# Fractional Midpoint-type Inequalities in Multiplicative Calculus

Zongrui Tan and Tingsong Du*

***Abstract*—This study develops midpoint-type inequalities for multiplicative $\psi$-Hilfer fractional integrals. We first establish a midpoint-type identity involving multiplicative $\psi$-Hilfer fractional integral operators, which forms the basis for deriving inequalities applicable to multiplicatively $M_\psi A$-$p$-functions. Furthermore, we also provide concrete numerical examples with graphical illustrations to validate the theoretical results and enhance their interpretability. Finally, we explore applications of these results to numerical quadrature formulas and special mean value estimations, demonstrating their practical utility.**

***Index Terms*—Multiplicative calculus, fractional integrals, multiplicative convexities, midpoint-type inequalities**

## I. Multiplicative calculus

In 1967, Grossman and Katz [1] introduced multiplicative calculus, also called non-Newtonian calculus, which replaces addition and subtraction with multiplication and division, enabling it to better address exponential change functions.

In contrast to Newton–Leibniz calculus, multiplicative calculus has a narrower application scope, as it applies exclusively to positive functions. Nevertheless, although mathematicians primarily used Cartesian coordinates to represent points in a plane, they also developed polar coordinates, which require a nonnegative radius. Despite these limitations, research on multiplicative calculus remains crucial. We believe that multiplicative calculus could provide innovative analytical tools for disciplines like economics, biology, and finance.

For the function $f$, its multiplicative derivative $f^*$ is given by:

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}. \tag{1}$$

Compared with (1), the conventional derivative concept is formulated below:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{2}$$

Notably, the difference and product in equation (2) are replaced by the quotient and power, respectively. Consequently, the equation (1) is referred to as the multiplicative derivative. The relationship between $f^*$ and $f'$ is given by:

$$f^*(x) = \exp\left\{[\ln f(x)]'\right\}. \tag{3}$$

Manuscript received March 31, 2025; revised June 16, 2025.

Zongrui Tan is a postgraduate student of the Department of Mathematics, College of Mathematics and Physics, China Three Gorges University, Yichang 443002, China (e-mail: zongruitan_ctgu@163.com).

*Tingsong Du is a professor of the Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, 443002, China (Corresponding author, phone: +8607176392618; e-mail: tingsongdu@ctgu.edu.cn).

The multiplicative integral for the function $f$, represented by $\int_a^b (f(x))^{\mathrm{d}x}$, is formally expressed as:

$$\int_a^b (f(x))^{\mathrm{d}x} = \exp\left\{\int_a^b \ln f(x)\mathrm{d}x\right\}. \tag{4}$$

Here, we present a biological case study to demonstrate the utility of multiplicative calculus: bacterial population growth modeled by ordinary differential equations. Ideally, the exponential growth equation below provides a foundational model for this phenomenon:

$$f'(t) = R(t)\times f(t), \tag{5}$$

where

- $f(t)$ is the size of the bacteria population.
- $f'(t)$ is the rate of variation in the bacterial population over time.
- $R(t)$ represents the intrinsic growth rate of the bacterial population.

Equation (5) models exponential bacterial population growth, where both the time variable $t$ and the growth rate of the bacterial population are strictly positive. In practice, factors like resource limitations require more complex models such as the logistic growth equation. Notably, (5) can be reformulated as:

$$\exp\left\{[\ln f(t)]'\right\} = \exp\left\{R(t)\right\}. \tag{6}$$

By using multiplicative calculus, the equation (6) transforms into:

$$f^*(t) = \exp\left\{R(t)\right\}. \tag{7}$$

The solution to equation (7) is presented by multiplicative integral as follows:

$$f(t) = \lambda\int_{t_0}^{t}\left(e^{R(t)}\right)^{\mathrm{d}t},\ \text{with } \lambda = f(t_0). \tag{8}$$

Therefore, this example showcases the practical value of multiplicative calculus in differential equations.

Convexity theory plays a significant role in mathematics and engineering sciences. The Hermite–Hadamard (HH) inequality, a key integral inequality for convex functions, is stated as:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}, \tag{9}$$

where the function $f:[a,b]\to\mathbb{R}$ exhibits convexity. The first and second inequalities in (9) are referred to as the midpoint-type and trapezoid-type inequalities, respectively.

We now recall the fractional integral operators.

**Definition 1.1:** [2] Let $f \in L^1[a,b]$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, the Riemann–Liouville (RL) fractional integral operators, namely $\mathcal{J}_{a^+}^{\alpha} f(x)$ and $\mathcal{J}_{b^-}^{\alpha} f(x)$, are defined as

$$\mathcal{J}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a,$$

and

$$\mathcal{J}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b,$$

where $\Gamma(\cdot)$ is the gamma function, which is expressed by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{d}t, \quad \mathrm{Re}(x) > 0,$$

with $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$.

In 2013, Sarikaya et al. [3] established the following fractional Hermite–Hadamard inequality.

**Theorem 1.1:** [3] If the function $f : [a,b] \to \mathbb{R}^+$ is convex on $[a,b]$ and $f \in L^1[a,b]$, then for any $\alpha > 0$, the following inequalities are satisfied:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\mathcal{I}_{a^+}^{\alpha} f(b) + \mathcal{I}_{b^-}^{\alpha} f(a)\right] \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned}$$

The $\psi$-Hilfer fractional integral operators, which generalize the RL-fractional integrals, are defined as:

**Definition 1.2:** [2] Let $\psi : [a,b] \to \mathbb{R}$ be a monotonically increasing function possessing a continuous derivative $\psi'(t)$. For $\alpha > 0$, the left-sided $\psi$-Hilfer fractional integral of a function $f$ with respect to another function $\psi$ on $[a,b]$ is defined by

$$\mathcal{I}_{a^+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) \left[\psi(x) - \psi(t)\right]^{\alpha-1} f(t)\mathrm{d}t,$$

and the right-sided one is defined by

$$\mathcal{I}_{b^-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) \left[\psi(t) - \psi(x)\right]^{\alpha-1} f(t)\mathrm{d}t.$$

Here $\Gamma(\cdot)$ is the gamma function.

The nonlocal properties and model capabilities of fractional calculus enable its widespread application in various disciplines, including engineering [4], computer science [5], and control science [6]. Recognizing the importance of fractional calculus, researchers have extensively studied fractional integral operators, such as $(k,h)$-RL-fractional integrals [7], $(k,s)$-RL-fractional integrals [8], generalized conformable fractional integrals [9], Hadamard fractional integrals [10], and others. Building on this work, scholars derived numerous fractional integral inequalities to establish error bounds for numerical integration formulas. Particularly, midpoint-type inequalities have been formulated by utilizing RL-fractional integrals [11], fractional $(p,q)$-integrals [12], and generalized fractional integrals [13], among others. Moreover, for further inequalities related to fractional calculus, see Refs. [14], [15], [16], [17] and [18].

In 2016, Abdeljawad and Grossman [19] introduced multiplicative RL-fractional integral operators.

**Definition 1.3:** [19] For $\alpha > 0$, the multiplicative RL-fractional integrals, namely ${}_a\mathcal{I}_*^{\alpha} f(x)$ and ${}_*\mathcal{I}_b^{\alpha} f(x)$, are outlined as follows:

$$\begin{aligned} &{}_a\mathcal{I}_*^{\alpha} f(x) \\ &= \exp\left\{\mathcal{I}_{a^+}^{\alpha} \ln f(x)\right\} \\ &= \exp\left\{\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \ln f(t)\mathrm{d}t\right\}, \quad x > a, \end{aligned}$$

and

$$\begin{aligned} &{}_*\mathcal{I}_b^{\alpha} f(x) \\ &= \exp\left\{\mathcal{I}_{b^-}^{\alpha} \ln f(x)\right\} \\ &= \exp\left\{\frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \ln f(t)\mathrm{d}t\right\}, \quad x < b, \end{aligned}$$

where the function $f$ defined on interval $[a,b]$ is positive.

In 2020, Budak and Özçelik [20] established endpoints- and midpoint-type inequalities for multiplicative RL-fractional integrals.

**Theorem 1.2:** [20] Given that $f : [a,b] \to (0,\infty)$ exhibits multiplicative convexity, it follows that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \left[{}_a\mathcal{I}_*^{\alpha} f(b) \cdot {}_*\mathcal{I}_b^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}} \\ &\le \sqrt{f(a)f(b)}, \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \left[{}_{\frac{a+b}{2}}\mathcal{I}_*^{\alpha} f(b) \cdot {}_*\mathcal{I}_{\frac{a+b}{2}}^{\alpha} f(a)\right]^{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}} \\ &\le \sqrt{f(a)f(b)}. \end{aligned}$$

Multiplicative calculus has significant application value in fields such as mathematical finance [21], biomedical sciences [22], and applied nonlinear models [23], among others. In the context of inequality theory, multiplicative calculus has been applied to derive bounds for integer-order inequalities in diverse forms, including Maclaurin-type [24], Radau-type [25], Ostrowski-type [26], Boole-type [27], HH-type [28], [29], midpoint- and trapezoidal-type [30], [31], and parametrized inequality [32], [33], and so on.

Inspired by merging multiplicative calculus and fractional calculus, researchers recently focused on fractional multiplicative calculus theory, analyzing bounds for multiplicative fractional inequalities. Notably, Budak and Özçelik [20] deduced HH-type inequalities through multiplicative RL-fractional integral operators, which aroused the interest of scholars. Subsequently, Merad et al. [34] and Boulares et al. [35] employed the same operators to formulate Maclaurin- and Bullen-type inequalities, respectively. Furthermore, Du and Long [36] introduced a multi-parameter fractional integral identity, which they used to derive three-point Newton–Cotes-type inequalities. Similarly, Almatrafi et al. [37] established a parametric integral identity, allowing them to obtain inequalities for one, two, and three-point quadrature formulas. Additionally, researchers have investigated inequalities involving different multiplicative fractional integral operators. Specifically, Fu et al. [38] studied multiplicative tempered fractional integrals, Peng et al. [39] examined multiplicative fractional integrals with exponential kernels, and Kashuri et al. [40]

explored multiplicative Sarikaya fractional integrals, with each deriving corresponding fractional HH-type inequalities. Recently, Zhang et al. [41] introduced multiplicative $k$-RL-fractional integrals and used them to establish HH- and Newton-type inequalities for multiplicative $(P, m)$-convex functions. For further study of multiplicative fractional integrals, we recommend several publications [42], [43], [44] and [45], as well as the references they cite.

Motivated by prior work, this paper aims to explore midpoint-type inequalities for multiplicatively $M_\psi A$-$p$-functions that involve multiplicative $\psi$-Hilfer fractional integrals. To achieve these goals, the paper is structured as follows: After Sec. II, Sec. III presents a multiplicative $\psi$-Hilfer fractional integral identity, from which we derive some multiplicative fractional midpoint-type inequalities for multiplicatively $M_\psi A$-$p$-functions. Additionally, we provide several illustrative instances to confirm the accuracy of these inequalities. In Sec. IV, we apply the obtained results to numerical quadrature and special mean estimates. Finally, Sec. V summarizes the key findings of the study.

## II. Multiplicative calculus

In 2008, Bashirov et al. [46] introduced $^*$integral operators, and deduced the following properties.

**Proposition 2.1:** [46] Given that the positive functions $f$ and $g$ are $^*$integrable on $[a, b]$, and $a \le c \le b$, the following properties clearly hold:

$$(i)\ \int_a^b ((f(x))^p)^{\mathrm{d}x} = \left(\int_a^b (f(x))^{\mathrm{d}x}\right)^p, \quad p \in \mathbb{R},$$

$$(ii)\ \int_a^b (f(x)g(x))^{\mathrm{d}x} = \int_a^b (f(x))^{\mathrm{d}x} \cdot \int_a^b (g(x))^{\mathrm{d}x},$$

$$(iii)\ \int_a^b \left(\frac{f(x)}{g(x)}\right)^{\mathrm{d}x} = \frac{\int_a^b (f(x))^{\mathrm{d}x}}{\int_a^b (g(x))^{\mathrm{d}x}},$$

$$(iv)\ \int_a^b (f(x))^{\mathrm{d}x} = \int_a^c (f(x))^{\mathrm{d}x} \cdot \int_c^b (f(x))^{\mathrm{d}x},$$

$$(v)\ \int_a^a (f(x))^{\mathrm{d}x} = 1,$$

$$(vi)\ \int_a^b (f(x))^{\mathrm{d}x} = \left(\int_b^a (f(x))^{\mathrm{d}x}\right)^{-1}.$$

The concept of the multiplicative derivative, also termed $^*$differentiable functions and first proposed by Bashirov et al. in Ref. [46], establishes a specific mathematical correlation between $f^*$ and $f'$, which can be formally expressed as follows:

$$f^*(x) = \exp\Big\{ [\ln f(x)]' \Big\} = \exp\left\{\frac{f'(x)}{f(x)}\right\}.$$

For further properties related to the $^*$differentiability of the function $f$, please refer to Ref. [46].

The following result naturally arises from the previous discussion.

**Proposition 2.2:** Provided that the function $f : I \subset \mathbb{R} \to \mathbb{R}^+$ is $^*$differentiable, it holds that $f^* \ge 1$ when $f$ is an increasing function.

*Proof:* Applying the relationship between $f^*$ and $f'$ leads to the result. ∎

Finally, we review the integration by parts formula for $^*$integral operators.

**Theorem 2.1:** [47] Assuming the function $f : [a, b] \to \mathbb{R}^+$ is $^*$differentiable, and considering that $g : [a, b] \to \mathbb{R}$ and $h : J \subset \mathbb{R} \to [a, b]$ are differentiable, the following equation can be derived:

$$\int_a^b \left(\left[f^*(h(x))\right]^{g(x)h'(x)}\right)^{\mathrm{d}x} = \frac{\left[f(h(b))\right]^{g(b)}}{\left[f(h(a))\right]^{g(a)}} \cdot \frac{1}{\int_a^b \left(\left[f(h(x))\right]^{g'(x)}\right)^{\mathrm{d}x}}.$$

## III. Main results

This part aims to obtain some bounds for midpoint-type inequalities from the perspective of the multiplicative $\psi$-Hilfer fractional integral operators. For this, we need to revisit the concepts of the multiplicative $\psi$-Hilfer fractional integral operators and the multiplicatively $M_\psi A$-$p$-functions.

**Definition 3.1:** [48] Let $\psi$ be a monotonically increasing function on $(a, b)$ with a continuous derivative $\psi'(t)$ within the interval. For $\alpha > 0$, the multiplicative left-sided $\psi$-Hilfer fractional integral of a function $f$ with regard to another function $\psi$ on $[a, b]$ is defined as

$$\begin{aligned}&{}_a\mathcal{I}_*^{\alpha;\psi} f(x)\\ &= \exp\{\mathcal{I}_{a^+}^{\alpha;\psi} \ln f(x)\}\\ &= \exp\left\{\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)\left[\psi(x)-\psi(t)\right]^{\alpha-1} \ln f(t)\mathrm{d}t\right\},\end{aligned}$$

and the multiplicative right-sided one is defined as

$$\begin{aligned}&{}_*\mathcal{I}_b^{\alpha;\psi} f(x)\\ &= \exp\{\mathcal{I}_{b^-}^{\alpha;\psi} \ln f(x)\}\\ &= \exp\left\{\frac{1}{\Gamma(\alpha)}\int_x^b \psi'(t)\left[\psi(t)-\psi(x)\right]^{\alpha-1} \ln f(t)\mathrm{d}t\right\}.\end{aligned}$$

**Remark 3.1:** The functional parameter $\psi$ plays a pivotal role in multiplicative $\psi$-Hilfer fractional integrals, allowing them to transform into diverse classes of multiplicative fractional integral operators.

$(i)$ Setting $\psi(\nu) = \nu$, we can obtain the multiplicative RL-fractional integrals [19].

$(ii)$ Setting $\psi(\nu) = \ln\nu$ and $b > a > 0$, we can get the multiplicative Hadamard fractional integrals:

$$\begin{aligned}{}_a\mathcal{H}_*^{\alpha} f(x) &= \exp\{\mathcal{H}_{a^+}^{\alpha} \ln f(x)\}\\ &= \exp\left\{\frac{1}{\Gamma(\alpha)}\int_a^x (\ln x - \ln t)^{\alpha-1} \ln f(t)\frac{\mathrm{d}t}{t}\right\},\end{aligned}$$

and

$$\begin{aligned}{}_*\mathcal{H}_b^{\alpha} f(x) &= \exp\{\mathcal{H}_{b^-}^{\alpha} \ln f(x)\}\\ &= \exp\left\{\frac{1}{\Gamma(\alpha)}\int_x^b (\ln t - \ln x)^{\alpha-1} \ln f(t)\frac{\mathrm{d}t}{t}\right\},\end{aligned}$$

which are defined in [45, Definition 3.1] by setting $k = 1$.

$(iii)$ Setting $\psi(\nu) = -\frac{1}{\nu}$, we can deduce the "multiplicative Harmonic fractional integrals":

$$\begin{aligned}{}_a\mathcal{R}_*^{\alpha} f(x) &= \exp\{\mathcal{R}_{a^+}^{\alpha} \ln f(x)\}\\ &= \exp\left\{\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{\ln f(t)}{(x-t)^{1-\alpha} t^{\alpha+1}}\mathrm{d}t\right\},\end{aligned}$$

and

$$
\begin{aligned}
{}_*\mathcal{R}_b^\alpha f(x) &= \exp\{\mathcal{R}_{b^-}^\alpha \ln f(x)\} \\
&= \exp\left\{\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{\ln f(t)}{(t-x)^{1-\alpha}t^{\alpha+1}}\mathrm{d}t\right\}.
\end{aligned}
$$

$(iv)$ Setting $\psi(\nu)=\frac{\nu^\rho}{\rho}$, where $\rho>0$, we can acquire the multiplicative Katugampola fractional integrals [42]:

$$
\begin{aligned}
{}_a\mathcal{K}_*^\alpha f(x) &= \exp\{\mathcal{K}_{a^+}^\alpha \ln f(x)\} \\
&= \exp\left\{\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1}\ln f(t)\mathrm{d}t\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
{}_*\mathcal{K}_b^\alpha f(x) &= \exp\{\mathcal{K}_{b^-}^\alpha \ln f(x)\} \\
&= \exp\left\{\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^\rho - x^\rho)^{\alpha-1} t^{\rho-1}\ln f(t)\mathrm{d}t\right\}.
\end{aligned}
$$

**Definition 3.2:** [48] Suppose $\psi : I \subset \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function. A function $f : I \to \mathbb{R}^+$ is called a multiplicatively $M_\psi A$-$p$-function, if the inequality

$$
f\Big(\psi^{-1}\big(t\psi(x)+(1-t)\psi(y)\big)\Big) \le f(x)f(y)
$$

is satisfied for all $x, y \in I$ and $t \in [0,1]$.

**Remark 3.2:** The multiplicatively $M_\psi A$-$p$-functions can generalize distinct multiplicative convex functions by varying the functional parameter $\psi$.

$(i)$ Setting $\psi(\nu)=\nu$, it can be obtained that the definition of multiplicative $P$-convex functions [49]:

$$
f\left(tx+(1-t)y\right) \le f(x)f(y).
$$

$(ii)$ Setting $\psi(\nu)=\ln\nu$, it can be gotten that the definition of "GG-$P$-convex functions":

$$
f\left(x^t y^{1-t}\right) \le f(x)f(y),
$$

which is defined in [50, Definition 2.7] by taking $h(t)=1$.

$(iii)$ Setting $\psi(\nu)=-\frac{1}{\nu}$, it can be deduced that the definition of "multiplicative Harmonic $P$-convex functions":

$$
f\left(\frac{xy}{ty+(1-t)x}\right) \le f(x)f(y).
$$

$(iv)$ Setting $\psi(\nu)=\frac{\nu^\rho}{\rho}$, it can be acquired that the definition of "multiplicative $\rho p$-convex functions":

$$
f\left(\left[tx^\rho+(1-t)y^\rho\right]^{\frac{1}{\rho}}\right) \le f(x)f(y).
$$

We now introduce a pivotal lemma that supports the subsequent theoretical analysis and discussion.

**Lemma 3.1:** Given that the function $f : [a,b] \to \mathbb{R}^+$ is $^*$differentiable, and assuming that $f^*$ is multiplicatively integrable on $[a,b]$, with $\psi$ is a strictly increasing function whose derivative $\psi'$ is continuous on $(a,b)$, the following multiplicative $\psi$-Hilfer fractional integral identity holds:

$$
\begin{aligned}
&\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi} f(b)\cdot {}_*\mathcal{I}_b^{\alpha;\psi} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}} \\
&= \int_0^{\frac12}\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha}{2}A(t;\psi(\nu))}\right)^{\mathrm{d}t} \\
&\times \int_{\frac12}^1\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha-1}{2}A(t;\psi(\nu))}\right)^{\mathrm{d}t} \\
&\times \int_0^{\frac12}\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}t\psi(a)\\+(1-t)\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha}{2}D(t;\psi(\nu))}\right)^{\mathrm{d}t} \\
&\times \int_{\frac12}^1\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}t\psi(a)\\+(1-t)\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha-1}{2}D(t;\psi(\nu))}\right)^{\mathrm{d}t}
\end{aligned}
$$

where

$$
A\big(t;\psi(\nu)\big) := \left[\psi^{-1}\big((1-t)\psi(a)+t\psi(b)\big)\right]'
$$

and

$$
D\big(t;\psi(\nu)\big) := \left[\psi^{-1}\big(t\psi(a)+(1-t)\psi(b)\big)\right]'.
$$

*Proof:* We define the following notations to streamline the discussion:

$$
T_1 = \int_0^{\frac12}\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha}{2}A(t;\psi(\nu))}\right)^{\mathrm{d}t}
$$

$$
T_2 = \int_{\frac12}^1\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha-1}{2}A(t;\psi(\nu))}\right)^{\mathrm{d}t}
$$

$$
T_3 = \int_0^{\frac12}\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}t\psi(a)\\+(1-t)\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha}{2}D(t;\psi(\nu))}\right)^{\mathrm{d}t}
$$

and

$$
T_4 = \int_{\frac12}^1\left(\left[f^*\left(\psi^{-1}\left(\begin{array}{c}t\psi(a)\\+(1-t)\psi(b)\end{array}\right)\right)\right]^{\frac{t^\alpha-1}{2}D(t;\psi(\nu))}\right)^{\mathrm{d}t}.
$$

Utilizing integration by parts for multiplicative integrals, as described in Theorem 2.1, we have that:

$$
T_1 = \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\left(\frac12\right)^{\alpha+1}}}{\exp\left\{\int_0^{\frac12}\frac{\alpha t^{\alpha-1}}{2}\ln f\left(\psi^{-1}\left(\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right)\right)\mathrm{d}t\right\}}. \quad (10)
$$

Similarly, we can get that

$$T_2 = \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\frac{1}{2}-\left(\frac{1}{2}\right)^{\alpha+1}}}{\exp\left\{\int_{\frac{1}{2}}^{1}\frac{\alpha t^{\alpha-1}}{2}\ln f\left(\psi^{-1}\left(\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right)\right)\mathrm{d}t\right\}} \quad (11)$$

$$T_3 = \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\left(\frac{1}{2}\right)^{\alpha+1}}}{\exp\left\{\int_{0}^{\frac{1}{2}}\frac{\alpha t^{\alpha-1}}{2}\ln f\left(\psi^{-1}\left(\begin{array}{c}t\psi(a)\\+(1-t)\psi(b)\end{array}\right)\right)\mathrm{d}t\right\}} \quad (12)$$

and

$$T_4 = \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\frac{1}{2}-\left(\frac{1}{2}\right)^{\alpha+1}}}{\exp\left\{\int_{\frac{1}{2}}^{1}\frac{\alpha t^{\alpha-1}}{2}\ln f\left(\psi^{-1}\left(\begin{array}{c}t\psi(a)\\+(1-t)\psi(b)\end{array}\right)\right)\mathrm{d}t\right\}}. \quad (13)$$

By multiplying both sides of equalities (10) and (11), we obtain that

$$T_1 \times T_2 = \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\frac{1}{2}}}{\exp\left\{\int_{0}^{1}\frac{\alpha t^{\alpha-1}}{2}\ln f\left(\psi^{-1}\left(\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right)\right)\mathrm{d}t\right\}}. \quad (14)$$

By using the variable change $x=\psi^{-1}\big((1-t)\psi(a)+t\psi(b)\big)$, we derive that

$$\begin{aligned} &T_1 \times T_2 \\ &= \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\frac{1}{2}}}{\exp\left\{\int_a^b \frac{\alpha}{2}\left[\frac{\psi(x)-\psi(a)}{\psi(b)-\psi(a)}\right]^{\alpha-1}\frac{\psi'(x)\ln f(x)}{\psi(b)-\psi(a)}\mathrm{d}x\right\}} \\ &= \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\frac{1}{2}}}{\exp\left\{\begin{array}{c}\frac{\Gamma(\alpha+1)}{2\left[\psi(b)-\psi(a)\right]^{\alpha}}\frac{1}{\Gamma(\alpha)}\\ \times\int_a^b \psi'(x)\left[\psi(x)-\psi(a)\right]^{\alpha-1}\ln f(x)\mathrm{d}x\end{array}\right\}} \\ &= \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\frac{1}{2}}}{\left[{}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^{\alpha}}}}. \end{aligned} \quad (15)$$

In analogy with the preceding steps, we can deduce that

$$T_3 \times T_4 = \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)^{\frac{1}{2}}}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^{\alpha}}}}. \quad (16)$$

From equalities (15) and (16), we can readily infer that

$$\begin{aligned} &T_1 \times T_2 \times T_3 \times T_4 \\ &= \frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\cdot {}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^{\alpha}}}}. \end{aligned} \quad (17)$$

The proof is completed here. ■

**Corollary 3.1:** Under the condition that $\psi$ assumes certain functions in Lemma 3.1, the following conclusions can be reached:

$(i)$ Setting $\psi(\nu)=\nu$, we can derive the following multiplicative RL-fractional integral identity:

$$\begin{aligned} &\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_a\mathcal{I}_*^{\alpha}f(b)\cdot {}_*\mathcal{I}_b^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[b-a]^{\alpha}}}} \\ &= \int_0^{\frac{1}{2}}\left(\left[f^*\big((1-t)a+tb\big)\right]^{\frac{t^{\alpha}}{2}(b-a)}\right)^{\mathrm{d}t} \\ &\times \int_{\frac{1}{2}}^{1}\left(\left[f^*\big((1-t)a+tb\big)\right]^{\frac{t^{\alpha}-1}{2}(b-a)}\right)^{\mathrm{d}t} \\ &\times \int_0^{\frac{1}{2}}\left(\left[f^*\big(ta+(1-t)b\big)\right]^{\frac{t^{\alpha}}{2}(a-b)}\right)^{\mathrm{d}t} \\ &\times \int_{\frac{1}{2}}^{1}\left(\left[f^*\big(ta+(1-t)b\big)\right]^{\frac{t^{\alpha}-1}{2}(a-b)}\right)^{\mathrm{d}t}. \end{aligned}$$

$(ii)$ Setting $\psi(\nu)=\ln\nu$, we can achieve the following multiplicative Hadamard fractional integral identity:

$$\begin{aligned} &\frac{f\left(\sqrt{ab}\right)}{\left[{}_a\mathcal{H}_*^{\alpha}f(b)\cdot {}_*\mathcal{H}_b^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\ln b-\ln a]^{\alpha}}}} \\ &= \int_0^{\frac{1}{2}}\left(\left[f^*\left(a^{1-t}b^{t}\right)\right]^{\frac{t^{\alpha}}{2}(\ln b-\ln a)a^{1-t}b^{t}}\right)^{\mathrm{d}t} \\ &\times \int_{\frac{1}{2}}^{1}\left(\left[f^*\left(a^{1-t}b^{t}\right)\right]^{\frac{t^{\alpha}-1}{2}(\ln b-\ln a)a^{1-t}b^{t}}\right)^{\mathrm{d}t} \\ &\times \int_0^{\frac{1}{2}}\left(\left[f^*\left(a^{t}b^{1-t}\right)\right]^{\frac{t^{\alpha}}{2}(\ln a-\ln b)a^{t}b^{1-t}}\right)^{\mathrm{d}t} \\ &\times \int_{\frac{1}{2}}^{1}\left(\left[f^*\left(a^{t}b^{1-t}\right)\right]^{\frac{t^{\alpha}-1}{2}(\ln a-\ln b)a^{t}b^{1-t}}\right)^{\mathrm{d}t}. \end{aligned}$$

$(iii)$ Setting $\psi(\nu)=-\frac{1}{\nu}$, we can attain the following "multiplicative Harmonic fractional integral" identity:

$$\begin{aligned} &\frac{f\left(\frac{2ab}{a+b}\right)}{\left[{}_a\mathcal{R}_*^{\alpha}f(b)\cdot_* \mathcal{R}_b^{\alpha}f(a)\right]^{\frac{(ab)^{\alpha}\Gamma(\alpha+1)}{2[b-a]^{\alpha}}}} \\ &= \int_0^{\frac{1}{2}}\left(\left[f^*\left(\frac{ab}{ta+(1-t)b}\right)\right]^{\frac{t^{\alpha}}{2}\frac{(b-a)ab}{[ta+(1-t)b]^2}}\right)^{\mathrm{d}t} \\ &\times \int_{\frac{1}{2}}^{1}\left(\left[f^*\left(\frac{ab}{ta+(1-t)b}\right)\right]^{\frac{t^{\alpha}-1}{2}\frac{(b-a)ab}{[ta+(1-t)b]^2}}\right)^{\mathrm{d}t} \\ &\times \int_0^{\frac{1}{2}}\left(\left[f^*\left(\frac{ab}{(1-t)a+tb}\right)\right]^{\frac{t^{\alpha}}{2}\frac{(a-b)ab}{[(1-t)a+tb]^2}}\right)^{\mathrm{d}t} \\ &\times \int_{\frac{1}{2}}^{1}\left(\left[f^*\left(\frac{ab}{(1-t)a+tb}\right)\right]^{\frac{t^{\alpha}-1}{2}\frac{(a-b)ab}{[(1-t)a+tb]^2}}\right)^{\mathrm{d}t}. \end{aligned}$$

$(iv)$ Setting $\psi(\nu)=\frac{\nu^\rho}{\rho}$, where $\rho>0$, we can export the following multiplicative Katugampola fractional integral identity:

$$\frac{f\left(\left(\frac{a^\rho+b^\rho}{2}\right)^{\frac{1}{\rho}}\right)}{\left[{}_a\mathcal{K}_*^\alpha f(b)\cdot_* \mathcal{K}_b^\alpha f(a)\right]^{\frac{\rho^\alpha\Gamma(\alpha+1)}{2[b^\rho-a^\rho]^\alpha}}}$$
$$=\int_0^{\frac{1}{2}}\left(\left[f^*\left([(1-t)a^\rho+tb^\rho]^{\frac{1}{\rho}}\right)\right]^{\nabla_1(t;a,b,\rho)}\right)^{dt}$$
$$\times\int_{\frac{1}{2}}^1\left(\left[f^*\left([(1-t)a^\rho+tb^\rho]^{\frac{1}{\rho}}\right)\right]^{\nabla_2(t;a,b,\rho)}\right)^{dt}$$
$$\times\int_0^{\frac{1}{2}}\left(\left[f^*\left([ta^\rho+(1-t)b^\rho]^{\frac{1}{\rho}}\right)\right]^{\nabla_3(t;a,b,\rho)}\right)^{dt}$$
$$\times\int_{\frac{1}{2}}^1\left(\left[f^*\left([ta^\rho+(1-t)b^\rho]^{\frac{1}{\rho}}\right)\right]^{\nabla_4(t;a,b,\rho)}\right)^{dt},$$

where

$$\nabla_1(t;a,b,\rho)=\frac{t^\alpha}{2}\frac{(b^\rho-a^\rho)\left[(1-t)a^\rho+tb^\rho\right]^{\frac{1-\rho}{\rho}}}{\rho}$$
$$\nabla_2(t;a,b,\rho)=\frac{t^\alpha-1}{2}\frac{(b^\rho-a^\rho)\left[(1-t)a^\rho+tb^\rho\right]^{\frac{1-\rho}{\rho}}}{\rho}$$
$$\nabla_3(t;a,b,\rho)=\frac{t^\alpha}{2}\frac{(a^\rho-b^\rho)\left[ta^\rho+(1-t)b^\rho\right]^{\frac{1-\rho}{\rho}}}{\rho}$$

and

$$\nabla_4(t;a,b,\rho)=\frac{t^\alpha-1}{2}\frac{(a^\rho-b^\rho)\left[ta^\rho+(1-t)b^\rho\right]^{\frac{1-\rho}{\rho}}}{\rho}.$$

The subsequent theorems require three special functions: beta functions, incomplete confluent hypergeometric functions, and incomplete hypergeometric functions. We therefore review these function classes.

**Definition 3.3:** [2] For any positive numbers $\sigma$ and $\vartheta$, the beta function is expressed as

$$B(\sigma,\vartheta)=\int_0^1 t^{\sigma-1}(1-t)^{\vartheta-1}dt,$$

where $\Gamma(\cdot)$ denotes the Euler gamma function.

**Definition 3.4:** [51] The integral form of the incomplete confluent hypergeometric function is provided by

$$ {}_1F_1([\sigma,\vartheta;y],z) =\frac{1}{B(\sigma,\vartheta-\sigma)}\int_0^y t^{\sigma-1}(1-t)^{\vartheta-\sigma-1}e^{zt}dt,$$

where $\mathrm{Re}(\vartheta)>\mathrm{Re}(\sigma)>0$ and $B(\cdot,\cdot)$ is the beta function.

**Definition 3.5:** [51] Under the conditions $\kappa>\vartheta>0$ and $|x|<1$, the mathematical expression for the incomplete hypergeometric function is given by

$$ {}_2F_1(\sigma,[\vartheta,\kappa;y],x) =\frac{1}{B(\vartheta,\kappa-\vartheta)}\int_0^y t^{\vartheta-1}(1-t)^{\kappa-\vartheta-1}(1-xt)^{-\sigma}dt,$$

where $B(\cdot,\cdot)$ is the beta function.

Making use of Lemma 3.1 and taking into account that $f^*$ is a multiplicatively $M_\psi A$-$p$-function, we establish the following theorem.

**Theorem 3.1:** Let $f:[a,b]\to\mathbb{R}^+$ be an increasing and $^*$differentiable function, and let $\psi$ be a continuous and strictly increasing function, whose derivative $\psi'$ is continuous on $(a,b)$. If $f^*$ is a multiplicatively $M_\psi A$-$p$-function on $[a,b]$, then for $\alpha>0$ the following inequality holds:

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\cdot {}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right| \le\left[f^*(a)f^*(b)\right]^{\Upsilon_1(t;\psi(\nu))},$$

where

$$\Upsilon_1(t;\psi(\nu)):=\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)\\ \times\left(\begin{array}{c}|A(t;\psi(\nu))|\\ +|D(t;\psi(\nu))|\end{array}\right)\end{array}\right]dt$$

and $A(t;\psi(\nu))$, $D(t;\psi(\nu))$ are given in Lemma 3.1.

*Proof:* By invoking Lemma 3.1 and using the property of multiplicative integrals, it can be derived that

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\cdot {}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right|$$
$$=\left|\exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}A(t;\psi(\nu))\times\\ \ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\end{array}\right]dt\right\}\right|$$
$$\times\left|\exp\left\{\int_{\frac{1}{2}}^1\left[\begin{array}{c}\left(\frac{t^\alpha-1}{2}\right)A(t;\psi(\nu))\times\\ \ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\end{array}\right]dt\right\}\right|$$
$$\times\left|\exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}D(t;\psi(\nu))\times\\ \ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\end{array}\right]dt\right\}\right|$$
$$\times\left|\exp\left\{\int_{\frac{1}{2}}^1\left[\begin{array}{c}\left(\frac{t^\alpha-1}{2}\right)D(t;\psi(\nu))\times\\ \ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\end{array}\right]dt\right\}\right|$$
$$\le\exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}\left|A(t;\psi(\nu))\right|\times\\ \left|\ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\right|\end{array}\right]dt\right\}$$
$$\times\exp\left\{\int_{\frac{1}{2}}^1\left[\begin{array}{c}\left|\frac{t^\alpha-1}{2}\right|\left|A(t;\psi(\nu))\right|\times\\ \left|\ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\right|\end{array}\right]dt\right\}$$
$$\times\exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}\left|D(t;\psi(\nu))\right|\times\\ \left|\ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\right|\end{array}\right]dt\right\}$$
$$\times\exp\left\{\int_{\frac{1}{2}}^1\left[\begin{array}{c}\left|\frac{t^\alpha-1}{2}\right|\left|D(t;\psi(\nu))\right|\times\\ \left|\ln f^*\left(\psi^{-1}\left(\begin{array}{c}t\psi(a)\\ +(1-t)\psi(b)\end{array}\right)\right)\right|\end{array}\right]dt\right\}.$$

By employing the monotonically increasing property of the positive function $f$, we deduce that

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_{a}\mathcal{I}_{*}^{\alpha;\psi}f(b)\cdot{}_{*}\mathcal{I}_{b}^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^{\alpha}}}}\right|$$

$$\leq \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}\left|A\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_{\frac{1}{2}}^{1}\left[\begin{array}{c}\left(\frac{1-t^\alpha}{2}\right)\left|A\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}\left|D\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_{\frac{1}{2}}^{1}\left[\begin{array}{c}\left(\frac{1-t^\alpha}{2}\right)\left|D\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$= \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}\left|A\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha}{2}\right)\left|D\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\frac{t^\alpha}{2}\left|D\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha}{2}\right)\left|A\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$= \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)\left|A\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)\left|D\left(t;\psi(\nu)\right)\right|\times\\ \ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\end{array}\right]\mathrm{d}t\right\}. \tag{18}$$

Leveraging the condition that $f^*$ is a multiplicatively $M_\psi A$-$p$-function on $[a,b]$, we have that

$$\ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right) \leq \ln\left[f^*(a)f^*(b)\right] \tag{19}$$

and

$$\ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right) \leq \ln\left[f^*(a)f^*(b)\right]. \tag{20}$$

Through the application of the inequalities (19) and (20) to the inequality (18), we arrive at the following result:

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_{a}\mathcal{I}_{*}^{\alpha;\psi}f(b)\cdot{}_{*}\mathcal{I}_{b}^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^{\alpha}}}}\right|$$

$$\leq \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)\left|A\left(t;\psi(\nu)\right)\right|\\ \times\ln\left[f^*(a)f^*(b)\right]\end{array}\right]\mathrm{d}t\right\}$$

$$\times \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)\left|D\left(t;\psi(\nu)\right)\right|\\ \times\ln\left[f^*(a)f^*(b)\right]\end{array}\right]\mathrm{d}t\right\}$$

$$= \exp\left\{\begin{array}{l}\ln\left[f^*(a)f^*(b)\right]\\ \times\displaystyle\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)\\ \times\left[\begin{array}{c}\left|A\left(t;\psi(\nu)\right)\right|\\ +\left|D\left(t;\psi(\nu)\right)\right|\end{array}\right]\end{array}\right]\mathrm{d}t\end{array}\right\}. \tag{21}$$

Hence, the proof of the theorem is complete. ■

**Corollary 3.2:** By setting $\psi(\nu)=\nu$ in Theorem 3.1, we can derive the following inequality for multiplicative $P$-convex functions involving the multiplicative RL-fractional integrals.

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_{a}\mathcal{I}_{*}^{\alpha}f(b)\cdot_{*}\mathcal{I}_{b}^{\alpha}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[b-a]^{\alpha}}}}\right| \leq \left[f^*(a)f^*(b)\right]^{\frac{(b-a)\left(\alpha-1+2^{1-\alpha}\right)}{2(\alpha+1)}}.$$

*Proof:* The required result can be deduced from the subsequent calculation:

$$\Upsilon_1(t;\nu)=\int_0^{\frac{1}{2}}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)\left[2(b-a)\right]\mathrm{d}t = \frac{(b-a)\left(\alpha-1+2^{1-\alpha}\right)}{2(\alpha+1)}. \tag{22}$$

The proof is now concluded. ■

**Example 3.1:** Let $f(x)=\exp\left\{\frac{1}{3}x^3\right\}$ for any $x\in[0,\infty)$. We can infer that the function $f^*(x)=\exp\left\{x^2\right\}$ exhibits multiplicative $P$-convexity. By taking $a=0$ and $b=1$, it then follows that the inequality presented in Corollary 3.2 transforms into

$$\exp\left\{\frac{1}{24}-\frac{\alpha}{6}\left[\frac{6}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{1}{\alpha+3}\right]\right\} \leq \exp\left\{\frac{\alpha-1+2^{1-\alpha}}{2(\alpha+1)}\right\}. \tag{23}$$

TABLE I: Numerical values of the inequality (23) for $f(x)=\exp\left\{\frac{1}{3}x^3\right\}$ in Example 3.1

| $\alpha$ | Left term | Right term |
|---|---|---|
| 0.1 | 0.90537 | 1.55134 |
| 0.2 | 0.91664 | 1.48012 |
| 0.3 | 0.92790 | 1.42701 |
| 0.4 | 0.93662 | 1.38686 |
| 0.5 | 0.94334 | 1.35627 |
| 0.6 | 0.94849 | 1.33289 |
| 0.7 | 0.95242 | 1.31504 |
| 0.8 | 0.95539 | 1.30151 |
| 0.9 | 0.95760 | 1.29140 |

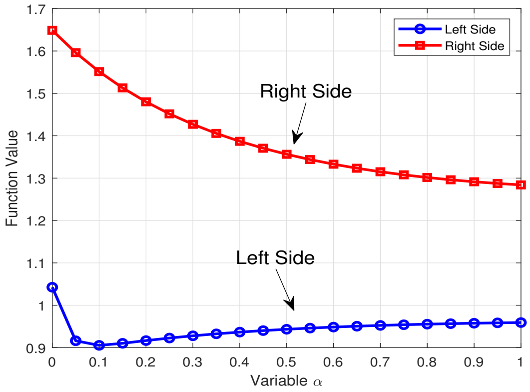


Fig. 1: Visualization graphic of Example 3.1 for $\alpha \in (0, 1]$

Figure 1 and Table I show that the quantity on the left is significantly less than the quantity on the right, validating the theoretical result presented in Corollary 3.2.

**Corollary 3.3:** For $b > a > 0$, if we choose $\psi(\nu) = \ln\nu$ in Theorem 3.1, then the following inequality concerning multiplicative Hadamard fractional integrals can be deduced for GG-$P$-convex functions.

$$\left| \frac{f\left(\frac{a+b}{2}\right)}{\left[{}_a\mathcal{H}_*^\alpha f(b) \cdot_* \mathcal{H}_b^\alpha f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\ln b - \ln a]^\alpha}}} \right| \leq \left[f^*(a) f^*(b)\right]^{\Upsilon_1(t;\ln\nu)},$$

where

$$\begin{aligned}
&\Upsilon_1(t;\ln\nu) \\
&= \frac{b-a}{2} + \frac{a(\ln b - \ln a)}{2(\alpha+1)} \\
&\times \left[ \begin{array}{l} {}_1F_1\left(\left[1, \alpha+1; \frac{1}{2}\right], \ln\frac{b}{a}\right) \\ -{}_1F_1\left(\left[\alpha+1, \alpha+2; \frac{1}{2}\right], \ln\frac{b}{a}\right) \end{array} \right] \\
&- \frac{b(\ln b - \ln a)}{2(\alpha+1)} \left[ \begin{array}{l} {}_1F_1\left(\left[1, \alpha+1; \frac{1}{2}\right], \ln\frac{a}{b}\right) \\ -{}_1F_1\left(\left[\alpha+1, \alpha+2; \frac{1}{2}\right], \ln\frac{a}{b}\right) \end{array} \right].
\end{aligned}$$

*Proof:* The predicted outcome is derived from the following fact:

$$\begin{aligned}
&\Upsilon_1(t;\ln\nu) \\
&= \int_0^{\frac{1}{2}} \left[ \begin{array}{l} \left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)(\ln b - \ln a) \\ \times\left(a^{1-t}b^t + a^t b^{1-t}\right) \end{array} \right] dt \\
&= \frac{\ln b - \ln a}{2} \left[ \begin{array}{l} a\int_0^{\frac{1}{2}} (1-(1-t)^\alpha + t^\alpha)\left(\frac{b}{a}\right)^t dt \\ +b\int_0^{\frac{1}{2}} (1-(1-t)^\alpha + t^\alpha)\left(\frac{a}{b}\right)^t dt \end{array} \right] \\
&= \frac{b-a}{2} + \frac{a(\ln b - \ln a)}{2(\alpha+1)} \\
&\times \left[ \begin{array}{l} {}_1F_1\left(\left[1, \alpha+1; \frac{1}{2}\right], \ln\frac{b}{a}\right) \\ -{}_1F_1\left(\left[\alpha+1, \alpha+2; \frac{1}{2}\right], \ln\frac{b}{a}\right) \end{array} \right] \\
&- \frac{b(\ln b - \ln a)}{2(\alpha+1)} \left[ \begin{array}{l} {}_1F_1\left(\left[1, \alpha+1; \frac{1}{2}\right], \ln\frac{a}{b}\right) \\ -{}_1F_1\left(\left[\alpha+1, \alpha+2; \frac{1}{2}\right], \ln\frac{a}{b}\right) \end{array} \right]. \quad (24)
\end{aligned}$$

The proof is finalized. ■

**Corollary 3.4:** If we consider $\psi(\nu) = -\frac{1}{\nu}$ in Theorem 3.1, then the following inequality involving "multiplicative Harmonic fractional integral operators" can be derived for multiplicatively Harmonic $P$-convex functions.

$$\left| \frac{f\left(\frac{a+b}{2}\right)}{\left[{}_a\mathcal{R}_*^\alpha f(b) \cdot_* \mathcal{R}_b^\alpha f(a)\right]^{\frac{(ab)^\alpha\Gamma(\alpha+1)}{2[b-a]^\alpha}}} \right| \leq \left[f^*(a) f^*(b)\right]^{\Upsilon_1\left(t;-\frac{1}{\nu}\right)},$$

where

$$\begin{aligned}
&\Upsilon_1\left(t;-\frac{1}{\nu}\right) \\
&= \frac{b-a}{2} + \frac{a(b-a)}{2b(\alpha+1)} \\
&\times \left( \begin{array}{l} {}_2F_1\left(2, \left[\alpha+1, \alpha+2; \frac{1}{2}\right], \frac{b-a}{b}\right) \\ -{}_2F_1\left(2, \left[1, \alpha+2; \frac{1}{2}\right], \frac{b-a}{b}\right) \end{array} \right) \\
&+ \frac{b(b-a)}{2a(\alpha+1)} \left( \begin{array}{l} {}_2F_1\left(2, \left[\alpha+1, \alpha+2; \frac{1}{2}\right], \frac{a-b}{a}\right) \\ -{}_2F_1\left(2, \left[1, \alpha+2; \frac{1}{2}\right], \frac{a-b}{a}\right) \end{array} \right).
\end{aligned}$$

*Proof:* The following fact gives rise to the expected result:

$$\begin{aligned}
&\Upsilon_1\left(t;-\frac{1}{\nu}\right) \\
&= \frac{ab(b-a)}{2} \int_0^{\frac{1}{2}} \left[ \begin{array}{l} (1-(1-t)^\alpha + t^\alpha) \\ \times\left( \begin{array}{l} \frac{1}{[ta+(1-t)b]^2} \\ +\frac{1}{[(1-t)a+tb]^2} \end{array} \right) \end{array} \right] dt \\
&= \frac{ab(b-a)}{2} \int_0^{\frac{1}{2}} \left[ \begin{array}{l} (1-(1-t)^\alpha + t^\alpha) \\ \times\left( \begin{array}{l} \frac{1}{b^2}\left[1-\frac{b-a}{b}t\right]^{-2} \\ +\frac{1}{a^2}\left[1-\frac{a-b}{a}t\right]^{-2} \end{array} \right) \end{array} \right] dt \\
&= \frac{b-a}{2} + \frac{a(b-a)}{2b(\alpha+1)} \\
&\times \left( \begin{array}{l} {}_2F_1\left(2, \left[\alpha+1, \alpha+2; \frac{1}{2}\right], \frac{b-a}{b}\right) \\ -{}_2F_1\left(2, \left[1, \alpha+2; \frac{1}{2}\right], \frac{b-a}{b}\right) \end{array} \right) \\
&+ \frac{b(b-a)}{2a(\alpha+1)} \left( \begin{array}{l} {}_2F_1\left(2, \left[\alpha+1, \alpha+2; \frac{1}{2}\right], \frac{a-b}{a}\right) \\ -{}_2F_1\left(2, \left[1, \alpha+2; \frac{1}{2}\right], \frac{a-b}{a}\right) \end{array} \right). \quad (25)
\end{aligned}$$

This marks the end of the proof. ■

**Corollary 3.5:** If we set $\psi(\nu) = \frac{\nu^\rho}{\rho}$ with $\rho > 0$ in Theorem 3.1, then the following inequality including multiplicative Katugampola fractional integrals can be obtained for multiplicatively $\rho p$-convex functions.

$$\left| \frac{f\left(\frac{a+b}{2}\right)}{\left[{}_a\mathcal{K}_*^\alpha f(b) \cdot_* \mathcal{K}_b^\alpha f(a)\right]^{\frac{\rho^\alpha\Gamma(\alpha+1)}{2[b^\rho - a^\rho]^\alpha}}} \right| \leq \left[f^*(a) f^*(b)\right]^{\Upsilon_1\left(t;\frac{\nu^\rho}{\rho}\right)},$$

where

$$\begin{aligned}
&\Upsilon_1\left(t;\frac{\nu^\rho}{\rho}\right) \\
&= \frac{b-a}{2} + \frac{a^{1-\rho}(b^\rho - a^\rho)}{2\rho(\alpha+1)} \\
&\times \left[ \begin{array}{l} {}_2F_1\left(\frac{\rho-1}{\rho}, \left[\alpha+1, \alpha+2; \frac{1}{2}\right], \frac{a^\rho-b^\rho}{a^\rho}\right) \\ -{}_2F_1\left(\frac{\rho-1}{\rho}, \left[1, \alpha+2; \frac{1}{2}\right], \frac{a^\rho-b^\rho}{a^\rho}\right) \end{array} \right] + \\
&\frac{b^{1-\rho}(b^\rho - a^\rho)}{2\rho(\alpha+1)} \left[ \begin{array}{l} {}_2F_1\left(\frac{\rho-1}{\rho}, \left[\alpha+1, \alpha+2; \frac{1}{2}\right], \frac{b^\rho-a^\rho}{b^\rho}\right) \\ -{}_2F_1\left(\frac{\rho-1}{\rho}, \left[1, \alpha+2; \frac{1}{2}\right], \frac{b^\rho-a^\rho}{b^\rho}\right) \end{array} \right].
\end{aligned}$$

*Proof:* The proof is akin to the approach used in Corollary 3.4. ■

On the assumption that $p$ and $q$ are conjugate exponents that obey the relation $p+q=pq$ with $p,q\in(1,\infty)$, we now state the theorem below.

**Theorem 3.2:** Suppose $f:[a,b]\to\mathbb{R}^+$ is an increasing function that is $^*$differentiable, and $\psi$ is a continuous and strictly increasing function with a continuous derivative $\psi'$ on $(a,b)$. If $[\ln f^*]^q$ is a $M_\psi A$-$p$-function on $[a,b]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we can derive that

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\cdot{}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right| \le [f^*(a)f^*(b)]^{\left(\frac{1}{2}\right)^{\frac{1}{q}}[\Upsilon_2(t;\psi(\nu))+\Upsilon_3(t;\psi(\nu))]},$$

where

$$\Upsilon_2(t;\psi(\nu))=\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|A(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}$$

$$\Upsilon_3(t;\psi(\nu))=\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|D(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}$$

and $A(t;\psi(\nu))$, $D(t;\psi(\nu))$ are given in Lemma 3.1.

*Proof:* Relying on the inequality (18) as stated in the proof of Theorem 3.1, and employing the Hölder's inequality, we can conclude that

$$\begin{aligned}&\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\cdot{}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right|\\ &\le\exp\left\{\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|A(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\times\\ \left[\int_0^{\frac{1}{2}}\left[\ln f^*\left(\psi^{-1}\left[\begin{array}{c}(1-t)\psi(a)\\+t\psi(b)\end{array}\right]\right)\right]^q\mathrm{d}t\right]^{\frac{1}{q}}\end{array}\right\}\\ &\times\exp\left\{\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|D(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\times\\ \left[\int_0^{\frac{1}{2}}\left[\ln f^*\left(\psi^{-1}\left[\begin{array}{c}(1-t)\psi(b)\\+t\psi(a)\end{array}\right]\right)\right]^q\mathrm{d}t\right]^{\frac{1}{q}}\end{array}\right\}.\end{aligned}\tag{26}$$

Exploiting the condition that $[\ln f^*]^q$ is a $M_\psi A$-$p$-function on $[a,b]$, we infer that

$$\begin{aligned}&\left[\ln f^*\left(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\right)\right]^q\\ &\le[\ln f^*(a)]^q+[\ln f^*(b)]^q\end{aligned}\tag{27}$$

and

$$\begin{aligned}&\left[\ln f^*\left(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\right)\right]^q\\ &\le[\ln f^*(a)]^q+[\ln f^*(b)]^q.\end{aligned}\tag{28}$$

By applying the inequalities (27) and (28) to the inequality (26), we can arrive at the conclusion that

$$\begin{aligned}&\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\cdot{}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right|\\ &\le\exp\left\{\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|A(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\\ \times\left(\int_0^{\frac{1}{2}}[\ln f^*(a)]^q+[\ln f^*(b)]^q\,\mathrm{d}t\right)^{\frac{1}{q}}\end{array}\right\}\\ &\times\exp\left\{\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|D(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\\ \times\left(\int_0^{\frac{1}{2}}[\ln f^*(a)]^q+[\ln f^*(b)]^q\,\mathrm{d}t\right)^{\frac{1}{q}}\end{array}\right\}\\ &=\exp\left\{\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|A(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\\ \times\left(\frac{1}{2}[\ln f^*(a)]^q+\frac{1}{2}[\ln f^*(b)]^q\right)^{\frac{1}{q}}\end{array}\right\}\\ &\times\exp\left\{\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|D(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\\ \times\left(\frac{1}{2}[\ln f^*(a)]^q+\frac{1}{2}[\ln f^*(b)]^q\right)^{\frac{1}{q}}\end{array}\right\}.\end{aligned}\tag{29}$$

Leveraging the inequality $M^\tau+N^\tau\le(M+N)^\tau$, which is valid for any $M\ge0, N\ge0$ with $\tau\ge1$, we have that

$$\begin{aligned}&\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi}f(b)\cdot{}_*\mathcal{I}_b^{\alpha;\psi}f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right|\\ &\le\exp\left\{\begin{array}{c}\left(\frac{1}{2}\right)^{\frac{1}{q}}\ln[f^*(a)f^*(b)]\\ \times\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|A(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\end{array}\right\}\\ &\times\exp\left\{\begin{array}{c}\left(\frac{1}{2}\right)^{\frac{1}{q}}\ln[f^*(a)f^*(b)]\\ \times\left(\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)^p\\ \times|D(t;\psi(\nu))|^p\end{array}\right]\mathrm{d}t\right)^{\frac{1}{p}}\end{array}\right\}.\end{aligned}\tag{30}$$

Here ends the proof. ■

**Corollary 3.6:** By setting $\psi(\nu)=\nu$ in Theorem 3.2, we can obtain the following inequality for $P$-convex functions that involve multiplicative RL-fractional integrals.

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_a\mathcal{I}_*^\alpha f(b)\cdot_*\mathcal{I}_b^\alpha f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[b-a]^\alpha}}}\right| \le[f^*(a)f^*(b)]^{(b-a)\left(\frac{1}{2}\right)^{\frac{1}{q}}\Delta(\alpha,p)},$$

where

$$\Delta(\alpha,p)=\left(\int_0^{\frac{1}{2}}(1-(1-t)^\alpha+t^\alpha)^p\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

**Remark 3.3:** The numerical evaluation of $\Delta(\alpha, p)$, as described in Corollary 3.6, can be performed for certain parameter values. The results are illustrated in Table II below.

TABLE II: Numerical evaluations of the integrals $\Delta(\alpha, 2)$, $\Delta(\alpha, 4)$ and $\Delta(\alpha, 6)$ by Matlab

| $\alpha$ | $\Delta(\alpha,2)$ | $\Delta(\alpha,4)$ | $\Delta(\alpha,6)$ |
|---|---|---|---|
| 0.1 | 0.62461 | 0.75018 | 0.80134 |
| 0.2 | 0.56556 | 0.69302 | 0.75052 |
| 0.3 | 0.52226 | 0.65448 | 0.71810 |
| 0.4 | 0.48996 | 0.62728 | 0.69590 |
| 0.5 | 0.46559 | 0.60747 | 0.67999 |
| 0.6 | 0.44707 | 0.59274 | 0.66826 |
| 0.7 | 0.43296 | 0.58166 | 0.65946 |
| 0.8 | 0.42226 | 0.57328 | 0.65282 |
| 0.9 | 0.41419 | 0.56699 | 0.64783 |

Finally, our study centers on the scenario where $q \in (0, 1]$, and we begin by revisiting the following Lemma.

**Lemma 3.2:** [48] Let $0 < q \leq 1$ and let $f : [a, b] \to [1, \infty)$ such that $f^q$ is a $M_\psi A$-$p$-function on $[a, b]$.

(i) If $0 < q \leq \frac{1}{2}$, then

$$f\left(\psi^{-1}\left((1-t)\psi(a) + t\psi(b)\right)\right) \leq q2^{\frac{1}{q}-1}\left[f(a) + f(b) + \left(\frac{2}{q} - 2\right)\sqrt{f(a)f(b)}\right].$$

(ii) If $\frac{1}{2} < q \leq 1$, then

$$f\left(\psi^{-1}\left((1-t)\psi(a) + t\psi(b)\right)\right) \leq f(a) + f(b) + \left(2^{\frac{1}{q}} - 2\right)\sqrt{f(a)f(b)}.$$

**Theorem 3.3:** Let $f : [a, b] \to \mathbb{R}^+$ be a function that is both increasing and $^*$differentiable, and let $\psi$ be a continuous and strictly increasing function with a continuous derivative $\psi'$ on $(a, b)$. If $[\ln f^*]^q$ is a $M_\psi A$-$p$-function on $[a, b]$ for $0 < q \leq 1$, then we have the following results.

$(i)$ For $0 < q \leq \frac{1}{2}$, we have

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi} f(b) \cdot {}_*\mathcal{I}_b^{\alpha;\psi} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right| \leq \exp\left\{\begin{array}{l} q2^{\frac{1}{q}-1}\Upsilon_1(t;\psi(\nu)) \\ \times\left[\begin{array}{l}\ln f^*(a) + \ln f^*(b) \\ +\left(\frac{2}{q}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right]\end{array}\right\}.$$

$(ii)$ For $\frac{1}{2} < q \leq 1$, we have

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi} f(b) \cdot {}_*\mathcal{I}_b^{\alpha;\psi} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right| \leq \exp\left\{\begin{array}{l} \Upsilon_1(t;\psi(\nu)) \\ \times\left[\begin{array}{l}\ln f^*(a) + \ln f^*(b) \\ +\left(2^{\frac{1}{q}}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right]\end{array}\right\},$$

where $\Upsilon_1(t;\psi(\nu))$ is given in Theorem 3.1.

*Proof:* Relying on the inequality (18) obtained through the proof of Theorem 3.1, we can acquire that

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi} f(b) \cdot {}_*\mathcal{I}_b^{\alpha;\psi} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right| \leq \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{l}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)|A(t;\psi(\nu))| \times \\ \ln f^*\left[\psi^{-1}\left(\begin{array}{l}(1-t)\psi(a) \\ +t\psi(b)\end{array}\right)\right]\end{array}\right]dt\right\} \times \exp\left\{\int_0^{\frac{1}{2}}\left[\begin{array}{l}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right)|D(t;\psi(\nu))| \times \\ \ln f^*\left[\psi^{-1}\left(\begin{array}{l}(1-t)\psi(b) \\ +t\psi(a)\end{array}\right)\right]\end{array}\right]dt\right\}. \quad (31)$$

**Case 1**. Provided $0 < q \leq \frac{1}{2}$, knowing that $[\ln f^*]^q$ is a $M_\psi A$-$p$-function, we can employ the inequality from part $(i)$ of Lemma 3.2 to derive that

$$\ln f^*\left(\psi^{-1}\left((1-t)\psi(a) + t\psi(b)\right)\right) \leq q2^{\frac{1}{q}-1}\left[\begin{array}{l}\ln f^*(a) + \ln f^*(b) \\ +\left(\frac{2}{q}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right] \quad (32)$$

and

$$\ln f^*\left(\psi^{-1}\left(t\psi(a) + (1-t)\psi(b)\right)\right) \leq q2^{\frac{1}{q}-1}\left[\begin{array}{l}\ln f^*(a) + \ln f^*(b) \\ +\left(\frac{2}{q}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right]. \quad (33)$$

The following result is derived by applying the inequalities (32) and (33) to the inequality (31):

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi} f(b) \cdot {}_*\mathcal{I}_b^{\alpha;\psi} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^\alpha}}}\right| \leq \exp\left\{\begin{array}{l} q2^{\frac{1}{q}-1}\times \\ \left[\begin{array}{l}\ln f^*(a) + \ln f^*(b) \\ +\left(\frac{2}{q}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right]\times \\ \int_0^{\frac{1}{2}}\left[\begin{array}{l}\left(\frac{1-(1-t)^\alpha+t^\alpha}{2}\right) \\ \times\left(\begin{array}{l}|A(t;\psi(\nu))| \\ +|D(t;\psi(\nu))|\end{array}\right)\end{array}\right]dt\end{array}\right\}. \quad (34)$$

It is possible to acquire the first inequality as stated in Theorem 3.3.

**Case 2**. Provided $\frac{1}{2} < q \leq 1$, knowing that $[\ln f^*]^q$ is a $M_\psi A$-$p$-function, we can employ the inequality from part $(ii)$ of Lemma 3.2 to deduce that

$$\ln f^*\left(\psi^{-1}\left((1-t)\psi(a) + t\psi(b)\right)\right) \leq \ln f^*(a) + \ln f^*(b) + \left(2^{\frac{1}{q}}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)} \quad (35)$$

and

$$\ln f^*\left(\psi^{-1}\left(t\psi(a) + (1-t)\psi(b)\right)\right) \leq \ln f^*(a) + \ln f^*(b) + \left(2^{\frac{1}{q}}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}. \quad (36)$$

Applying the inequalities (35) and (36) to the inequality (31), we derive that

$$\left|\frac{f\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)}{\left[{}_a\mathcal{I}_*^{\alpha;\psi} f(b)\cdot {}_*\mathcal{I}_b^{\alpha;\psi} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[\psi(b)-\psi(a)]^{\alpha}}}}\right| \le \exp\left\{ \begin{array}{c} \left[\begin{array}{c}\ln f^*(a)+\ln f^*(b)\\ +\left(2^{\frac{1}{q}}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right] \\ \times\int_0^{\frac{1}{2}}\left[\begin{array}{c}\left(\frac{1-(1-t)^{\alpha}+t^{\alpha}}{2}\right)\\ \times\left[\begin{array}{c}|A(t;\psi(\nu))|\\+|D(t;\psi(\nu))|\end{array}\right]\end{array}\right]\mathrm{d}t\end{array}\right\}. \tag{37}$$

Hence, the proof is finished here. ■

**Corollary 3.7:** By setting $\psi(\nu)=\nu$ in Theorem 3.3, we can derive the following inequality for $P$-convex functions that involve multiplicative RL-fractional integrals:

$(i)$ For $0<q\le\frac{1}{2}$, we have

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_a\mathcal{I}_*^{\alpha} f(b)\cdot_* \mathcal{I}_b^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[b-a]^{\alpha}}}}\right| \le \exp\left\{\begin{array}{c} q2^{\frac{1}{q}-1}\left[\begin{array}{c}\ln f^*(a)+\ln f^*(b)\\ +\left(\frac{2}{q}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right] \\ \times\frac{(b-a)\left(\alpha-1+2^{1-\alpha}\right)}{2(\alpha+1)}\end{array}\right\}.$$

$(ii)$ For $\frac{1}{2}<q\le 1$, we have

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\left[{}_a\mathcal{I}_*^{\alpha} f(b)\cdot_* \mathcal{I}_b^{\alpha} f(a)\right]^{\frac{\Gamma(\alpha+1)}{2[b-a]^{\alpha}}}}\right| \le \exp\left\{\begin{array}{c} \left[\begin{array}{c}\ln f^*(a)+\ln f^*(b)\\ +\left(2^{\frac{1}{q}}-2\right)\sqrt{\ln f^*(a)\ln f^*(b)}\end{array}\right] \\ \times\frac{(b-a)\left(\alpha-1+2^{1-\alpha}\right)}{2(\alpha+1)}\end{array}\right\}.$$

To visually showcase and validate the correctness of Corollary 3.7, a specific illustrated example is given below.

**Example 3.2:** Considering $f(t)=\exp\left\{\frac{q}{2+q}t^{\frac{2}{q}+1}\right\}$ for all $t\in[0,\infty)$. We can infer that the function $(\ln f^*(t))^q=t^2$ possesses $P$-convexity. By setting $a=0$ and $b=1$ in the inequalities established by Corollary 3.7, we can arrive at the conclusions detailed below.

(i) For $0<q\le\frac{1}{2}$, we have

$$\exp\left\{\frac{q}{4+2q}\left(\left(\frac{1}{2}\right)^{\frac{2}{q}}-\alpha\left[\begin{array}{c}\int_0^1(1-t)^{\alpha-1}t^{\frac{2}{q}+1}\mathrm{d}t\\ +\int_0^1 t^{\frac{2}{q}+\alpha}\mathrm{d}t\end{array}\right]\right)\right\} \le \exp\left\{q2^{\frac{1}{q}-1}\left[\frac{\alpha-1+2^{1-\alpha}}{2(\alpha+1)}\right]\right\}.$$

(ii) For $\frac{1}{2}<q\le 1$, we have

$$\exp\left\{\frac{q}{4+2q}\left(\left(\frac{1}{2}\right)^{\frac{2}{q}}-\alpha\left[\begin{array}{c}\int_0^1(1-t)^{\alpha-1}t^{\frac{2}{q}+1}\mathrm{d}t\\ +\int_0^1 t^{\frac{2}{q}+\alpha}\mathrm{d}t\end{array}\right]\right)\right\} \le \exp\left\{\frac{\alpha-1+2^{1-\alpha}}{2(\alpha+1)}\right\}.$$

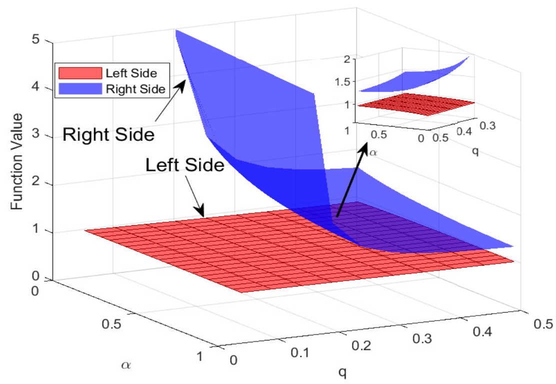


Fig. 2: Visualization graphics for $q\in\left(0,\frac{1}{2}\right]$

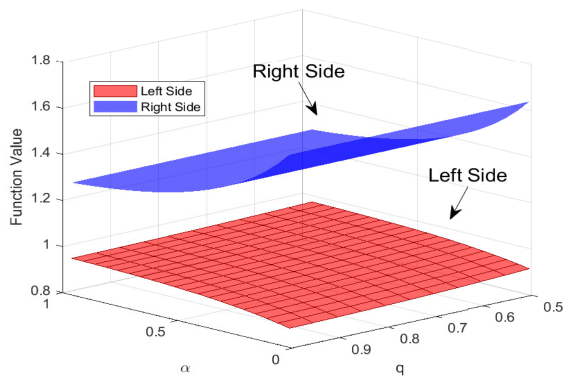


Fig. 3: Visualization graphics for $q\in\left(\frac{1}{2},1\right]$

It is apparent from Figures 2 and 3 that the left-hand value is less than the right-hand value, aligning with the theoretical result given in Corollary 3.7.

**Remark 3.4:** It is worth noting that Theorem 3.3 and Theorem 3.1 are interconnected, although they are derived using two completely different methods. Specifically, by setting $q=1$ in Theorem 3.3, we can regain Theorem 3.1.

**Remark 3.5:** In Theorems 3.2–3.3, by choosing specific forms of the function $\psi(\nu)$, we can obtain three different midpoint-type inequalities:

(1)Setting $\psi(\nu)=\ln\nu$, we get the midpoint-type inequalities for multiplicative Hadamard fractional integrals, which are applicable to GA-$P$-convex functions.

(2)Setting $\psi(\nu)=-\frac{1}{\nu}$, we get the midpoint-type inequalities for multiplicative Harmonic fractional integrals, which are applicable to Harmonic $P$-convex functions.

(3)Setting $\psi(\nu)=\frac{\nu^{\rho}}{\rho}$, we get the midpoint-type inequalities for multiplicative Katugampola fractional integrals, which are applicable to $\rho p$-convex functions.

## IV. Applications

This section is divided into two subsections, each of which addresses different applications stemming from our derived results. Subsection IV-A centers on quadrature formulas, while Subsection IV-B discusses special means.

### A. Applications to quadrature formulas

By dividing the interval $[a, b]$ into $n$ subintervals $[\mu_i, \mu_{i+1}]$ for $i = 0, 1, 2, 3, ..., n-1$, we get a partition $d : a = \mu_0 < \mu_1 < ... < \mu_{n-1} < \mu_n = b$. Subsequently, we examine the quadrature formula in the context of multiplicative calculus, as outlined below:

$$\int_a^b (f(x))^{dx} = \frac{\mathcal{N}(f,d)}{\mathcal{R}(f,d)}, \quad \mathcal{R} > 0, \tag{38}$$

where $\mathcal{N}(f,d)$ and $\mathcal{R}(f,d)$ respectively represent the approximate value and the corresponding approximation error of the integral $\int_a^b (f(x))^{dx}$ when employing different numerical integration methods. Here, we examine the midpoint quadrature formula.

$$\mathcal{N}(f,d) = \prod_{i=0}^{n-1} \left[ f\left(\frac{\mu_i + \mu_{i+1}}{2}\right)\right]^{\mu_{i+1}-\mu_i}. \tag{39}$$

The integral formulations outlined above allow us to obtain the following error estimation.

**Proposition 4.1:** Assuming all conditions in Corollary 3.2 hold, we set $\alpha = 1$ to derive the following error estimate for the midpoint quadrature formula.

$$|\mathcal{R}(f,d)| \le \prod_{i=0}^{n-1} [f^*(\mu_i) f^*(\mu_{i+1})]^{\frac{(\mu_{i+1}-\mu_i)^2}{4}}.$$

*Proof:* Applying the result obtained by setting $\alpha = 1$ in Corollary 3.2 on the subinterval $[\mu_i, \mu_{i+1}] \in [a, b]$, $i = 0, 1, \cdots, n-1$, we can conclude that:

$$\left| \frac{\left[ f\left(\frac{\mu_i+\mu_{i+1}}{2}\right)\right]^{\mu_{i+1}-\mu_i}}{\int_{\mu_i}^{\mu_{i+1}} (f(x))^{dx}} \right| \le [f^*(\mu_i) f^*(\mu_{i+1})]^{\frac{(\mu_{i+1}-\mu_i)^2}{4}}. \tag{40}$$

By multiplying $i$ from 0 up to $n-1$, we can derive the following error estimate:

$$\begin{aligned} |\mathcal{R}(f,d)| &= \left| \frac{\mathcal{N}(f,d)}{\int_a^b (f(x))^{dx}} \right| \\ &= \left| \prod_{i=0}^{n-1} \frac{\left[ f\left(\frac{\mu_i+\mu_{i+1}}{2}\right)\right]^{\mu_{i+1}-\mu_i}}{\int_{\mu_i}^{\mu_{i+1}} (f(x))^{dx}} \right| \\ &= \prod_{i=0}^{n-1} \left| \frac{\left[ f\left(\frac{\mu_i+\mu_{i+1}}{2}\right)\right]^{\mu_{i+1}-\mu_i}}{\int_{\mu_i}^{\mu_{i+1}} (f(x))^{dx}} \right| \\ &\le \prod_{i=0}^{n-1} [f^*(\mu_i) f^*(\mu_{i+1})]^{\frac{(\mu_{i+1}-\mu_i)^2}{4}}. \end{aligned} \tag{41}$$

This concludes the proof. ■

**Proposition 4.2:** Under the assumptions of Corollary 3.6, we set $\alpha = 1$ and $p = 2$ to derive the following error estimate for the midpoint quadrature formula.

$$|\mathcal{R}(f,d)| \le \prod_{i=0}^{n-1} [f^*(\mu_i) f^*(\mu_{i+1})]^{\frac{(\mu_{i+1}-\mu_i)^2}{2\sqrt{3}}}.$$

*Proof:* Relying on Corollary 3.6, the proof proceeds in a manner analogous to Proposition 4.1. ■

### B. Applications to special means

We consider two particular types of special means as illustrated below:

(1)The arithmetic mean:
$A(a_1, a_2, ..., a_n) = \frac{a_1+a_2+...+a_n}{n}$.

(2)The $\eta$-logarithmic mean:
$L_\eta(a,b) = \left(\frac{b^{\eta+1}-a^{\eta+1}}{(\eta+1)(b-a)}\right)^{\frac{1}{\eta}}$,

where $a, b > 0, a \neq b$ and $\eta \in \mathbb{R}\backslash\{-1, 0\}$.

**Proposition 4.3:** Given $a, b \in \mathbb{R}$ with $0 \le a < b$, it follows that

$$\begin{aligned} &\exp\left\{A^4(a,b) - L_4^4(a,b)\right\} \\ &\quad \le \exp\left\{4A(-a,b)\,A\left(a^3, b^3\right)\right\}. \end{aligned}$$

*Proof:* Setting $\alpha = 1$ in Corollary 3.2 and applying it to the function $f(t) = \exp\{t^4\}$ with $t \in \mathbb{R}^+$, we arrive at the desired result. ■

**Proposition 4.4:** For $a, b \in \mathbb{R}$ with $0 \le a < b$, we deduce that:

$$\begin{aligned} &\exp\left\{\frac{q}{q+1}\left[A^{\frac{1}{q}+1}(a,b) - L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(a,b)\right]\right\} \\ &\quad \le \exp\left\{\frac{2\sqrt{3}}{3} A(-a,b)\,A\left(a^{\frac{1}{q}}, b^{\frac{1}{q}}\right)\right\}. \end{aligned}$$

*Proof:* Setting $\alpha = 1$ and $p = 2$ in Corollary 3.6 and applying it to the function $f(t) = \exp\left\{\frac{q}{q+1} t^{\frac{1}{q}+1}\right\}$ with $t \in \mathbb{R}^+$, we arrive at the desired conclusion. ■

## V. Conclusions

This study presents midpoint-type inequalities for multiplicatively $M_\psi A$-$p$-functions using multiplicative $\psi$-Hilfer fractional integrals. To the best of our knowledge, the investigation of midpoint-type inequalities through the application of multiplicative $\psi$-Hilfer fractional integrals has not been reported in the existing literature. As a result, our work broadens the theoretical scope of multiplicative calculus, particularly regarding multiplicative fractional integrals.

The methodologies presented in this work can be extended to other inequalities, such as Maclaurin-type [24], Milne-type [52], Bullen-type [53], [54], Hadamard-type [55], [56], Ostrowski-type [57], [58], among others. Moreover, researchers could employ various multiplicative fractional integrals, including multiplicative $(k, s)$-fractional integrals [59], multiplicative $k$-RL-fractional integrals [41], and multiplicative fractional integrals having exponential kernels [39], to establish new midpoint-type inequalities. This represents an interesting field for future research.

## REFERENCES


[1] M. Grossman and R. Katz, *Non-Newtonian Calculus: A Self-contained, Elementary Exposition of the Authors' Investigations*, Lee Press, Pigeon Cove, MA, 1972.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[3] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities," *Math. Comput. Model.*, vol. 57, no. 9–10, pp. 2403–2407, 2013.

[4] L. N. Li, Y. Z. Li, and Q. D. Huang, "Fractional order Chebyshev cardinal functions for solving two classes of fractional differential equations," *Engineering Letters*, vol. 30, no. 1, pp. 208–213, 2022.

[5] A. K. Alomari and Y. Massoun, "Numerical solution of time fractional coupled Korteweg-de Vries equation with a Caputo fractional derivative in two parameters," *IAENG International Journal of Computer Science*, vol. 50, no. 2, pp. 388–393, 2023.

[6] K. Kankhunthodi, V. Kongratana, A. Numsomran, and V. Tipsuwanporn, "Self-balancing robot control using fractional-order PID controller," in *Proceedings of the International MultiConference of Engineers and Computer Scientists 2019*, 13–15 March, Hong Kong, pp. 77–82, 2019.

[7] Q. J. Cheng and C. Y. Luo, "Analytical properties, fractal dimensions and related inequalities of $(k, h)$-Riemann–Liouville fractional integrals," *J. Comput. Appl. Math.*, vol. 450, Art. ID 115999, 19 pages, 2024.

[8] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris, and F. Ahmad, "$(k, s)$-Riemann–Liouville fractional integral and applications," *Hacet. J. Math. Stat.*, vol. 45, no. 1, pp. 77–89, 2016.

[9] T. U. Khan and M. A. Khan, "Generalized conformable fractional operators," *J. Comput. Appl. Math.*, vol. 346, pp. 378–389, 2019.

[10] W. W. Liu and L. S. Liu, "Properties of Hadamard fractional integral and its application," *Fractal Fract.*, vol. 6, Art. ID 670, 28 pages, 2022.

[11] H. Budak and P. Agarwal, "New generalized midpoint type inequalities for fractional integral," *Miskolc Math. Notes*, vol. 20, no. 2, pp. 781–793, 2019.

[12] P. Neang, K. Nonlaopon, J. Tariboon, S.K. Ntouyas, and P. Agarwal, "Some trapezoid and midpoint type inequalities via fractional $(p, q)$-calculus," *Adv. Differ. Equ.*, vol. 2021, Art. ID 333, 22 pages, 2021.

[13] F. Hezenci and H. Budak, "Generalized fractional midpoint type inequalities for co-ordinated convex functions," *Filomat*, vol. 37, no. 17, pp. 4103–4124, 2023.

[14] B. Benaissa, N. Azzouz, and H. Budak, "Hermite–Hadamard type inequalities for new conditions on $h$-convex functions via $\psi$-Hilfer integral operators," *Anal. Math. Phys.*, vol. 14, Art. ID 35, 20 pages, 2024.

[15] S. S. Dragomir, "Hermite–Hadamard type inequalities for generalized Riemann–Liouville fractional integrals of $h$-convex functions," *Math. Meth. Appl. Sci.*, vol. 44, pp. 2364–2380, 2021.

[16] M. A. Latif, "Refinements and applications of Hermite–Hadamard-type inequalities using Hadamard fractional integral operators and GA-convexity," *Mathematics*, vol. 12, Art. ID 442, 18 pages, 2024.

[17] H. M. Srivastava, S. K. Sahoo, P. O. Mohammed, D. Baleanu, and B. Kodamasingh, "Hermite–Hadamard type inequalities for interval-valued preinvex functions via fractional integral operators," *Int. J. Comput. Intell. Syst.*, vol. 15, Art. ID 8, 12 pages, 2022.

[18] T. C. Zhou and T. S. Du, "On the reverse Minkowski's, reverse Hölder's and other fractional integral inclusions arising from interval-valued mappings," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 4, pp. 1294–1307, 2023.

[19] T. Abdeljawad and M. Grossman, "On geometric fractional calculus," *J. Semigroup Theory Appl.*, vol. 2016, Art. ID 2, 14 pages, 2016.

[20] H. Budak and K. Özçelik, "On Hermite–Hadamard type inequalities for multiplicative fractional integrals," *Miskolc Math. Notes*, vol. 21, no. 1, pp. 91–99, 2020.

[21] A.E. Bashirov, E. Mısırlı, Y. Tandoğdu, and A. Özyapıcı, "On modeling with multiplicative differential equations," *Appl. Math.-J. Chin. Univ. Ser. B*, vol. 26, no. 4, pp. 425–438, 2011.

[22] L. Florack and H. V. Assen, "Multiplicative calculus in biomedical image analysis," *J. Math. Imaging Vis.*, vol. 42, no. 1, pp. 64–75, 2012.

[23] G. Singh, S. Bhalla, and R. Behl, "A multiplicative calculus approach to solve applied nonlinear models," *Math. Comput. Appl.*, vol. 2023, Art. ID 28, 11 pages, 2023.

[24] B. Meftah, "Maclaurin type inequalities for multiplicatively convex functions," *Proc. Amer. Math. Soc.*, vol. 151, no. 5, pp. 2115–2125, 2023.

[25] A. Berkane, B. Meftah, and A. Lakhdari, "Right-Radau-type inequalities for multiplicative differentiable $s$-convex functions," *J. Appl. Math. Informatics*, vol. 42, no. 4, pp. 785–800, 2024.

[26] B. Meftah, A. Lakhdari, S. Wedad, and C. D. Benchettah, "Companion of Ostrowski inequality for multiplicatively convex functions," *Sahand Commun. Math. Anal.*, vol. 21, no. 2, pp. 289–304, 2024.

[27] A. Mateen, Z. Y. Zhang, M. A. Ali, and M. Fečkan, "Generalization of some integral inequalities in multiplicative calculus with their computational analysis," *Ukrainian Math. J.*, vol. 76, no. 10, pp. 1027–3190, 2024.

[28] M. A. Ali, M. Abbas, Z. Y. Zhang, I. B. Sial, and R. Arif, "On integral inequalities for product and quotient of two multiplicatively convex functions," *Asian Res. J. Math.*, vol. 12, no. 3, pp. 1–11, 2019.

[29] S. Özcan, A. Uruš, and S. I. Butt, "Hermite–Hadamard-type inequalities for multiplicative harmonic $s$-convex functions," *Ukrainian Math. J.*, vol. 76, no. 9, pp. 1364–1382, 2024.

[30] S. Khan and H. Budak, "On midpoint and trapezoid type inequalities for multiplicative integrals," *Mathematica*, vol. 64, no. 87, pp. 95–108, 2022.

[31] J. Q. Xie, M. A. Ali, and T. Sitthiwirattham, "Some new midpoint and trapezoidal type inequalities in multiplicative calculus with applications," *Filomat*, vol. 37, no. 20, pp. 6665–6675, 2023.

[32] A. Mateen, Z. Y. Zhang, M. Toseef, and M. A. Ali, "A new version of Boole's formula type inequalities in multiplicative calculus with application to quadrature formula," *Bull. Belg. Math. Soc. Simon Stevin*, vol. 31, no. 4, pp. 541–562, 2024.

[33] A. Frioui, B. Meftah, A. Shokri, A. Lakhdari, and H. Mukalazi, "Parametrized multiplicative integral inequalities," *Adv. Continuous Disc. Models*, vol. 2024, Art. ID 12, 18 pages, 2024.

[34] M. Merad, B. Meftah, A. Moumen, and M. Bouye, "Fractional Maclaurin-type inequalities for multiplicatively convex functions," *Fractal Fract.*, vol. 7, Art. ID 879, 13 pages, 2023.

[35] H. Boulares, B. Meftah, A. Moumen, R. Shafqat, H. Saber, T. Alraqad, and E. E. Ali, "Fractional multiplicative Bullen-type inequalities for multiplicative differentiable functions," *Symmetry*, vol. 15, Art. ID 451, 12 pages, 2023.

[36] T. S. Du and Y. Long, "The multi-parameterized integral inequalities for multiplicative Riemann–Liouville fractional integrals," *J. Math. Anal. Appl.*, vol. 541, no. 1, Art. ID 128692, 41 pages, 2025.

[37] M. B. Almatrafi, W. Saleh, A. Lakhdari, F. Jarad, and B. Meftah, "On the multiparameterized fractional multiplicative integral inequalities," *J. Inequal. Appl.*, vol. 2024, Art. ID 52, 27 pages, 2024.

[38] H. Fu, Y. Peng, and T. S. Du, "Some inequalities for multiplicative tempered fractional integrals involving the $\lambda$-incomplete gamma functions," *AIMS Math.*, vol. 6, no. 7, pp. 7456–7478, 2021.

[39] Y. Peng, H. Fu, and T. S. Du, "Estimations of bounds on the multiplicative fractional integral inequalities having exponential kernels," *Commun. Math. Stat.*, vol. 12, no. 2, pp. 187–211, 2024.

[40] A. Kashuri, S. K. Sahoo, M. Aljuaid, M. Tariq, and M. D. L. Sen, "Some new Hermite–Hadamard type inequalities pertaining to generalized multiplicative fractional integrals," *Symmetry*, vol. 15, Art. ID 868, 14 pages, 2023.

[41] L. L. Zhang, Y. Peng, and T. S. Du, "On multiplicative Hermite–Hadamard- and Newton-type inequalities for multiplicatively $(P, m)$-convex functions," *J. Math. Anal. Appl.*, vol. 534, Art. ID 128117, 39 pages, 2024.

[42] M. A. Ali, M. Feckan, C. Promsakon, and T. Sitthiwirattham, "A new approach of generalized fractional integrals in multiplicative calculus and related Hermite–Hadamard-type inequalities with applications," *Math. Slovaca*, vol. 74, no. 6, pp. 1445–1456, 2024.

[43] D. Khan, S. I. Butt, and Y. Seol, "Analysis on multiplicatively $(P, m)$-superquadratic functions and related fractional inequalities with applications," *Fractals*, vol. 33, Art. ID 2450129, 20 pages, 2025.

[44] Y. Peng and T. S. Du, "Fractional Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively $P$-functions," *Filomat*, vol. 37, no. 28, pp. 9497–9509, 2023.

[45] Z. Y. Zhou and T. S. Du, "Analytical properties and related inequalities derived from multiplicative Hadamard $k$-fractional integrals," *Chaos Solitons Fractals*, vol. 189, Art. ID 115715, 36 pages, 2024.

[46] A. E. Bashirov, E. M. Kurpınır, and A. Özyapıcı, "Multiplicative calculus and its applications," *J. Math. Anal. Appl.*, vol. 337, no. 1, pp. 36–48, 2008.

[47] M. A. Ali, H. Budak, M. Z. Sarikaya, and Z. Y. Zhang, "Ostrowski and Simpson type inequalities for multiplicative integrals," *Proyecciones*, vol. 40, no. 3, pp. 743–763, 2021.

[48] Z. R. Tan and T. S. Du, "Analytical properties on multiplicatively $M_\psi A$-$p$-functions and related multiplicative $\psi$-Hilfer fractional trapezoid-type inequalities," *Fractals*, vol. 33, Art. ID 2550059, 27 pages, 2025.

[49] H. Kadakal, "Multiplicatively $P$-functions and some new inequalities," *New Trends Math. Sci.*, vol. 6, no. 4, pp. 111–118, 2018.

[50] F. Safdar, M. A. Noor, and K. I. Noor, "$\varphi$-Geometrically Log $h$-Convex Functions," *Punjab Univ. J. Math. (Lahore)*, vol. 51, no. 9, pp. 71–83, 2019.

[51] M.A. Özarslan and C. Ustaoğlu, "Some incomplete hypergeometric functions and incomplete Riemann–Liouville fractional integral operators," *Mathematics*, vol. 7, Art. ID 483, 18 pages, 2019.

[52] A. Lakhdari, H. Budak, M. U. Awan, and B. Meftah, "Extension of Milne-type inequalities to Katugampola fractional integrals," *Bound. Value Probl.*, vol. 2024, Art. ID 100, 16 pages, 2024.

[53] T. S. Du, C. Y. Luo, and Z. Cao, "On the Bullen-type inequalities via generalized fractional integrals and their applications," *Fractals*, vol. 29, Art. ID 2150188, 20 pages, 2021.

[54] D. Zhao, M. A. Ali, H. Budak, and Z. Y. He, "Some Bullen-type inequalities for generalized fractional integrals," *Fractals*, vol. 31, no. 4, Art. ID 2340060, 11 pages, 2023.

[55] S. I. Butt, S. Yousaf, A. O. Akdemir, and M. A. Dokuyucu, "New Hadamard-type integral inequalities via a general form of fractional integral operators," *Chaos Solitons Fractals*, vol. 148, Art. ID 111025, 14 pages, 2021.

[56] Z. R. Yuan and T. S. Du, "The multipoint-based Hermite–Hadamard Inequalities for fractional integrals with exponential kernels," *IAENG International Journal of Applied Mathematics*, vol. 54, no. 6, pp. 1013–1025, 2024.

[57] M. Vivas-Cortez, M. Samraiz, A. Ullah, S. Iqbal, and M. Mukhtar, "A modified class of Ostrowski-type inequalities and error bounds of Hermite–Hadamard inequalities," *J. Inequal. Appl.*, vol. 2023, Art. ID 130, 27 pages, 2023.

[58] W. B. Sun and H. Y. Wan, "New local fractional Hermite–Hadamard-type and Ostrowski-type inequalities with generalized Mittag-Leffler kernel for generalized $h$-preinvex functions," *Demonstr. Math.*, vol. 57, no. 1, Art. ID 20230128, 27 pages, 2024.

[59] X. H. Zhang, Y. Peng, and T. S. Du, "$(k, s)$-fractional integral operators in multiplicative calculus," *Chaos Solitons Fractals*, vol. 195, Art. ID 116303, 25 pages, 2025.