Properties of the Space of Group Valued Continuous Functions

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***Abstract*—In this paper, we establish necessary and sufficient conditions for countable fan tightness and countable strong fan tightness of the space $C_p(T,G)$ based on Menger property and the Rothberger property, respectively. Additionally, we explore the relationship between countable fan tightness, Reznichenko and Hurewicz properties for the space $C_p(T,G)$. Moreover, we demonstrated that the Menger property is preserved under G equivalence of topological spaces. As a key contribution, we present a general result concerning the fan tightness of $C_p(T,G)$ and the Hurewicz number of the space T^n for every natural number n. Finally, we investigate the monolithicity of the space $C_p(T,G)$.**

***Index Terms*—Topological group, Selection principles, Menger property, Rothberger property, Hurewicz property, Reznichenko property, Countable fan tightness, Countable strong fan tightness, Fan tightness, Hurewicz number, G-equivalence, Monolithicity.**

I. Introduction

IN 1992, Arkhangelski [1] introduced a theory called C_p theory for topological function spaces. Subsequently, many mathematicians made considerable efforts to enhance C_p theory, giving it the sophistication and elegance it currently has. Arkhangelsky's PhD student, Tkachuk, has written a wide range of books [2], [3], [4], [5], which serve as a vast compilation of many findings related to C_p theory. In these book series, Tkachuk has produced a large collection of open problems that serve as an attractive stimulus for further research, not only to advance C_p theory but also to meet the challenges of other branches of mathematics. Building on this line of inquiry, a recent monograph by McCoy [6] highlights the broad aspects of the C_p theory, and provides information about its more general implications. His study includes a comprehensive exploration of many properties related to the space of continuous functions from one topological space to another, culminating in the examination of uniform, fine, and graph topology. More recently, in 2023, Mishra and Bhaumik [7] and Aaliya and Mishra [8], [9], [10] studied properties of topological function spaces under Cauchy convergence topology and regular topology, respectively. These studies also give an idea for further studies on topological function space by applying the new structure.

A topological group provides a natural setting where algebraic operations (group structure) interact smoothly with topology (continuity). This unification allows the application of topological methods to group theory and vice versa. Recently in 2024, the research papers [25], [26] studied about topological transformation groups and this concept has sparked a surge of research across various disciplines, including engineering, medicine, economics and environmental sciences.

However, such a study was initiated in 2010 by Shakhmatov and Spevak [11], considering a new structure on topological function spaces as topological groups. In this paper [11] authors defined point wise convergence topology on the space of group-valued continuous functions (denoted by $C_p(T,G)$) and further studied on good numbers of properties of such space along with preservation of properties during some equivalencies like G equivalence and $\mathbb{T}$ equivalence. In 2011, Kocinac [12] extended the closure type properties of the function spaces $C_p(T)$ to the function space $C_p(T,G)$.

We aim to investigate various topological properties of the space $C_p(T,G)$, including Menger, Rothberger, Hurewicz, Reznichenko properties as well as countable fan tightness, countable strong fan tightness, fan tightness, and monolithicity. Additionally, we establish that under certain conditions, G-equivalence preserves the Menger property.

We are following most of the notation and terminology from [14] and [11], unless we state otherwise. The study assumes that all topological spaces under consideration are Tychonoff, meaning that they are both completely regular and satisfy the T_1 separation axiom.

The selection principle refers to a guiding principle that affirms the feasibility of obtaining mathematically significant objects by selecting elements from predetermined sequences of sets. Selection theory primarily involves the characterization of covering properties, measure theoretic properties, category theoretic properties and local properties within topological spaces with a particular emphasis on function spaces. These theories provide a framework for understanding and analyzing the behavior of sets and functions with respect to these properties. Frequently, employing selection theory to characterize mathematical properties presents a challenging endeavor, which often yields fresh insights into the distinctive nature of the property being studied. In this paper we are going to apply the results of selection principles to discuss the tightness of the $C_p(T,G)$ space.

In order to better understand of our results we are going to briefly recall some important concepts such as selection principles, Menger, Hurewicz, Rothberger and Reznichenko properties but for detail readers can refer to the following paper [15], [16]. Also note that throughout this paper G refers to arbitrary topological group.

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II. Preliminaries

Definition II.1 (Selection principles)**.** *[17], Section 1.1, Let $\mathcal{P}$ and $\mathcal{Q}$ are the collection of the subsets of an infinite set I. Let $S_1(\mathcal{P}, \mathcal{Q})$ and $S_f(\mathcal{P}, \mathcal{Q})$ are the two selection hypotheses defined by*

- *(i) Notation $S_1(\mathcal{U}, \mathcal{V})$ denote the statement: Corresponding to every sequence $(P_n) \in \mathcal{P}$ we can find a sequence $(Q_n) \in P_n$ and $\{Q_n : n \in \mathbb{N}\} \in \mathcal{Q}$.*
- *(ii) Notation $S_f(\mathcal{P}, \mathcal{P})$ denote the statement: Corresponding to every sequence $(P_n) \in \mathcal{P}$ we can find a sequence $(P_n) \subset Q_n$ with $\bigcup_{n=1}^{\infty} Q_n \in \mathcal{Q}$ and P_n is finite for every n.*

Definition II.2 (Menger space)**.** *[13], Definition 1.2, A space T is Menger if for every open covers (P_k), $k \in \omega$ of T there exists a sequence $(Q_k) \subset P_k$ with $\bigcup\{Q_k : k \in \omega\}$ is a cover of T and Q_k is finite.*

Definition II.3 (Countable fan tightness space)**.** *[12], Section 1.1, If $S_f(\Omega_t, \Omega_t)$ holds for each $t \in T$, then T has countable fan tightness (CFT) where $\Omega_t = \{K \subset T\backslash\{t\} : t \in \overline{K}\}$.*

Example II.1. *The space $C_p(T, G)$ has CFT, where $T = [0, 1]$ is set with usual topology and $G = S^1$ is unit circle in the complex plane with the usual topology.*

Definition II.4 (ω cover)**.** *[12], Section 1.1, A ω cover $\mathcal{P}$ is an open cover of a space T and if for every finite $K \subset T$ we can find a $P \in \mathcal{P}$ such that $K \subset P$ and $T \notin \mathcal{P}$.*

Definition II.5 (Hurewicz covering property)**.** *[12], Section 1.1, If for any sequence (P_k) of open covers of T we can identify a sequence (Q_k) with Q_k being a finite subset of P_k for all $k \in \mathbb{N}$ and every $t \in T$ belongs to $\bigcup Q_k$ for all but finitely many k, then the space T satisfies the Hurewicz covering property.*

Definition II.6 (Rothberger space)**.** *[13], Section 2, A space T satisfies Rothberger property if for each $(\mathcal{P}_k)$, $k \in \omega$ of open covers of T, we can find some $Q_k \in \mathcal{P}_k$. Here $\{P_k : k \in \omega\}$ is a cover of T.*

The following implications are true for any arbitrary space.

$\sigma -$ compact space $\Rightarrow$ Hurewicz space $\Rightarrow$ Menger space $\Rightarrow$ Lindelöf space

Rothberger space $\Rightarrow$ Menger space

Definition II.7 (Network weight of a space)**.** *[1], Section 2, Let $\mathcal{D}$ be a collection of subsets of a space T. Then $\mathcal{D}$ is a network of T if for each $t \in T$ and for $O \in \tau(T)$ containing t we can identify some $P \in \mathcal{D}$ such that $t \in P \subset O$.*
Network weight is the least cardinality of a network in T. It is denoted by $nw(T)$.

Definition II.8 (Weight of a space)**.** *[1], Section 2, Weight of a space T is the smallest cardinal number κ with $\mathcal{B}$ is a base for the space T, then $\kappa \leq |\mathcal{B}|$ and it is denoted by $w(T)$.*

Note that if $w(X) = \omega$, then X is a space with a countable base (it is a separable metric space). Also if $nw(X) = \omega$, then X is a space with a countable network.

Definition II.9 ($\tau\, monolithic$ space)**.** *[1], Section 6, Let τ denotes the cardinal number with infinite cardinality. Then a space T is said to be $\tau\, monolithic$ if S is a subspace of T with $|S| \leq \tau$, $nw(\overline{S}) \leq \tau$.*

Definition II.10 (Monolithic space)**.** *[1], Section 6, A monolithic space is $\tau\, monolithic$ for each infinite cardinal τ.*

Example II.2. *$\mathbb{N}$ with Cofinite topology is a monolithic space.*

Example II.3. *$C_p([0, 1])$ is a monolithic space.*

Definition II.11 (Stable space)**.** *[1], Section 6, Let Y be a continuous image of X. Then a space X is called τ stable space if for every Y, $iw(Y) < \tau$ ($iw(Y)$ denotes the minimal weight of all spaces onto which Y can be condensed). A stable space is τ stable for each infinite cardinal τ.*

Note that all compact topological spaces are stable.

Definition II.12. *[11], Definition 2.2, Let T be any arbitrary space and $K \subset T$ be closed. Then T is*

- *(i) G regular, provided that corresponding to each K in T and for each t in $T\backslash K$, there exists a function h in $C_p(T, G)$ and $a \neq e$ in G with $h(t) = a$ for $t \in T\backslash K$ and $h(t) = e$ for all $t \in K$.*
- *(ii) G^* regular, provided that there exists an element $a \neq e$ in G such that corresponding to each K in T and for t in $T\backslash K$, we can find a function h in $C_p(T, G)$ such that $h(t) = a$ and $h(t) = e$ for all $t \in K$.*
- *(iii) G^{**} regular, provided that corresponding to each K in T, for each t in $T\backslash K$ and for every a in G, we can find $h \in C_p(T, G)$ such that $h(t) = a$ and $h(t) = e$ for every $t \in K$.*

III. Tightness properties of the space $C_p(T, G)$

In 1986, [18] Arkhangelskii find an equivalent condition for countable fan tightness of the space $C_p(X)$ in term of Menger property of the space X. In continuation of this study we are going to generalize this result for the space $C_p(X, G)$ under the certain condition. In this section we are going to find an equivalent condition for the space $C_p(T, G)$ to be CFT (countable strong fan tightness) in term of Menger (Rothberger) property of the base space T. Following results motivates us to find such conditions.

Theorem III.1. *[12], Corollary 2.4, Let G be a metric and T be a space satisfies G^*-regularity. The subsequent assertions are equivalent.*

1. *The space $C_p(T, G)$ holds countable fan tightness.*
2. *T satisfies $S_{fin}(\Omega, \Omega)$.*

Theorem III.2. *[12],Theorem 2.5, Let G be a metric and T be a space satisfies G^*-regularity. The subsequent assertions are equivalent.*

1. *The space $C_p(T, G)$ possesses countable strong fan tightness.*
2. *T satisfies $S_1(\Omega, \Omega)$.*

Definition III.1. *[12], Section 2, A ω Lindelof space is a space T that has a countable ω-subcover for every ω-cover of T. Equivalently, T^n are Lindelof for all $n \in \mathbb{N}$.*

Theorem III.3. *[19], theorem 14, Let T be a ω Lindelof space, then T satisfies $S_f(\Omega, \Omega^{gp})$ if and only if T^n has Hurewicz property for every natural number n.*

Theorem III.4. *Let G be a metric and T be an ω-lindeloff space satisfies G^*-regularity. Then the subsequent assertions are equivalent.*

1) *$C_p(T, G)$ satisfies countable fan tightness*
2) *T^n holds Menger property for every natural number n.*

Proof: Let $C_p(T, G)$ has CFT, then from Theorem (III.1), X satisfies $S_f(\Omega, \Omega)$. So from Theorem (III.3), X^n has Hurewicz property for every $n \in \mathbb{N}$. Since Hurewicz property implies Menger property for every topological space [13, Definition 1.2], X^n has Menger property for every $n \in \mathbb{N}$.

Conversely assume that X^n has Menger property for every $n \in \mathbb{N}$. Then from [[19], theorem 14] X satisfies $S_f(\Omega, \Omega)$. Therefore from theorem (III.1), $C_p(T, G)$ has CFT. ■

Definition III.2 (ϕ small set)**.** *[20], Section 3, Let $\phi =< \mathcal{H}_k, k \in \omega >$ be an open covers of the space T. Then $A \subseteq T$ is ϕ small if for every $k \in \omega$ there are $j \in \omega$ and sets $H_i \in \mathcal{H}_{K+i}$ with $A \subset \cap\{G_i : i < j\}$.*

Let us consider the following property($*$) that was introduced by Gerlits and Nagy in [20, Theorem 5]: If $\phi =< \mathcal{G}_n, n \in \omega >$ is an open covers of T, then T can be represented as a union of countably many ϕ small sets.

Theorem III.5. *[19], Theorem 19, For an ω Lindelof space T, T has property $S_1(\Omega, \Omega^{gp})$ if and only if T^n has property($*$) for $n \in \mathbb{N}$.*

Using the above theorems we can generalize the result of [21, Theorem 1], in C_p theory that $C_p(X)$ has countable strong fan tightness if and only if every finite power of X is Rothberger (equivalent to property C'' in [21]).

Theorem III.6. *Let G be a metric and T be an ω-lindeloff space satisfies G^*-regularity. The subsequent assertions are equivalent.*

1) *$C_p(T, G)$ holds countable strong fan tightness*
2) *Z^n satisfies Rothberger property for every natural number n.*

Proof: Suppose that (1) holds, then from theorem (III.2), T satisfies $S_1(\Omega, \Omega)$. Then from theorem (III.5), since T is a ω Lindelof space T satisfies property ($*$). In a ω Lindelof space property($*$) implies Rothberger property. Therefore T^n satisfies Rothberger property for $n \in \mathbb{N}$.

Conversely assume that (2) holds. In a ω Lindelof space Rothberger property implies property($*$), so from Theorem (III.5), T satisfies $S_1(\Omega, \Omega)$. Then from Theorem (III.2), $C_p(T, G)$ has countable strong fan tightness. ■

Now we will generalize another result on C_p theory that $C_p(X)$ has countable fan tightness and the Reznichenko property if and only if X^n have the Hurewicz property for $n \in \mathbb{N}$ ([19], theorem 21). To prove the following result we need the following lemma from ([12], Lemma 2.1).

Lemma III.1. *[12], Lemma 2.1, Suppose that T satisfies G^*-regularity, then there exists an element $a \neq e$ in G such that for every $O \in \tau(T)$ and every non-empty finite set $K \subset O$, there exists $h_{K,O} \in C_p(T, G)$ satisfying $h_{K,O}(K) \subseteq \{e\}$ and $h_{K,O}(T \backslash O) \subseteq \{a^{-1}\}$.*

Theorem III.7. *Let G be a metric and T be an ω-Lindelof space satisfies G^* regularity. The subsequent assertions are equivalent.*

1) *$C_p(T, G)$ has CFT and Reznichenko property*
2) *Z^n has Hurewicz covering property for every natural number n.*

Proof: Suppose that (i) holds. Consider the sequence (W_i) of ω covers of T. From the definition (II.4), corresponding to a finite subset K of X we can identify $W(i, K) \in W_i$ with $K \subset W(i, K)$. Then according to the lemma (III.1) there exists a in G and h in $C_p(T, G)$ depends on K and $W(i, K)$ such that $h(K) = \{e\}$, and $h(T \backslash W) \subset \{a^{-1}\}$. Let us consider $A_k = \{h : W \in W_i, h$ depends on K and $W(i, K)\}$. This implies for all $k \in \mathbb{N}$, $h_e \in \overline{A_k}$. Since $C_p(T, G)$ have Reznichenko property, for each $k \in \mathbb{N}$ we can identify a sequence of finite subsets B_k of A_k such that corresponding to every neighbourhood V of h_e, $V \cap B_n \neq \phi$, except for finitely many $k \in \mathbb{N}$. Let $V_i = \{W(i, K) : h \in B_k\}$. Then $V_i \subset W_i$ for every $i \in \mathbb{N}$. Let $E \subseteq T$ be finite. Now we will show that $E \subset V$ for some $V \in V_k$, except for finitely many $k \in \mathbb{N}$. Consider the neighborhood $N(D, O_k)$ of h_e such that $N(D, O_k) \cap B_k \neq \phi$ for $k \geq k_0$ and for some $k_0 \in \mathbb{N}$. Let $h \in N(D, O_k)$, that is for $t \in D$, $h(t) \in O_k$. Therefore $D \subset W(i, K) \in V_i$. This proves (2).

Conversely assume that X^n has Hurewicz property for every natural number n. Let $C_k : k \in \mathbb{N}$ be the countable local base at the identity e in G. Consider the sequence I_k of subsets in $C_p(T, G)$ with $h_e \in \bigcap_{k=1}^{\infty} \overline{I_k}$. Let $D \subseteq T$ be finite. Then the neighborhood $N(D, O_1)$ of h_e satisfies $N(D, O_1) \cap I_1 \neq \phi$. Take $h_{D(1)} \in I_1$. The continuity of $h_{D(1)}$ guarantees that for every $t \in D$ we can choose $U_t \in \tau(T)$ with $h_{D(1)}(U_t) \subset C_1$ and set $U_{D(1)} = \bigcup_{t \in D} U_t$. Then the collection $\mathcal{U}_1 = \{U_{D(1)} : D \subset T$ is finite $\}$. Clearly $\mathcal{U}_1$ is a ω cover of T. Similar way we can construct $\mathcal{U}_n$ for every natural number $n \geq 2$. Since T satisfies Hurewicz property and also by using theorem (III.3), for any $n \in \mathbb{N}$ we can find finite subsets $\mathcal{K}_n$ of $\mathcal{U}_n$ with every finite subset of T is a member of $\mathcal{K}_n$ except for finitely many n. Now without effecting the generality, we can suppose that $\mathcal{K}_i \cap \mathcal{K}_j = \phi$, for $i, j \in \mathbb{N}$. This implies $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$ is a groupable ω cover of T. Suppose that for $n \in \mathbb{N}$, let $\mathcal{K}_n = \{U_{D(1,n)}, U_{D(2,n)}, \ldots U_{D(k,n)}\}$. For $n \in \mathbb{N}$, let $M_n \subset I_n$ be the set of all functions in I_n having the property $U_{D(i,n)} \in \mathcal{K}_n, i \leq k_n$. Then $f_{D(i)}(U_t) \subset C_i$. This implies $C_p(T, G)$ satisfies Reznichenko property. Countable tightness of $C_p(T, G)$ will implies from theorem (III.1) and (III.3). ■

IV. Fan tightness of $C_p(X, G)$ and Hurewicz number of X

We define fan tightness and Hurewicz number from [24] by Shou Lin.

Definition IV.1. *[24], Section 1, Let $a \in T$. Then $vet(T, a) = \omega + min\{\epsilon :$ for each $\delta < \epsilon$ and corresponding to each collection of subsets $\{A_\delta\}$ of T having $a \in \bigcap \overline{A_\delta}$*

there exists a subset $B_\delta \subset A_\delta$ satisfies $|B_\delta| < \epsilon$ such that $a \in \overline{\bigcup B_\delta}\}$. Then fan tightness $vet(T)$ is given by $vet(T) = sup\{vet(T,a) : a \in T\}$.

Note that, a space T has CFT if and only if $vet(T) = \omega$.

Definition IV.2. *[24], Section 1, Let $\mathcal{C}$ be the collection of compact subsets of T and α be the network of $\mathcal{C}$, which is closed with respect to the finite unions and closed subsets. Then we say that a collection of subsets of T is an α cover if every member of α is contained in some member of this collection.*

Definition IV.3. *[24], Section 1, A k cover is an α cover and it is equal to $\mathcal{C}$.*

Definition IV.4. *[24], Section 1, An α cover is same as the set of all finite subsets of T, then it is called ω cover.*

Definition IV.5. *[24], Section 1, We define $\alpha H(T) = \omega + min\{\epsilon :$ for each collection $\{\mathcal{U}_\delta\}, \delta < \epsilon$ of open α-covers of T we can identify a subset $\mathcal{B}_\delta \subset \mathcal{U}_\delta$ with $|\mathcal{B}_\delta| < \epsilon$ for each $\delta < \epsilon$ such that $\bigcup \mathcal{B}_\delta$ is an α cover of $T\}$. We call $\alpha H(T)$ as a α Hurewicz number. If α consists of the singleton sets of T, then it is called Hurewicz number of T. A space X is said to be a Hurewicz space if $H(T) = \omega$.*

Theorem IV.1. *Let G be a metric and T satisfies G^*-regularity. Then $vet(C_p(T,G)) = Sup\{H(T^n)\}$, $n \in N$.*

Proof: Assume that $vet(C_p(T,G)) = \epsilon$. We need to show that $Sup\{H(T^n)\} = \epsilon$, for every natural number n. To see this, for $\delta < \epsilon$ and for the space T^n, consider the collection of open covers $\{\mathcal{U}_\delta\}$. For $\delta < \epsilon$, Now we define a property $P(n,\delta)$ for the collection $\mathcal{V}$ of subsets of T that for every $\{V_i\}_{i=1}^n \subset \mathcal{V}$ there exists $U \in \mathcal{U}_\delta$ satisfies $\Pi_{i=1}^n V_i \subset U$. Let $I(n,\delta)$ represents the family of all finite open sets satisfies the property $P(n,\delta)$. Now for each $\mathcal{V} \in I(n,\delta)$ and for $V \in \mathcal{V}$ we can identify $U \in \mathcal{U}_\delta$ with $V \subset U$. By applying lemma (III.1) we can find a $a \in G$ and $h(V,\delta) \in C_p(T,G)$ such that $h(V,\delta)(V) = \{e\}$ and $h(V,\delta)(T\backslash U) \subset \{a^{-1}\}$. Let us define $F_\mathcal{V} = \{h(V,\delta) \in C_p(T,G) : V \in \mathcal{V}$ and $\mathcal{V} \in I(n,\delta)\}$. Now we claim that for $\delta < \epsilon$, this implies set $A_\delta = \bigcup F_\mathcal{V}$ is a dense subset of $C_p(T,G)$.
For a finite subset E of T and an open set O containing $e \in G$, we denote $W(E,O)$ be a neighbourhood of h_e in $C_p(T,G)$. Since E is finite, we can identify $\mathcal{W} \in I(n,\delta)$. That is $\mathcal{W}$ satisfies property $P(n,\delta)$. Then $E \subset \bigcup \mathcal{W}$. Now take $t \in E$, we define the set $V_t = \bigcap\{W \in \mathcal{W} : t \in W\}$. Let $\mathcal{V}$ be the collection sets V_t. It is evident that the family $\mathcal{V}$ satisfies the property $P(n,\delta)$ and $E \subset \bigcup \mathcal{V}$. Also we observed that $\mathcal{W}$ satisfies property $P(n,\delta)$. Since $V(t_i)$ is a subset of W_i, we have $\Pi_{i=1}^n V(t_i) \subset U$. Choose a function h' in $C_p(T,G)$ with $h'(E) = h(E)$ and $h'(T\backslash U) = a^{-1}$. Then $h' \in F_\mathcal{V} \subset A_\delta$, so $W(E,O) \bigcap A_\delta \neq \phi$. Therefore A is a dense in $C_p(T,G)$.
Now fix a $c \in G$, then consider a function $f_c \in C(T,G)$ with $f_c(T) = c$. Then for $\delta < \epsilon$, $f_c \in \bigcap \overline{A_\delta}$. From the definition of fan tightness of T, for each $\delta < \epsilon$ there is a subset $B_\delta \subset A_\delta$ with $|B_\delta| < \epsilon$ and $f_g \in \overline{\bigcup B_\delta}$. Then we can identify a subset $J(n,\delta)$ of $I(n,\delta)$ with $|J(n,\delta)| < \epsilon$ such that $B_\delta \subset \bigcup\{F_\mathcal{V} : \mathcal{V} \in J(n,\delta)\}$. Since $J(n,\delta)$ satisfies property $P(n,\delta)$, we have for $\mathcal{V} \in J(n,\delta)$ and for each $\psi = (V_1, V_2 \ldots V_n) \in \mathcal{V}^n$, take $M_\psi \in \mathcal{U}_\delta$ such that $\Pi_{i=1}^n V_i \subset M_\psi$. Put $\mathcal{M}_\delta = \{M_\psi : \psi \in \mathcal{V}^n, \mathcal{V} \in J(n,\delta)\}$. Obviously $|\mathcal{M}_\delta| < \epsilon$ and $\mathcal{M}_\delta \subset \mathcal{U}_\delta$. Now we claim that $\bigcup \mathcal{M}_\delta$ covers T.
Let $(t_1, t_2, \ldots, t_n) \in T^n$ and N be an open set containing of $h_c \in C_p(T,G)$. Since $h_c \in \overline{\bigcup B_\delta}$, we can find $\delta < \epsilon$ such that $N \cap B_\delta$ is non empty. This implies there exists $\mathcal{V} \in I(n,\delta)$ such that $N \cap F_\mathcal{V} \neq \phi$. Let $z \in N \cap F_\mathcal{V}$. Then $z(T\backslash U) = c^{-1}$ and $z(t_i) \in N$, for every i between 1 and n. Now for i between 1 and n we can choose $V_i \in \mathcal{V}$ with $t_i \in V_i$. That is we can find $M_\psi \in \mathcal{M}_\delta$ with $(t_1, t_2, \ldots, t_n) \in \Pi_{i=1}^n V_i \subset M_\psi$. So $(t_1, t_2, \ldots, t_n) \in \bigcup(\bigcup_{\delta<\epsilon} \mathcal{M}_\delta)$. Hence $H(T^n) \leq vet(C_p(T,G))$.

Conversely assume that $Sup\{H(T^n)\} = \epsilon$. Let (O_k) be a sequence of decreasing local base at e in G. For $\delta < \epsilon$, let $\{A_\delta\}$ be a collection of subsets of $C_p(T,G)$ with $h_e \in \bigcap \overline{A_\delta}$. For each finite set K of T and $\delta < \epsilon$ the neighborhood $W(K,O_1)$ of h_e has non-empty intersection with A_δ. Choose $h(t,\delta) \in W(F,O_1) \bigcap A_\delta$. Since $h(t,\delta)$ is continuous, for $t_i \in F$ we can identify an open set $V(t_i)$ with $h(K,\delta)(V(t_i)) \subset O_1$. Let $U(t,\delta) = \Pi_{i=1}^n V(t_i)$ be a neighborhood of $t = (t_1, t_2, \ldots, t_n) \in T^n$. Then $\mathcal{U}(n,\delta) = \{U(t,\delta) : t \in T^n\}$ covers T^n and also note that for each $(y_1, y_2, \ldots, y_n) \in U(t,\delta)$, $h(t,\delta)(y_i) \in O_i$.

Case (i) Suppose $\epsilon > \omega$. Since $H(T^n) \leq \epsilon$, for each $\delta < \epsilon$ we can identify a collection of subsets $\{S(n,\delta)\}$ in T^n with $|S(n,\delta)| < \epsilon$ such that $\bigcup S(n,\delta)$ covers T^n. Note that $S(n,\delta) = \{U(t,\delta) : t \in S(n,\delta)\}$. Now define for each $\delta < \epsilon$, $B(n,\delta) = \{h(t,\delta) : t \in S(n,\delta)\}$ and $B_\delta = \bigcup_{n=1}^{\infty} B(n,\delta)$. Then $B_\delta \subset A_\delta$ with $|B_\delta| < \epsilon$, and $h \in \overline{\bigcup B_\delta}$. Let $W(E,O)$ be a basic neighborhood of $h_e \in C_p(T,G)$. Then $\delta < \epsilon$ such that $(y_1, y_2, \ldots, y_n) \in \bigcup S(n,\delta)$ and $x \in S(n,\epsilon)$ such that $(y_1, y_2, \ldots, y_n) \in U(t,\delta)$. So $h(t,\delta) \in B(n,\delta)$ and $h(t,\delta) \in O_n$ for each $i \leq n$. So $h(t,\delta) \in W(E,O)$ (Here $E = \{y_1, y_2, \ldots, y_n\}$). That is $h(t,\delta) \in W(E,O) \cap B_\delta$. Therefore $f_e \in \overline{\bigcup B_\delta}$.
Case 2. Suppose $\epsilon = \omega$. That is replace δ with a natural number $k \geq n$. Then choose $B_k = \bigcup_{n=1}^k B(n,k)$ and follow case 1. The proof will be immediate. Therefore $vet(C_p(T,G)) = Sup\{H(T^n)\}$. ∎

V. Preservation of Menger property on G-equivalence

Definition V.1. *[11], Definition 1.2(ii), Two topological spaces X and Y are said to be G equivalent if $C_p(X,G) \cong C_p(Y,G)$.*

If $G = \mathbb{T}$, the circle group $\mathbb{R}\backslash\mathbb{Z}$, then G equivalence can be viewed as $\mathbb{T}$ equivalence.

Theorem V.1. *[11], Theorem 10.7, $\mathbb{T}$ equivalence preserves pseudo compactness.*

Lemma V.1. *Suppose X and Y are $\mathbb{T}$ equivalent. If X is a Cech complete Menger space, then Y is also a Menger space.*

Proof: Let X be a Cech complete Menger space. Then from [[22], theorem 1.2], X is a σ compact space. Since

X and Y are $\mathbb{T}$ equivalent and from (V.1), σ compactness preserves $\mathbb{T}$ equivalence, we can say that Y is also a Menger space. ■

Definition V.2. *If G can be embedded as a subgroup of a compact group, then it is called precompact.*

Theorem V.2. *[11], Corollary 10.5, If G is precompact and Abelian, then $\mathbb{T}$-equivalence implies G-equivalence.*

By using above theorem and lemma, we can state the following result.

Theorem V.3. *Let G be precompact and Abelian. Suppose that X and Y are G equivalent. If X is a Cech complete Menger space, then Y is also a Menger space.*

Proof: Suppose X and Y are G equivalent. Take $G = \mathbb{T}$, then by Lemma (V.1), $\mathbb{T}$ equivalence preserves Menger property. From Theorem (V.2), in case of a precompact Abelian group $\mathbb{T}$ eqivalence implies G equivalence. Therefore Y is also a Menger space. ■

VI. Monolithicity on $C_p(X,G)$

Theorem VI.1. *Let T be a topological space and G be second countable. Then $nw(T) = nw(C_p(T,G))$*

Proof: To prove this, first we claim that $nw(C_p(T,G)) \leq nw(T)$. Fix a network $\mathcal{N}$ in T and a countable base $\mathcal{B}$ in G. Choose $M_1, M_2, M_3, \ldots, M_\kappa \in \mathcal{N}$ and $U_1, U_2, U_3, \ldots, U_\kappa \in \mathcal{B}$. We define $W(M_1, M_2, M_3, \ldots, M_\kappa, U_1, U_2, U_3, \ldots, U_\kappa) = \{f \in C_p(T,G) \colon f(M_i) \subset U_i, i = 1,2,3,\ldots,\kappa\}$.

Let $N' = \{W(M_1, M_2, M_3, \ldots, M_\kappa, U_1, U_2, U_3, \ldots, U_\kappa)\}$. Then we will show that N' is a network in $C_p(T,G)$. Since $|N'| \leq |N|$, it follows that $nw(C_p(T,G)) \leq nw(T)$.

Let $h \in C_p(T,G)$ and V be the open neighbourhood of h in $C_p(T,G)$ i.e. $V = \{g \in C_p(T,G) \colon g(t) \in U$ for some open set $U \in G\}$. Then, there exists open sets $U_1, U_2, U_3, \ldots, U_\kappa \in \mathcal{B}$ such that $U = \bigcup_{i=1}^{\kappa} U_i$. Since h is continuous, there exists $M_i \in \mathcal{N}$ with $h(M_i) \subset U_i$, for $i = 1,2,3,\ldots,\kappa$ i.e, $h \in N'$.

Now we claim that $N' \subset V$. To prove this, let $g \in N'$, then $g \in \{W(M_1, M_2, M_3, \ldots, M_\kappa, U_1, U_2, U_3, \ldots, U_\kappa\}$, i.e. $g \in \{f \in C_p(T,G) \colon f(M_i) \subset U_i, i = 1,2,3,\ldots,\kappa\}$. Let $U = \bigcup_{i=1}^{\kappa} U_i$. Clearly U is an open set and $g(t) \in U$, for each $t \in T$, i.e. $g \in V$. Thus N' is a network of $C_p(T,G)$. So $nw(C_p(T,G)) \leq nw(T)$. To get the reverse inequality we use the fact that $X \subset C_p(C_p(T,G))$. So $nw(T) \leq nw(C_p(C_p(T,G))) \leq nw(C_p(T,G))$. ■

In the above theorem the second countability of the topological group G is a sufficient condition. To see this consider the topological group $G = \mathbb{R}$ under addition with discrete topology. Every group together with discrete topology can be viewed as a topoloical group. An uncountable topological space with discrete topology cannot be a second countable space. So G does not satisfy second countablity.

If we let $T = \mathbb{Z}$ with the topology induced from the usual topology of $\mathbb{R}$ and $nw(T) = \aleph_0$, then the space $C_p(T,G)$ consists only of constant functions, and its cardinality is equal to that of $\mathbb{R}$, where $\mathbb{R}$ is considered with the discrete topology. Therefore, it does not have a countable network. Hence, $nw(C_p(T,G)) \neq \aleph_0$

Theorem VI.2. *Let $h \colon Z \to W$ be a map. Let $h^* \colon G^W \to G^Z$ be a map defined by $h^*(\phi)(z) = \phi(h(z))$ for $\phi \in G^W$, then h^* is continuous.*

Proof: To show that h^* is continuous, let V be an open set containing of $\psi \in G^Z$ and suppose that $\psi = h^*(\phi)$ for some $\phi \in G^W$. Then, $V = \{g \in G^Z \colon g(z) \in U$ for some open set $U \in G$ for each $z \in Z\}$. Since $\psi \in V$, we have $\psi(z) \in U$ for some open set U in G. Define $V' = \{h \in G^W : h(z) \in U\}$. Clearly V' is an open set in G^W.

We will prove that $h^*(V') \subset V$. Let $\theta \in h^*(V')$, then $\theta = h^*(g)$ for some $g \in V'$, i.e. $\theta(z) = g(h(z))$, and $g(w) \in U$ for some $w \in W$. That is, $\theta(z) \in U$. So $\theta \in V$. Therefore, h^* is continuous. ■

Theorem VI.3. *Let $h : Z \to W$ be a map. If $f(Z) = W$, then $h^* : G^W \to G^Z$ is a homeomorphism from G^W onto the closed subspace $h^*(G^W)$ of G^Z.*

Proof: Let $h(Z) = W$. To prove h^* is a homeomorphism, first we will show that h^* is a bijective function. To see this, let $\theta_1, \theta_2 \in G^W$ and assume that $\theta_1 \neq \theta_2$, then there exists $w \in W$ such that $\theta_1(w) \neq \theta_2(w)$. Since $h(Z) = W$, corresponding to each $w \in W$ there exists $z \in Z$ such that $h(z) = w$. So, $h^*(\theta_1)(z) = \theta_1(h(z)) = \theta_1(w) \neq \theta_2(w) = \theta_2(h(z)) = h^*(\theta_2)(z)$, i.e. h^* is one-one. Therefore, h^* is a bijective function from G^W to $h^*(G^W)$. In a similar way of the proof of Theorem (VI.2) we can easily show that $(h^*)^{-1}$ is continuous. Therefore, h^* is a homeomorphism from G^W to $h^*(G^W)$.

Next we will show that $h^*(G^W)$ is a closed set. To prove this, it is enough to show that $(h^*(G^W))^{\complement}$ is an empty set. Suppose $\theta \in (h^*(G^W))^{\complement}$. Then there does not exist $\psi \in G^W$ such that $h^*(\psi) = \theta$. That is, there does not exist $z \in Z$ such that $\psi(h(z)) = \theta(z)$. But it is a contradiction to our assumption that $h(Z) = W$. Hence $h^*(G^W)$ is a closed subspace of G^Z. ■

Note that in case of $G = \mathbb{R}$, G^* regular space can be replaced by completely regular.

Definition VI.1 (G quotient map)**.** *Let T be any arbitrary space, S be any arbitrary set and $f \colon T \to S$ be a onto map. Then G quotient topology on S is the strongest of all G^* regular topologies on S relative to which f is continuous. If the topology generated by a map $f : T \to S$ coincides with G quotient topology on S then it is called G quotient map.*

Example VI.1. *In case of $\mathbb{R}$ with usual topology the G quotient map will coincide with the $\mathbb{R}$ quotient map and topology generated by G quotient map is Real quotient topology.*

Theorem VI.4. *Let T be a compact space and G satisfies second axiom of countability. Then the space $C_p(T,G)$ is monolithic.*

Proof: To prove $C_p(T,G)$ is monolithic, let $D \subset C_p(T,G)$ and $|D| \leq \tau$. Let Δ be the diagonal product map of maps from D. Thus, $\Delta(t) = \{t_g = g(t) : g \in D\}$. Let $Y = \Delta(T)$, then S is a subspace of G^D. Therefore, $w(S) \leq |D| \leq \tau$.

Let S' be the points in S with the G quotient topology generated by the map Δ. Then define an identity map i from S' to S, making it a condensation map. So, $iw(S') \leq w(S) \leq \tau$. Since T is a compact space, it is stable, hence it is τ stable for every τ. Thus, we can find a continuous function from T to S', implying $nw(S') \leq \tau$. Also, from Theorem (VI.1), we have $nw(C_p(S', G)) = nw(S') \leq \tau$. Consider the map $\Delta\colon T \to S'$, which is evidently a G quotient map. Then define $\delta^*\colon C_p(S', G) \to C_p(T, G)$ such that $\delta^* = i^{-1} \circ \Delta$. Therefore, by Theorem (VI.3), $C_p(S', G)$ is homeomorphic to the closed subspace $K = \Delta^*(C_p(S', G))$ of $C_p(T, G)$.

We have $\Delta^* = i^{-1} \circ \Delta$, so $\Delta = i \circ \Delta^*$. Then for every $g \in D$, we can write $g = p_g \circ i \circ \Delta^*$, where p_g is a projection mapping from G^D to G. Since projection mapping on a topological group is continuous [23], $p_g \circ i$ is continuous from S' to G, thus $p_g \circ i \in C_p(S', G)$. This implies $g \in K$. Hence, $D \subset K$. Since K is closed, $\overline{D} \subset \overline{K} = K$. $nw(\overline{D}) \leq nw(K) = nw(C_p(S', G))$ as $C_p(S', G)$ is homeomorphic to K. Therefore, $nw(\overline{D}) \leq \tau$. Consequently, $C_p(T, G)$ is a monolithic space.

■

VII. Applications and Significance of the Work

Covering properties like menger, rothberger and hurewicz have concrete applications in functional analysis, dynamical systems, computer science, ramsey theory, mathematical physics, algebraic geometry, game theory and economics. These properties provide compactness like conditions that influence the structure and behavior of infinite mathematical objects across disciplines.

The results in the paper are helpful to enhance the study of C_p theory by generalizing the results from $C_p(T)$ to $C_p(T, G)$.

- Results in section III are generalizations of two main results in the topological function spaces related to tightness properties and covering properties. These results definitely helpful to solve problems related to the covering properties of function spaces.
- Result in section IV is a significant result that generalize the concept of fan tightness to the space $C_p(T, G)$.
- In section V we studied the G equivalence of Menger property and similar way we can study other properties like Rothberger property, Hurewicz properties under G equivalence.
- In section VI, Monolithicity is a compactness like property which is very useful to study C_p theory and for the theory of cardinal invariants. By studying this property in the space $C_p(T, G)$ will enhance the theory of function spaces, especially C_p theory.

VIII. Conclusion

The space $C_p(T, G)$ is a general case of $C_p(T)$ as we view $G = \mathbb{R}$. So all the properties discussed in $C_p(T)$ is relevant in case of $C_p(T, G)$ also. C_p theory serves as a bridge between topology, analysis, probability, and computation, influencing diverse areas of mathematics and science.

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