Farey Edge Graceful Labeling on Cycle, Star and Corona Product of Graph

Ajay Kumar, Neeraj Gupta, Ajendra Kumar, Suraj Tyagi, Vipin Kumar, Tarun Gupta *

Abstract—**Let $G(V, E)$ be a simple, connected and undirected graph. If each edge e_i is assigned a unique label from Farey sequence $\frac{a_i}{b_i} \in F_r$ injectively and each vertex is uniquely labeled as the sum of all adjacent edge labels $(b_i - a_i)$, then such labeling is called Farey edge graceful labeling. The maximum edge label $(b_i - a_i)$ among all edges is referred to as Farey edge strength of G. In this paper, we prove that C_r, $P_r \odot tK_1$, $P_r \odot K_2$, $K_p \odot K_1$ and $K_{1,r}$ are Farey edge graceful and we also determine Farey edge strength of these graph.**

Index Terms: Farey sequence, Farey edge graceful labeling, Farey edge strength, corona product of graph, star graph, cycle graph

I. INTRODUCTION

In 1988, Chartrand et. al. [2] identified a challenge in the field of graph labeling. The complex graphs with high vertex degrees could not be effectively labeled using exiting labeling methods. To address such limitations, they introduced a novel concept known as irregular labeling.

Definition: Let $G(V, E)$ be a graph without isolated vertices. Every edge of G is labeled with positive integer and each vertex is labeled with sum the of all adjacent edge labels. If all vertex have distinct labels then this labeling is called irregular labeling and maximum edge label is called irregularity strength of G.

Chartrand et. al. [2] proved that the irregularity strength of path P_n is $\frac{n}{2}$ if $n \equiv 0 \pmod 4$, $\frac{n+1}{2}$ if $n \equiv 1, 3 \pmod 4$, $\frac{n+2}{2}$ if $n \equiv 2 \pmod 4$ and complete graph K_n is 3.

Packiam et. al. [7] proved that irregularity strength of $C_n \odot K_2$ is $n + 1$, $C_n \odot K_3$ is $n + 1$, $C_n \odot mK_1$ is mn, $P_n \odot K_2$ is $n + 1$, $P_n \odot K_3$ is $n + 1$. Faudree et. al. [3] determined the irregularity strength of cycle graph and established the bounds of regular graph.

A decreasing sequence of the form $F_m = \{\frac{m_1}{m_2} : 0 \leq m_1 \leq m_2 \leq m,\ \gcd(m_1, m_2) = 1\}$ is called Farey sequence of order m [1], [8]. In 2023 Kumar et. al. [6] introduces the concept of Farey graceful labeling. They proved that caterpillars, hairy cycles, cycles, and paths are Farey graceful. Kumar et. al. [5] proved that $P_2 \times P_n$, complete bipartite graphs, frecracker trees are all Farey graceful. In 2024 Kumar et. al. [4] introduced a novel concept of Farey edge graceful labeling, which proves to be highly effective for labeling of complex graphs with high vertex degree. They proved that symmetric hairy cycle, helm, flower, wheel, T_k-graph and subdivision of star graphs are Farey edge graceful and determined their Farey edge strength.

Before proceeding to the next section, we formally define the following terms.

Definition: Let G and H be two graph of order n and m respectively. Taking n copies of graph H and i^{th} vertex of G connected every vertex of H by an edge. Then resultant graph is called corona product of G and H and denoted by $G \odot H$.

Definition: Let G be a simple, connected and undirected graph. If each edge e_i, $i \in \mathbb{N}$ of G is injectively labeled with Farey number $\frac{a_i}{b_i} \in F_r$ and each vertex is injectively labeled with sum of all adjacent edges i.e.

$$w_f(x_i) = \sum_{e_i^j = x_i y_j \in E(G)} (b_i - a_i)$$

is called Farey edge graceful graph [4]. The highest edge label $(b_i - a_i)$ is called Farey edge strength of graph G.

II. RESULTS

Theorem 0.1. I. C_r, $r \equiv 0 \pmod 4$ is Farey edge graceful and its Farey edge strength is $\frac{r+2}{2}$.

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II. C_r, $r \equiv 1 \pmod 4$ is Farey edge graceful and its Farey edge strength is $\frac{r+1}{2}$.

III. C_r, $r \equiv 2 \pmod 4$ is Farey edge graceful and its Farey edge strength is $\frac{r+2}{2}$.

IV. C_r, $r \equiv 3 \pmod 4$ is Farey edge graceful and its Farey edge strength is $\frac{r+3}{2}$.

Proof. Let $x_1, x_2, ..., x_r$ be the vertices of cycle C_r.

I. Define a function $f : E(C_r) \longrightarrow F_k$ where

$$k = \begin{cases} 7 & , \quad r = 4 \\ 11 & , \quad r = 8 \\ \frac{3r-4}{2} & , \quad \text{otherwise} \end{cases} \quad \text{such that:}$$

$$f(x_1x_2) = \frac{2}{3}$$

$$f(x_{2i}x_{2i+1}) = \frac{2i+2}{4i+3} \quad , \quad 1 \le i \le \frac{r}{4}$$

$$f(x_{2i+1}x_{2i+2}) = \frac{4i+3}{6i+4} \quad , \quad 1 \le i \le \frac{r}{4} - 1$$

$$f(x_{\frac{r}{2}+1+i}x_{\frac{r}{2}+2+i}) = \frac{1}{\frac{r}{2}+1-i} \quad , \quad 0 \le i \le \frac{r}{2} - 2$$

$$f(x_rx_1) = \frac{1}{2}$$

By using the definition of Farey edge graceful labeling, vertices are labeled by

$$w_f(x_i) = 2i \quad , \quad 1 \le i \le \frac{r}{2}$$

$$w_f(x_{r+1-i}) = 2i + 1 \quad , \quad 1 \le i \le \frac{r}{2}$$

We can observe that all the vertex weights are distinct. Hence, C_r, $r \equiv 0 \pmod 4$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., \frac{r+2}{2}\}$. Hence, Farey edge strength of C_r, $r \equiv 0 \pmod 4$ is $\frac{r+2}{2}$. (See Fig. 1)

II. Define a function $f : E(C_r) \longrightarrow F_k$ where

$$k = \begin{cases} 7 & , \quad r = 5 \\ 11 & , \quad r = 9 \\ \frac{3r-7}{2} & , \quad \text{otherwise} \end{cases} \quad \text{such that:}$$

$$f(x_1x_2) = \frac{2}{3}$$

$$f(x_{2i}x_{2i+1}) = \frac{2i+2}{4i+3} \quad , \quad 1 \le i \le \frac{r-1}{4}$$

$$f(x_{2i+1}x_{2i+2}) = \frac{4i+3}{6i+4} \quad , \quad 1 \le i \le \frac{r-5}{4}$$

$$f(x_{\frac{r+1}{2}+i}x_{\frac{r+3}{2}+i}) = \frac{1}{\frac{r+3}{2}-i} \quad , \quad 0 \le i \le \frac{r-3}{2}$$

$$f(x_rx_1) = \frac{1}{2}$$

By using the definition of Farey edge graceful labeling, vertices are labeled by

$$w_f(x_i) = 2i \quad , \quad 1 \le i \le \frac{r+1}{2}$$

$$w_f(x_{r+1-i}) = 2i + 1 \quad , \quad 1 \le i \le \frac{r-1}{2}$$

We can observe that all the vertex weights are distinct. Hence, C_r, $r \equiv 1 \pmod 4$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., \frac{r+1}{2}\}$. Hence, Farey edge strength of C_r, $r \equiv 1 \pmod 4$ is $\frac{r+1}{2}$.

III. Define a function $f : E(C_r) \longrightarrow F_{r+3}$ such as:

$$f(x_1x_2) = \frac{2}{3}$$

$$f(x_{2i}x_{2i+1}) = \frac{2i+2}{4i+3} \quad , \quad 1 \le i \le \frac{r-2}{4}$$

$$f(x_{2i+1}x_{2i+2}) = \frac{4i+3}{6i+4} \quad , \quad 1 \le i \le \frac{r-6}{4}$$

$$f(x_{\frac{r+4}{2}+i}x_{\frac{r+6}{2}+i}) = \frac{1}{\frac{r}{2}-i} \quad , \quad 0 \le i \le \frac{r-6}{2}$$

$$f(x_rx_1) = \frac{1}{2}$$

$$f(x_{\frac{r}{2}}x_{\frac{r+2}{2}}) = \frac{1}{\frac{r+4}{2}}$$

$$f(x_{\frac{r+2}{2}}x_{\frac{r+4}{2}}) = \frac{\frac{r+4}{2}}{r+3}$$

By using the definition of Farey edge graceful labeling,

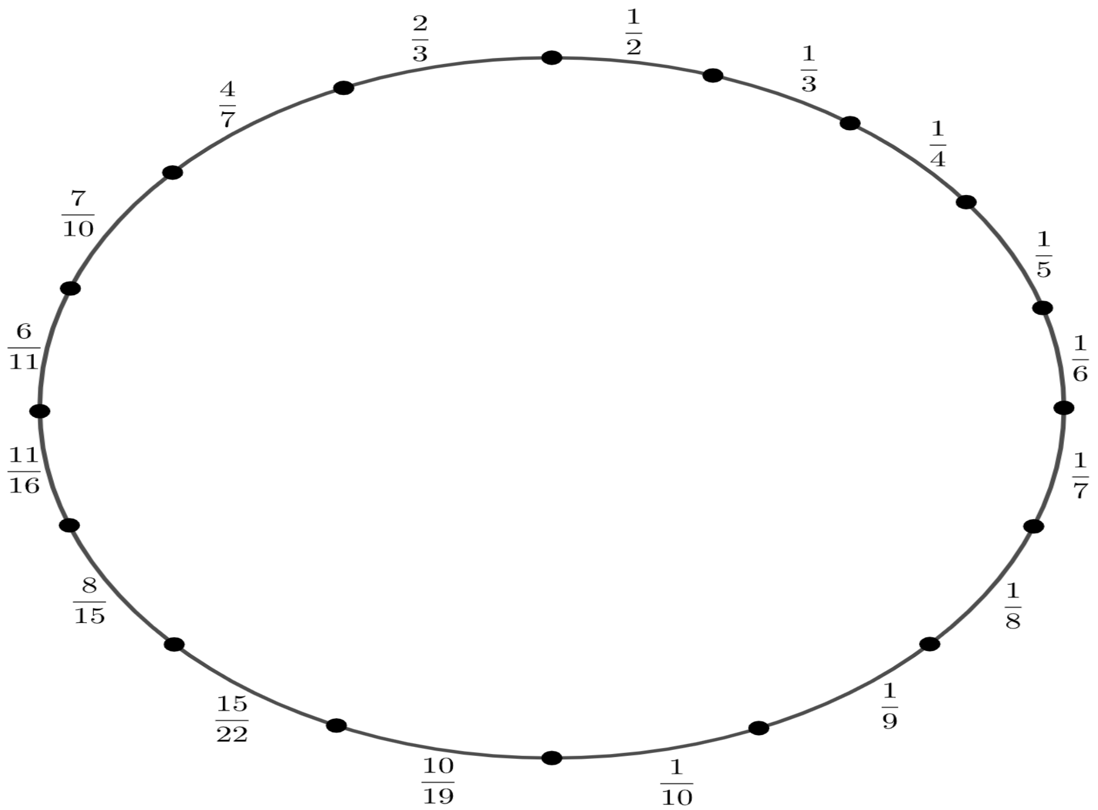

Fig. 1: Farey edge graceful labeling of C_{16}

vertices are labeled by

$$w_f(x_i) = 2i \quad , \quad 1 \le i \le \frac{r-2}{2}$$

$$w_f(x_{r+1-i}) = 2i + 1 \quad , \quad 1 \le i \le \frac{r-4}{2}$$

$$w_f(x_{\frac{r+4}{2}}) = r$$

$$w_f(x_{\frac{r}{2}}) = r + 1$$

$$w_f(x_{\frac{r+2}{2}}) = r + 2$$

We can observe that all the vertex weights are distinct. Hence, C_r, $r \equiv 2(\text{mod } 4)$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., \frac{r+2}{2}\}$. Hence, Farey edge strength of C_r, $r \equiv 2(\text{mod } 4)$ is $\frac{r+2}{2}$.

IV. Case I. When $r = 3$
Define a function $f : E(C_3) \longrightarrow F_4$ such as:

$$f(x_1x_2) = \frac{1}{2}$$

$$f(x_1x_3) = \frac{1}{3}$$

$$f(x_2x_3) = \frac{1}{4}$$

By using the definition of Farey edge graceful labeling, vertex are labeled by

$$w_f(x_i) = 2 + i \quad , \quad 1 \le i \le 3$$

We can observe that all the vertex weights are distinct. Hence, C_3 is Farey edge graceful graph.
Case II. When $r \neq 3$
Define a function $f : E(C_r) \longrightarrow F_{\frac{3r-1}{2}}$ such as:

$$f(x_{2i-1}x_{2i}) = \frac{2i+2}{4i+3} \quad , \quad 1 \le i \le \frac{r-3}{4}$$

$$f(x_{2i}x_{2i+1}) = \frac{4i+3}{6i+4} \quad , \quad 1 \le i \le \frac{r-3}{4}$$

$$f(x_{\frac{r+3}{2}+i}x_{\frac{r+5}{2}+i}) = \frac{1}{\frac{r+1}{2}-i} \quad , \quad 0 \le i \le \frac{r-5}{2}$$

$$f(x_rx_1) = \frac{1}{2}$$

$$f(x_{\frac{r-1}{2}}x_{\frac{r+1}{2}}) = \frac{1}{\frac{r+3}{2}}$$

$$f(x_{\frac{r+1}{2}}x_{\frac{r+3}{2}}) = \frac{1}{\frac{r+5}{2}}$$

By using the definition of Farey edge graceful labeling, vertices are labeled by

$$w_f(x_i) = 2i + 2 \quad , \quad 1 \le i \le \frac{r-3}{2}$$

$$w_f(x_{r+1-i}) = 2i + 1 \quad , \quad 1 \le i \le \frac{r-3}{2}$$

$$w_f(x_{\frac{r-1}{2}}) = r$$

$$w_f(x_{\frac{r+1}{2}}) = r + 2$$

$$w_f(x_{\frac{r+3}{2}}) = r + 1$$

We can observe that all the vertex weights are distinct. Hence, C_r, $r \equiv 3(\text{mod } 4)$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., \frac{r+3}{2}\}$. Hence, Farey edge strength of C_r, $r \equiv 3(\text{mod } 4)$ is $\frac{r+3}{2}$.

Theorem 0.2. The graph $P_r \odot tK_1$, $r \ge 2$ is Farey edge graceful and its Farey edge strength is rt. Proof. Let x_1, x_2, ..., x_r be the vertices of P_r and each vertex x_i of P_r is connected to the vertices y_i^1, y_i^2, ..., y_i^t by an edge. Then we get the graph $P_r \odot tK_1$. Define a function $f : E(P_r \odot tK_1) \longrightarrow F_{r^2t+1}$ such as:

$$f(x_i x_{i+1}) = \frac{rti+1}{rt(i+1)+1} \quad , \quad 1 \le i \le r-1$$

$$f(x_i y_i^j) = \frac{1}{r(j-1)+i+1} \quad , \quad 1 \le i \le r, \quad 1 \le j \le t$$

By using the definition of Farey edge graceful graph, vertices are labeled by

$$w_f(x_i) = \begin{cases} \frac{rt(t+1)+2it}{2} & , \quad i = 1, r \\ \frac{rt(t+3)+2it}{2} & , \quad 2 \le i \le r-1 \end{cases}$$

$$w_f(y_i^j) = r(j-1) + i \quad , \quad 1 \le i \le r, \quad 1 \le j \le t$$

We can observe that all vertex weights are distinct. Hence, $P_r \odot tK_1$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., rt\}$. Hence, Farey edge strength of $P_r \odot tK_1$ is rt. (See Fig. 2)

Theorem 0.3. The graph $P_r \odot K_2$, $r \ge 2$ is Farey edge graceful and its Farey edge strength is $r + 1$.
Proof. Let y_i^1, y_i^2 be the vertices of complete graph K_2 and P_r be the path graph defined in Theorem 0.2. If y_i^1, y_i^2 connected to each vertex x_i of P_r by an edge then we get $P_r \odot K_2$. Define a function $f : E(P_r \odot K_2) \longrightarrow F_k$ where

$$k = \begin{cases} 49 & , \quad r = 5 \\ r^2 + 4r + 4 & , \quad r \text{ is even} \\ 6r + 1 & , \quad r \text{ is odd } r \ne 5 \end{cases}$$

such that:

$$f(x_i y_i^1) = \frac{1}{i+1} \quad , \quad 1 \le i \le r$$

$$f(x_i y_i^2) = \frac{i+2}{2i+3} \quad , \quad 1 \le i \le r$$

$$f(x_1 x_2) = \begin{cases} \frac{4r+1}{5r+1} & , \quad r \text{ is odd} \\ \frac{4(r+1)+1}{5(r+1)+1} & , \quad r \text{ is even} \end{cases}$$

$$f(x_i x_{i+1}) = \frac{(r+1)(i+3)+1}{(r+1)(i+4)+1} \quad , \quad 2 \le i \le r-2$$

$$f(y_i^1 y_i^2) = \begin{cases} \frac{2}{3} & , \quad i = 1 \\ \frac{2i+1}{3i+1} & , \quad 2 \le i \le \lceil \frac{r+1}{2} \rceil \\ \frac{3i+4}{4i+5} & , \quad \lceil \frac{r+1}{2} \rceil < i \le n \end{cases}$$

$$f(x_{r-1} x_r) = \begin{cases} \frac{5r+1}{6r+1} & , \quad r \text{ is odd}, \quad r \ne 5 \\ \frac{43}{49} & , \quad r = 5 \\ \frac{(r+1)(r+2)+1}{(r+1)(r+3)+1} & , \quad r \text{ is even} \end{cases}$$

By using the definition of Farey edge graceful labeling, vertices are labeled by

$$w_f(y_i^1) = \begin{cases} 2i & , \quad 1 \le i \le \lceil \frac{r+1}{2} \rceil \\ 2i + 1 & , \quad \lceil \frac{r+1}{2} \rceil \le i \le r \end{cases}$$

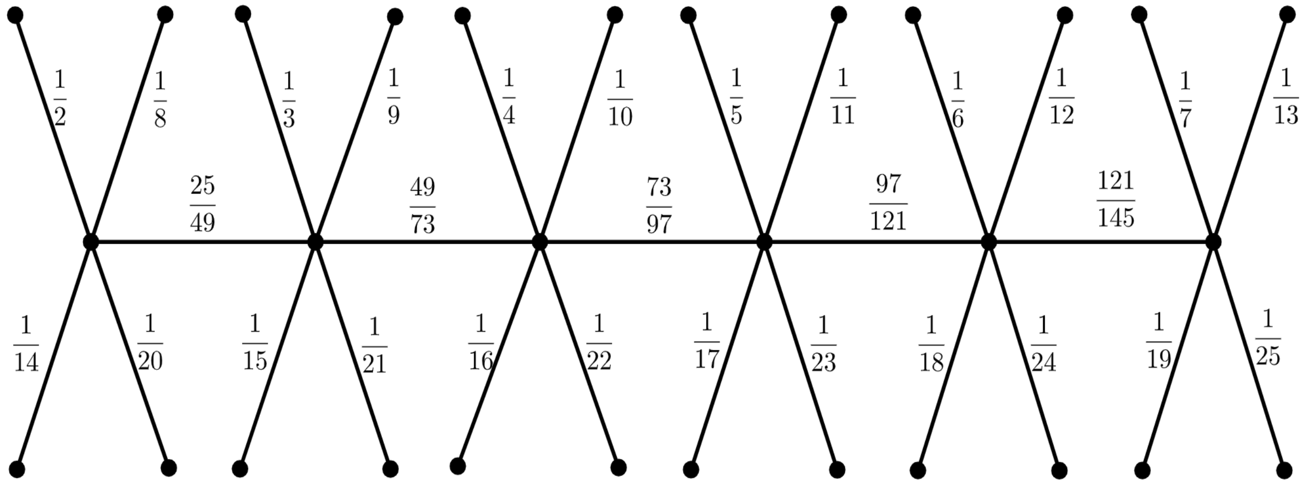

Fig. 2: Farey edge graceful labeling of $P_6 \odot 4K_1$

$$w_f(y_i^2) = \begin{cases} 2i+1 \quad , \quad 1 \le i \le \lceil \frac{r+1}{2} \rceil \\ 2i+2 \quad , \quad \lceil \frac{r+1}{2} \rceil \le i \le r \end{cases}$$

$$w_f(x_1) = \begin{cases} r+3 \quad , \quad \text{r is odd} \\ r+4 \quad , \quad \text{r is even} \end{cases}$$

$$w_f(x_2) = \begin{cases} 12 \quad , \quad r = 3 \\ 2r+6 \quad , \quad \text{r is odd, } r \ne 3 \\ 2r+7 \quad , \quad \text{r is even, } r \ge 4 \end{cases}$$

$$w_f(x_{r-1}) = \begin{cases} 4r \quad , \quad \text{r is odd, } r \ge 7 \\ 4r+1 \quad , \quad \text{r is even, } r \ge 4 \text{ or } r = 5 \end{cases}$$

$$w_f(x_r) = \begin{cases} 3r+1 \quad , \quad \text{r is odd, } r \ge 3,\ r \ne 5 \\ 3r+2 \quad , \quad \text{r is even, } r \ge 2 \text{ or } r = 5 \end{cases}$$

$$w_f(x_i) = 2(r+i)+3 \quad , \quad 3 \le i \le r-2$$

We can observe that all the vertex weights are distinct. Hence, $P_r \odot K_2$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., r+1\}$. Hence, Farey edge strength of $P_r \odot K_2$ is $r+1$. (See Fig. 3)

Theorem 0.4. The graph $K_p \odot K_1$, $p \ge 3$ is Farey edge graceful and its Farey edge strength is p.

Proof. Let x_1, x_2, ..., x_p be the vertices of complete graph K_p and y_1, y_2, ..., y_p be the pendant vertices. If x_i connected to y_i, $1 \le i \le p$ then we get the graph $K_p \odot K_1$. Define a function $f : E(K_p \odot K_1) \longrightarrow F_{p^3-3p^2+4p+1}$ such as:

$$f(x_i y_i) = \frac{1}{i+1} \quad , \quad 1 \le i \le p$$

For fixed j, $1 \le j \le p$

$$f(x_j x_i) = \frac{((j-1)p+i-2j+1)p+1}{((j-1)p+i-2j+2)p+1} \quad , \quad i \ne j,\ j+1 \le i \le p$$

By using the definition of Farey edge graceful labeling, vertices are labeled by

$$w_f(x_i) = (p-1)p + i \quad , \quad 1 \le i \le p$$

$$w_f(y_i) = i \quad , \quad 1 \le i \le p$$

We can observe that all the vertex weights are distinct. Hence, $K_p \odot K_1$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., p\}$. Hence, Farey edge strength of $K_p \odot K_1$ is p. (See Fig. 4)

Theorem 0.5. Star graph $K_{1,r}$ is Farey edge graceful and its Farey edge strength is r.

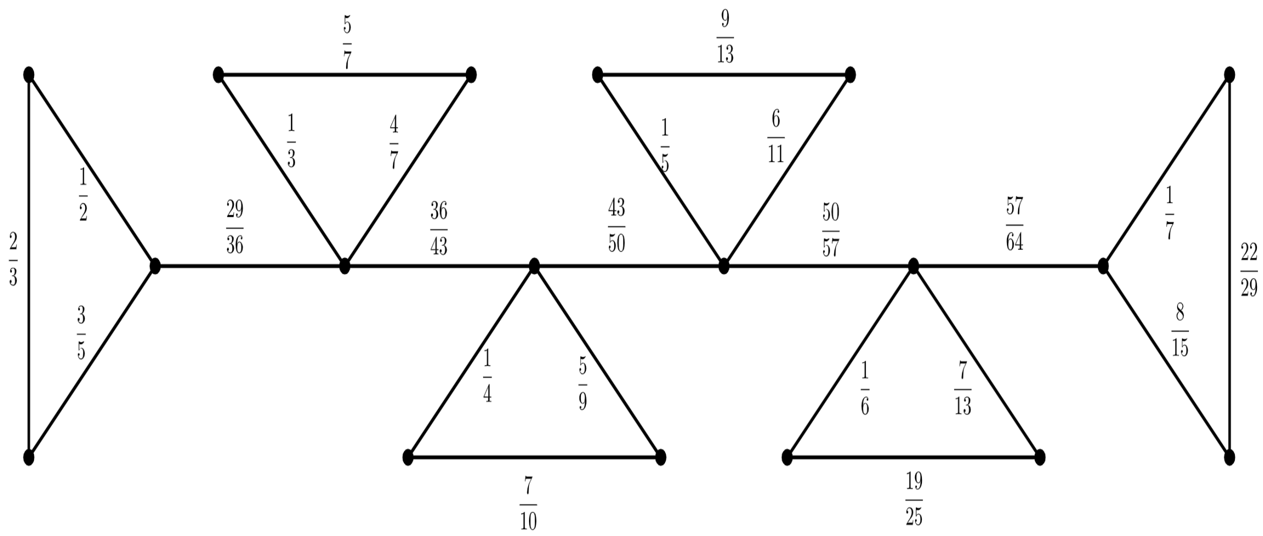

Fig. 3: Farey edge graceful labeling of $P_6 \odot K_2$

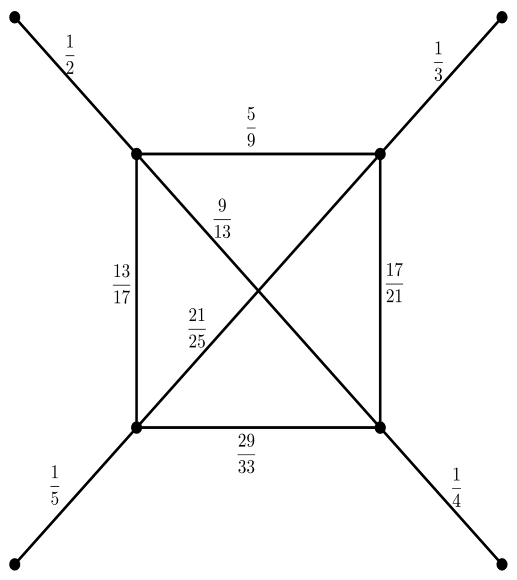

Fig. 4: Farey edge graceful labeling of $K_4 \odot K_1$

Proof. Let x_1, x_2, ..., x_r be the pendant vertices of $K_{1,r}$ and x is central vertex. We define a function $f : E(K_{1,r}) \longrightarrow F_{r+1}$ such that:

$$f(xx_i) = \frac{1}{i+1} \quad , \quad 1 \leq i \leq r$$

By using the definition of Farey edge graceful labeling, vertices are labeled by

$$f(x_i) = i \qquad , \quad 1 \leq i \leq r$$

$$f(x) = \frac{r(r+1)}{2}$$

We can observe that all the vertex weights are distinct. Hence, $K_{1,r}$ is Farey edge graceful graph.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we find the difference of numerator and denominator (i.e. $b_i - a_i$). Consequently, we get the label of each edge from the set $\{1, 2, ..., r\}$. Hence, Farey edge strength of $K_{1,r}$ is r. (See Fig. 6)

III. APPLICATION

Farey edge graceful labeling has promising applications in fuzzy set theory and probability theory due to the origin of its labels from the interval $[0, 1]$. Additionally, this labeling technique can be used in cryptography to enhance secure data transmission by using injective labeling property.

IV. CONCLUSION

In this paper, we work on a new concept of Farey edge graceful labeling and Farey edge strength for graphs. We

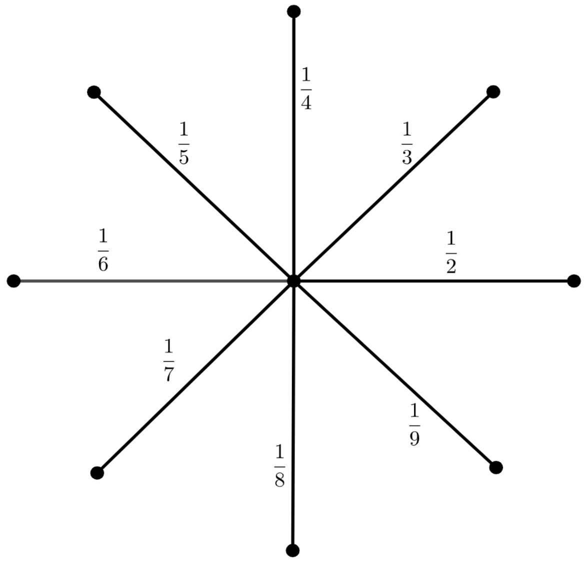

Fig. 5: Farey edge graceful labeling of $K_{1,8}$

prove that C_r, $P_r \odot tK_1$, $P_r \odot K_2$ and $K_p \odot K_1$, $K_{1,r}$ are farey edge graceful and we also determine their Farey edge strength.

References

[1] D. Burton, Elementary Number Theory, McGraw Hill, (2010).

[2] G. Chartrand, M. Jacobson, J. Lehel, O. Oellermann, S. Ruiz, and F. Saba, Irregular networks, Congr. Numer., 64 (1988) 187-192.

[3] R. J. Faudree, R. H. Schelp, M. S. Jacobson, and J. Lehel, Irregular networks, regular graphs and integer matrices with distinct row and column sums, Discrete Mathematics, 76(3) (1989) 223-240.

[4] A. Kumar, N. Gupta, A. Kumar, S. Tyagi and V. Kumar, On Farey Edge Graceful Labeling, IAENG International Journal of Applied Mathematics, 54(11) (2024) 2484-2490.

[5] A. Kumar, A. Kumar, S. Tyagi, and N. Gupta, Farey Graceful Labeling of Graphs and Its Applications, National Academy Science Letters, (2024) 1-4. https://doi.org/10.1007/s40009-024-01586-y.

[6] A. Kumar, D. Mishra and V. K. Srivastava, On Farey graceful labeling, National Academy of Science Letters 47(3) (2023) 303-308 https://doi.org/10.1007/s40009-023-01355-3.

[7] K. M. G. Packiam, T. Manimaran and A. Thuraiswamy, Irregularity strength of corona of two graphs, Theoretial Comput. Sci. Disc. Math., (2017) 175-181.

[8] I. Niven, H. S. Zuckerman and H. L. Montgomery, An introduction to Theory of Numbers, John Wiley Sons, (1991).