u})$ and $0 = \min \mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}(\ddot{u}) \leq \mathrm{SUP}\,\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)(\ddot{u}) \leq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) = \max \mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}(\ddot{u})$ for all $\ddot{u} \in \ddot{\mathfrak{S}}$.

Theorem 3.26. *Let $\widetilde{\kappa}$ be an element of HFS$(\ddot{\mathfrak{S}})$. Then the following are equivalent:*

(1) *$\widetilde{\kappa}$ is an $(\inf,\sup)$-HFii of $\ddot{\mathfrak{S}}$.*
(2) *$\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)$ and $\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)$ are HFiis of $\ddot{\mathfrak{S}}$ for all $\nabla \in \mathcal{P}([0,1])$.*
(3) *$\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}$ and $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}$ are HFiis of $\ddot{\mathfrak{S}}$.*
(4) *$\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{INF}}$ and $\mathcal{H}^{\widetilde{\kappa}}_{\mathrm{SUP}}$ are IvFiis of $\ddot{\mathfrak{S}}$.*

Proof: (1) $\Rightarrow$ (2) Let $\ddot{u},\ddot{v},\ddot{w} \in \ddot{\mathfrak{S}}$, $\nabla \in \mathcal{P}([0,1])$ and t are elments in $\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)(\ddot{v}), \mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)(\ddot{v})$. Then $\ddot{t} \in \nabla$ and $\ddot{t}\mathrm{INF}\,\widetilde{\kappa}(\ddot{v})$, $\ddot{t} \leq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v})$. By hypothesis (1) and Lemma 3.11, $\ddot{t} \leq \mathrm{INF}\,\widetilde{\kappa}(\ddot{v}) \leq \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w})$ and $\ddot{t} \leq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v}), \leq \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w})$. Thus, t are elments in $\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)(\ddot{u}\ddot{v}\ddot{w})$ and $\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)(\ddot{u}\ddot{v}\ddot{w})$.
Hence, $\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)(\ddot{v}) \subseteq \mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)(\ddot{u}\ddot{v}\ddot{w})$ and $\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)(\ddot{v}) \subseteq \mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)(\ddot{u}\ddot{v}\ddot{w})$. Therefore, $\mathcal{H}_{\mathrm{INF}}(\widetilde{\kappa};\nabla)$ and $\mathcal{H}_{\mathrm{SUP}}(\widetilde{\kappa};\nabla)$ are HFiis of $\ddot{\mathfrak{S}}$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Let $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$. Then $\text{INF}\,\widetilde{\kappa}(\ddot{v}) \in \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u})$. Similarly $\text{SUP}\,\widetilde{\kappa}(\ddot{v}) \in \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{v})$. By hypothesis (3), $\mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v})$ is an element of $\mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u}\ddot{v}\ddot{w})$. Similarly $\mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{v})$ is an element of $\mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{u}\ddot{v}\ddot{w})$. Thus,

$$\min \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u}\ddot{v}\ddot{w}) = \text{INF}\,^{\widetilde{\kappa}}(\ddot{u}\ddot{v}\ddot{w}) \geq \text{INF}\,\widetilde{\kappa}(\ddot{v}) = \min \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v}).$$

Similarly, we can show that $\max \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{u}\ddot{v}\ddot{w}) \geq \max \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{v})$.
Since $\min \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v}) = 0$ and $\min \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{v}) = 0$ for all $\ddot{v} \in \ddot{\mathfrak{S}}$ we have $\min \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u}\ddot{v}\ddot{w}) \geq \min \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v})$ and $\min \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{u}\ddot{v}\ddot{w}) \geq \min \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{v})$. Thus, $\mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v}) \precsim \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u}\ddot{v}\ddot{w})$ and $\mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{v}) \precsim \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{u}\ddot{v}\ddot{w})$. Thus, $\mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}$ and $\mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}$ are IvFiis of $\ddot{\mathfrak{S}}$.

(4) $\Rightarrow$ (1) Let $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$. Then $\mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v}) \precsim \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u}\ddot{v}\ddot{w})$ and $\mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{v}) \precsim \mathcal{H}^{\widetilde{\kappa}}_{\text{SUP}}(\ddot{u}\ddot{v}\ddot{w})$. Thus,

$$\text{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) = \max \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u}\ddot{v}\ddot{w}) \geq \max \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v})$$

and

$$\text{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) = \min \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{u}\ddot{v}\ddot{w}) \geq \min \mathcal{H}^{\widetilde{\kappa}}_{\text{INF}}(\ddot{v}).$$

By Lemma 3.11, $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$. ■

The following result is an immediate consequence of Lemma 3.4 and 3.26.

Corollary 3.27. *Let $\breve{\pi}$ be an IvFii of $\ddot{\mathfrak{S}}$. Then the following hold:*

(1) *$\mathcal{H}_{\text{INF}}(\breve{\pi}; \nabla)$ and $\mathcal{H}_{\text{SUP}}(\breve{\pi}; \nabla)$ are HFiis of $\ddot{\mathfrak{S}}$ for all $\nabla \in \mathcal{P}([0,1])$.*
(2) *$\mathcal{H}^{\breve{\pi}}_{\text{INF}}$ and $\mathcal{H}^{\breve{\pi}}_{\text{SUP}}$ are both a HFii and an IvFii of $\ddot{\mathfrak{S}}$.*

The following result is an immediate consequence of Lemma 3.7 and 3.26.

Corollary 3.28. *Let $\widetilde{\kappa} \in HFS^*(\ddot{\mathfrak{S}})$ be a HFii of $\ddot{\mathfrak{S}}$. Then the following hold:*

(1) *$\mathcal{H}_{\text{INF}}(\breve{\varsigma}; \nabla)$ and $\mathcal{H}_{\text{SUP}}(\breve{\varsigma}; \nabla)$ are HFiis of $\ddot{\mathfrak{S}}$ for all $\nabla \in \mathcal{P}([0,1])$ and for all $\breve{\varsigma} \in \text{IC}(\widetilde{\kappa})$.*
(2) *$\mathcal{H}^{\breve{\varsigma}}_{\text{INF}}$ and $\mathcal{H}^{\breve{\varsigma}}_{\text{SUP}}$ are both a HFii and an IvFii of $\ddot{\mathfrak{S}}$ for all $\breve{\varsigma} \in \text{IC}(\widetilde{\kappa})$.*

For a subset $\ddot{\mathfrak{K}}$ of $\ddot{\mathfrak{S}}$ and two elements $\triangle, \nabla \in \mathcal{P}([0,1])$, define the characteristic interval-valued fuzzy set (CIvFS) $\mathcal{CI}_{\ddot{\mathfrak{K}}} : \ddot{\mathfrak{S}} \to C([0,1])$, the characteristic hesitant fuzzy set (CHFS) $\mathcal{CH}_{\ddot{\mathfrak{K}}} : \ddot{\mathfrak{S}} \to \mathcal{P}([0,1])$ and hesitant fuzzy set $\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}} : \ddot{\mathfrak{S}} \to \mathcal{P}([0,1])$ by for all $\ddot{u} \in \ddot{\mathfrak{S}}$

$$\mathcal{CI}_{\ddot{\mathfrak{K}}}(\ddot{u}) = \begin{cases} \bar{1} & \text{if } \ddot{u} \in \ddot{\mathfrak{K}}, \\ \bar{0} & \text{otherwise}, \end{cases}$$

$$\mathcal{CH}_{\ddot{\mathfrak{K}}}(\ddot{u}) = \begin{cases} [0,1] & \text{if } \ddot{u} \in \ddot{\mathfrak{K}}, \\ \emptyset & \text{otherwise}, \end{cases}$$

$$\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{u}) = \begin{cases} \triangle & \text{if } \ddot{u} \in \ddot{\mathfrak{K}}, \\ \nabla & \text{otherwise}, \end{cases}$$

The following theorem we prove characterization of an interior ideal of a semigroup in terms of a HFS.

Theorem 3.29. *Let $\ddot{\mathfrak{K}}$ be a non-empty subset of $\ddot{\mathfrak{S}}$ and $\triangle, \nabla \in \mathcal{P}([0,1])$ with $\text{INF}\,\triangle < \text{INF}\,\nabla$ and $\text{SUP}\,\triangle < \text{SUP}\,\nabla$. Then $\ddot{\mathfrak{K}}$ is an interior ideal of $\ddot{\mathfrak{S}}$ if and only if $\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$.*

Proof: Suppose that $\ddot{\mathfrak{K}}$ is an interior ideal of $\ddot{\mathfrak{S}}$ and $\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}$ is not an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$. Then by Lemma 3.11, there exists $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$ such that $\text{INF}\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{v}) > \text{INF}\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{u}\ddot{v}\ddot{w})$ and $\text{SUP}\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{v}) > \text{SUP}\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{u}\ddot{v}\ddot{w})$. Thus, $\text{INF}\,\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{v}) = \text{INF}\,\nabla$ and $\text{SUP}\,\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{v}) = \text{SUP}\,\nabla$. It is a contradiction. Hence, $\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$.

For the converse, let $\ddot{v} \in \ddot{\mathfrak{K}}$ and $\ddot{u}, \ddot{w} \in \ddot{\mathfrak{S}}$. Then $\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{v}) = (\triangle, \nabla)$. Thus, $\text{INF}\,\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{v}) = \text{INF}\,\nabla$ and $\text{SUP}\,\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{v}) = \text{SUP}\,\nabla$. By assumption and Lemma 3.11, $\text{INF}\,\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{u}\ddot{v}\ddot{w}) \geq \text{INF}\,\nabla$ and $\text{SUP}\,\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}(\ddot{u}\ddot{v}\ddot{w}) \geq \text{SUP}\,\nabla$. Hence, $\ddot{u}\ddot{v}\ddot{w} \in \ddot{\mathfrak{K}}$. Therefore, $\ddot{\mathfrak{K}}$ is an interior ideal of $\ddot{\mathfrak{S}}$. ■

Theorem 3.30. *A non-empty subset $\ddot{\mathfrak{K}}$ of $\ddot{\mathfrak{S}}$ is an interior ideal of $\ddot{\mathfrak{S}}$ if and only if the CIvFs $\mathcal{CI}_{\ddot{\mathfrak{K}}}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$.*

Proof: It follows from Theorem 3.29. ■

Remark 3.31. *If $\ddot{\mathfrak{K}}$ is a subset of $\ddot{\mathfrak{S}}$, then the CHFS $\mathcal{CH}_{\ddot{\mathfrak{K}}}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$.*

Theorem 3.32. *Let $\ddot{\mathfrak{K}}$ be an non-empty subset of $\ddot{\mathfrak{S}}$. Then the following are equivalent:*

(1) *$\ddot{\mathfrak{K}}$ is an interior ideal of $\ddot{\mathfrak{S}}$.*
(2) *The CIvFS $\mathcal{CI}_{\ddot{\mathfrak{K}}}$ is a $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$.*
(3) *$\mathcal{X}^{(\triangle,\nabla)}_{\ddot{\mathfrak{K}}}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$ for all $\triangle, \nabla \in \mathcal{P}([0,1])$ with $\text{INF}\,\triangle < \text{INF}\,\nabla$ and $\text{SUP}\,\triangle < \text{SUP}\,\nabla$.*

IV. $(\inf, \sup)$-HESITANT FUZZY TRANSLATIONS

In this section, we defined of $(\inf, \sup)$-hesitant fuzzy translations of $(\inf, \sup)$-HFbis of semigroups and discussed the cencepts of extensions and intensions of $(\inf, \sup)$-HFbis.

For a HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{S}}$, let $\ddot{\mathfrak{K}}^{\widetilde{\kappa}}_{\text{INF}} := 1 - \text{INF}\,\{\text{INF}\,\widetilde{\kappa}(\ddot{u}) \mid \ddot{u} \in \ddot{\mathfrak{S}}\}$ and $\ddot{\mathfrak{K}}^{\widetilde{\kappa}}_{\text{SUP}} := 1 - \text{SUP}\,\{\text{SUP}\,\widetilde{\kappa}(\ddot{u}) \mid \ddot{u} \in \ddot{\mathfrak{S}}\}$.

Let $\ddot{t}$ be are elements in $[0, \ddot{\mathfrak{K}}^{\widetilde{\kappa}}_{\text{INF}}]$ and $[0, \ddot{\mathfrak{K}}^{\widetilde{\kappa}}_{\text{SUP}}]$. Then we say that a HFS g on $\ddot{\mathfrak{S}}$ is INF-hesitnat fuzzy $\ddot{t}^{+}$-traslation (INF-HFT$^{+}_{\ddot{t}}$) of $\ddot{\mathfrak{S}}$ if $\text{INF}\,\widetilde{\kappa}(\ddot{u}) + \ddot{t}$ for all $\ddot{u} \in \ddot{\mathfrak{S}}$ and SUP-hesitnat fuzzy $\ddot{t}^{+}$-traslation (SUP-HFT$^{+}_{\ddot{t}}$) of $\ddot{\mathfrak{S}}$ if $\text{SUP}\,\widetilde{\kappa}(\ddot{u}) + \ddot{t}$ for all $\ddot{u} \in \ddot{\mathfrak{S}}$ respectively. Then $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFT$^{+}_{0}$ of $\ddot{\mathfrak{S}}$.

Theorem 4.1. *Let $\widetilde{\kappa}$ be an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$ and t be are elements in $[0, \ddot{\mathfrak{K}}^{\widetilde{\kappa}}_{\text{INF}}]$ and $[0, \ddot{\mathfrak{K}}^{\widetilde{\kappa}}_{\text{SUP}}]$. Then every INF-HFT$^{+}_{\ddot{t}}$ and SUP-HFT$^{+}_{\ddot{t}}$ of $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$.*

Proof: Suppose that $\tilde{\rho}$ are INF-HFT$^{+}_{\ddot{t}}$, SUP-HFT$^{+}_{\ddot{t}}$ of $\widetilde{\kappa}$ and let $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$. Then

$$\text{INF}\,\tilde{\rho}(\ddot{u}\ddot{v}\ddot{w}) = \text{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) + \ddot{t} \geq \text{INF}\,\widetilde{\kappa}(\ddot{v}) + \ddot{t} = \text{INF}\,\widetilde{\kappa}(\ddot{v})$$

and

$$\text{SUP}\,\tilde{\rho}(\ddot{u}\ddot{v}\ddot{w}) = \text{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) + \ddot{t} \geq \text{SUP}\,\widetilde{\kappa}(\ddot{v}) + \ddot{t} = \text{SUP}\,\widetilde{\kappa}(\ddot{v}).$$

Thus, by Theorem 3.11, $\tilde{\rho}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$. ■

Theorem 4.2. *Let $\widetilde{\kappa}$ be an HFii of $\ddot{\mathfrak{S}}$ such that it is an INF-HFT$^{+}_{\ddot{t}}$ is INF-HFii of $\ddot{\mathfrak{S}}$ for some $\ddot{t} \in [0, \ddot{\mathfrak{K}}^{\widetilde{\kappa}}_{\text{INF}}]$. Then $\widetilde{\kappa}$ is a INF-HFii of $\ddot{\mathfrak{S}}$.*

Proof: Suppose that a INF -HFT$_{\ddot{t}}^{+}$ $\tilde{\rho}$ of $\widetilde{\kappa}$ is a INF -HFii of $\ddot{\mathfrak{S}}$ when $\ddot{t} \in [0, \ddot{\mathfrak{K}}_{\mathrm{INF}}^{\widetilde{\kappa}}]$. Then for all $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$,

$$\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) = \mathrm{INF}\,\tilde{\rho}(\ddot{u}\ddot{v}\ddot{w}) - \ddot{t} \geq \mathrm{INF}\,\tilde{\rho}(\ddot{v}) - \ddot{t} = \mathrm{INF}\,\tilde{\rho}(\ddot{v}).$$

Thus, by Theorem 3.11, $\widetilde{\kappa}$ is an INF -HFii of $\ddot{\mathfrak{S}}$. ■

Theorem 4.3. *Let $\widetilde{\kappa}$ be a HFii of $\ddot{\mathfrak{S}}$ such that it is an* SUP *-HFT$_{\ddot{t}}^{+}$ is* SUP *-HFii of $\ddot{\mathfrak{S}}$ for some $\ddot{t} \in [0, \ddot{\mathfrak{K}}_{\mathrm{SUP}}^{\widetilde{\kappa}}]$. Then $\widetilde{\kappa}$ is a* SUP *-HFii of $\ddot{\mathfrak{S}}$.*

Proof: It follows from Theorem 4.2. ■

Theorem 4.4. *Let $\widetilde{\kappa}$ be a HFS on $\ddot{\mathfrak{S}}$ and $\ddot{t} \in [0, \ddot{\mathfrak{K}}_{\mathrm{INF}}^{\widetilde{\kappa}}]$. Then an* INF *-HFT$_{\ddot{t}}^{+}$ of $\widetilde{\kappa}$ is an* INF *-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa}; \ddot{m} - \ddot{t})$ either empty or an interior ideal of $\ddot{\mathfrak{S}}$ for all $m \in [t, 1]$.*

Proof: $(\Rightarrow)$ By Theorem 3.17. (1).

$(\Leftarrow)$ Let ρ be an INF -HFT$_{\ddot{t}}^{+}$ of $\widetilde{\kappa}$ and $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$. Choose $\ddot{m} := \mathrm{INF}\,\rho(\ddot{v})$. Then $\ddot{m} - \ddot{t} = \mathrm{INF}\,\tilde{\rho}(\ddot{v}) - \ddot{t} = \mathrm{INF}\,\widetilde{\kappa}(\ddot{v})$. Thus, $\ddot{v} \in \mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa}; \ddot{m} - \ddot{t})$. By assumption, $\ddot{u}\ddot{v}\ddot{w} \in \mathfrak{U}_{\mathrm{INF}}(\widetilde{\kappa}; m - \ddot{t})$. Hence, $\mathrm{INF}\,\tilde{\rho}(\ddot{u}\ddot{v}\ddot{w}) = \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) + \ddot{t} \geq \ddot{m} = \mathrm{INF}\,\tilde{\rho}(\ddot{v})$. By Theorem 3.11, $\widetilde{\kappa}$ is an INF -HFii of $\ddot{\mathfrak{S}}$. ■

Theorem 4.5. *Let $\widetilde{\kappa}$ be a HFS on $\ddot{\mathfrak{S}}$ and $\ddot{t} \in [0, \ddot{\mathfrak{K}}_{\mathrm{INF}}^{\widetilde{\kappa}}]$. Then an* INF *-HFT$_{\ddot{t}}^{+}$ of $\widetilde{\kappa}$ is an* INF *-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{L}_{\mathrm{INF}}(\widetilde{\kappa}; \ddot{m} - \ddot{t})$ either empty or an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{m} \in [t, 1]$.*

Proof: $(\Rightarrow)$ By Theorem 3.17. (2).

$(\Leftarrow)$ It follow from Theorem 4.4. ■

Theorem 4.6. *Let $\widetilde{\kappa}$ be a HFS on $\ddot{\mathfrak{S}}$ and $\ddot{t} \in [0, \ddot{\mathfrak{K}}_{\mathfrak{h}}]$. Then a* SUP *-HFT$_{\ddot{t}}^{+}$ of $\mathfrak{h}$ is a* SUP *-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{U}_{\mathcal{SUP}}(\widetilde{\kappa}; m - \ddot{t})$ either empty or an interior ideal of $\ddot{\mathfrak{S}}$ for all $m \in [t, 1]$.*

Proof: $(\Rightarrow)$ By Theorem 3.20. (1).

$(\Leftarrow)$ Let $\tilde{\rho}$ be a SUP -HFT$_{\ddot{t}}^{+}$ of $\widetilde{\kappa}$ and $\ddot{u}, \ddot{v}, \ddot{w} \in \ddot{\mathfrak{S}}$. Choose $\ddot{m} := \mathrm{INF}\,\rho(\ddot{v})$. Then $\ddot{m} - \ddot{t} = \mathrm{SUP}\,\rho(\ddot{v}) - \ddot{t} = \mathrm{SUP}\,\widetilde{\kappa}(\ddot{v})$. Thus, $\ddot{v} \in \mathfrak{U}_{\mathrm{SUP}}(\widetilde{\kappa}; \ddot{m} - \ddot{t})$. By assumption, $\ddot{u}\ddot{v}\ddot{w} \in \mathfrak{U}_{\mathrm{SUP}}(\widetilde{\kappa}; \ddot{m} - \ddot{t})$. Hence, $\mathrm{SUP}\,\tilde{\rho}(\ddot{u}\ddot{v}\ddot{w}) = \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}\ddot{v}\ddot{w}) + \ddot{t} \geq \ddot{m} = \mathrm{SUP}\,\tilde{\rho}(\ddot{v})$. By Theorem 3.11, $\widetilde{\kappa}$ is a SUP -HFii of $\ddot{\mathfrak{S}}$. ■

Theorem 4.7. *Let $\widetilde{\kappa}$ be a HFS on $\ddot{\mathfrak{S}}$ and $\ddot{t} \in [0, \ddot{\mathfrak{K}}_{\mathfrak{h}}]$. Then a $\mathcal{SUP}$-HFT$_{\ddot{t}}^{+}$ of $\widetilde{\kappa}$ is a* SUP *-HFii of $\ddot{\mathfrak{S}}$ if and only if $\mathfrak{L}_{\mathrm{SUP}}(\widetilde{\kappa}; m - \ddot{t})$ either empty or an interior ideal of $\ddot{\mathfrak{S}}$ for all $\ddot{m} \in [\ddot{t}, 1]$.*

Proof: $(\Rightarrow)$ By Theorem 3.20. (2).

$(\Leftarrow)$ It follow from Theorem 4.6. ■

For a HFS $\widetilde{\kappa}$ on $\ddot{\mathfrak{S}}$ define $\mp\widetilde{\kappa} := \mathrm{SUP}\,\{\mathrm{INF}\,\widetilde{\kappa}(\ddot{u}) \mid \ddot{u} \in \ddot{\mathfrak{S}}\}$ and $\pm\widetilde{\kappa} = \mathrm{INF}\,\{\mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) \mid \ddot{u} \in \ddot{\mathfrak{S}}\}$.

For $\ddot{t}$ are element of $[0, \mp\widetilde{\kappa}]$ a HFS g of $\ddot{\mathfrak{S}}$ is said to be INF -hesitant fuzzy $\ddot{t}^{-}$-translation (INF -HFT$_{\ddot{t}^{-}}$) of $\widetilde{\kappa}$ if $\mathrm{INF}\,\rho(\ddot{u}) = \mathrm{INF}\,\widetilde{\kappa}(\ddot{u}) - \ddot{t}$ for all $\ddot{u} \in \ddot{\mathfrak{S}}$. Then $\widetilde{\kappa}$ is an INF -HFT$_{0^{-}}$ of $\widetilde{\kappa}$.

For t are element of $[0, \pm\widetilde{\kappa}]$ a HFS g of $\ddot{\mathfrak{S}}$ is said to be SUP -hesitant fuzzy t^{-}-translation (SUP -HFT$_{\ddot{t}^{-}}$) of $\widetilde{\kappa}$ if $\mathrm{SUP}\,\rho(\ddot{u}) = \mathrm{SUP}\,\widetilde{\kappa}(\ddot{u}) - \ddot{t}$ for all $\ddot{u} \in \ddot{\mathfrak{S}}$. Then $\widetilde{\kappa}$ is a SUP -HFT$_{0^{-}}$ of $\widetilde{\kappa}$.

Theorem 4.8. *Let $\widetilde{\kappa}$ be an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$ and t be are elements in $[0, \mp\widetilde{\kappa}]$ and $[0, \pm\widetilde{\kappa}]$. Then every* INF *-HFT$_{\ddot{t}^{-}}$ and* SUP *-HFT$_{\ddot{t}^{-}}$ of $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFii of $\ddot{\mathfrak{S}}$.*

Proof: It follows from Theorem 4.1. ■

Theorem 4.9. *Let $\widetilde{\kappa}$ be a HFS on $\ddot{\mathfrak{S}}$ such that its* INF *-HFT$_{\ddot{t}^{-}}$ is an* INF *-HFii of $\ddot{\mathfrak{S}}$ for some $\ddot{t} \in [0, \mp\widetilde{\kappa}]$. Then $\widetilde{\kappa}$ is an* INF *-HFii of $\ddot{\mathfrak{S}}$*

Proof: It follows from Theorem 4.2. ■

Theorem 4.10. *Let $\widetilde{\kappa}$ be a HFS on $\ddot{\mathfrak{S}}$ such that its* SUP *-HFT$_{\ddot{t}^{-}}$ is a* SUP *-HFii of $\ddot{\mathfrak{S}}$ for some $\ddot{t} \in [0, \pm\widetilde{\kappa}]$. Then $\widetilde{\kappa}$ is a* SUP *-HFii of $\ddot{\mathfrak{S}}$*

Proof: It follows from Theorem 4.3. ■

V. Conclusion

In the present paper, we have introduce the concept of $(\inf, \sup)$-HFiis, which is generalal concept of IvFi in a semigroup, and discuss its so properties. As important study results, $(\inf, \sup)$-HFiis have been characterized in terms of sets, FSs, HFSs, IvFSs, NFSs, PFSs and BFSs. Furthermore, we use concepts of $(\inf, \sup)$-HFiis and IvFiis to investigate characterizations of interior ideals and Fiis.

In our future study of BCK/BCI-algebras and other algebras, the following objectives considered:

(1) to introduce and study $(\inf, \sup)$-type of HFSs based on H-ideals and p-ideals of BCK/BCI-algebras,
(2) to introduce and study $(\inf, \sup)$-type of HFSs based on ideals, interior ideals and bi-ideals of ternary semigroups, Γ-semigroups and LA-semigroups,
(3) to introduce and study $(\inf, \sup)$-type of HFSs baded on substructures of BE-algebras KU-algebras, JU-algebras and IUP-algebras.

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