Incident Vertex Pi Coloring of Corona Product of Some Graphs: Path with Complete Graph, Path with Star Graph and Star Graph with Path

Sunil B. Thakare, *IAENG*, *Member*, Haribhau R. Bhapkar and Archana Bhange

Abstract—**In a graph theory, the Pi coloring of a graph is the method of assigning colors to its vertices using a power set colors, ensuring that no repeating pattern occurs. This defines the function from set X is a collection of subsets of vertices having some common characteristics or properties to $P(C)$, a power set of distinct colors C. The minimum number of colors required for Pi coloring is called the Pi chromatic number of the graph. A specialized type, Incident Vertex Pi extending this concept by defining coloring function on pairs of incident vertices of an edge of graph. This function assigns colors from a set X is collection of pairs of incident vertices of an edge to $P(C)$, the power set of distinct colors C and additionally each ordered pair of an incident vertices receives distinct color. The minimum number of colors required for Incident Vertex Pi coloring is called the Incident Vertex Pi chromatic number of a graph. In this paper, we discussed the Incident Vertex Pi chromatic number for the Corona Product of various graph families including P_2 with K_m, P_3 with K_m, C_3 with K_m, P_n with S_m, C_n with S_m, T_n with S_m, S_n with P_2.**

Index Terms (*Key words*)— **Pi Coloring, Incident Vertex Pi Coloring, Corona Product, Cycle, Path, Star graph, Complete graph.**

AMS Subject Classification: 05C15, 05C69, 05C70.

I. INTRODUCTION

THE graph G is defined as an order pair ($V(G)$, $E(G)$), where vertex set $V(G)$ is a set of points in the plane and an edge set $E(G)$ is an ordered pair of vertices connected by a line or a curve in the plane. Graph coloring is one of the most extensive topics in Graph Theory with significant global interest due to wide range of realistic applications [2], [3]. Over the time, mathematicians developed various coloring techniques to address difficult real-life problems. For detailed survey of graph coloring methods, refer to [1], [2], and [3]. Thakare and Bhapkar [5] invented the concept of Pi-Coloring and Incident Vertex Pi-Coloring in graph theory. The least number of colors required for Pi-Coloring, and Incident Vertex Pi-Coloring are known as Pi Chromatic number and Incident Vertex Pi Chromatic number respectively. In this coloring methods, function assigns color to specific element of a graph based on common properties in the set X of elements of a graph to the power set P(C) of distinct colors C, ensuring that each subset receives distinct color from the power set of available color.

The vertex coloring of Star graph $S_{1,n}$ is 2 [1], while its edge coloring is n [1]. The Total coloring of the Star graph families $S_{1,n}$ is n+1 [9] [10]. Similarly, the Incident Vertex Pi coloring of Star graph $S_{1,n}$ is n+1 [5]. For the Complete graph (K_n), the vertex coloring of is n, and the edge coloring is n, if n is an odd or n−1, if n is even. Total coloring of Complete graph (K_n) is n, if n is an odd or n+1, if n is even [10]. The Incident Vertex Pi coloring of Complete graph (K_n) is n [5].

In this study, we analyze the Incident Vertex Pi chromatic number for the Corona product of two graphs selected from the graph families: Path, Cycle, Star graph, and Complete graph.

II. DEFINITIONS

In Graph Theory, various types of graph products have been defined. In this paper, we focused on the Corona Product graphs as described below and analyze its Incident Vertex Pi chromatic number. For fundamental definitions, concepts in Graph Theory and overview of graph coloring methods, please refer to [1], [2], [3], [13], [14].

Definition 2.1: (Corona product)

Let A and B be two simple connected graphs such that the size of graph A is n and its vertex set is $\{v_1,v_2,...v_n\}$. The Corona product of graph A and B symbolized by A Θ B is graph constructed by taking one copy of graph A and n copies of B denoted by $B_1, B_2, ... B_n$ and join each vertex u from copy B_i to vertex v_i of graph A, for i = 1, 2...n [4], [6], [7], [8].

Definition 2.2: (Path graph)

A graph with n vertices $\{u_1,...u_n\}$ and $n-1$ edges $\{(u_1,u_2),(u_2,u_3),...(u_{n-1},u_n)\}$ is known as Path graph represented by P_n.

Definition 2.3: (Cycle graph)

A graph with n vertices $\{u_1,...u_n\}$ and n edges $\{(u_1,u_2),(u_2,u_3), ...(u_{n-1},u_n),(u_n,u_1)\}$ is known as Cycle graph represented C_n.

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Definition 2.4: (Complete graph)

A complete graph represented by K_n is the family of graphs on n vertices such that each vertex is adjacent to all other $n - 1$ vertices. It has $n(n - 1)/2$ number of edges.

Definition 2.5: (Star graph)

A Star graph represented by S_n, it is constructed by joining n pendent vertices to a single apex vertex. It consists $n + 1$ vertices and n edges with the apex vertex having the maximum degree of n.

Definition 2.6: (Pi Coloring)

Let $G = (V, E)$ be a simple connected graph where V is vertex set, E is edge set, and X is a collection of subsets of elements of graph G having some common properties. If there exists a function $f : X \rightarrow P(C)$, where C is a set of colors and $P(C)$ is its power set, such that $f(X_p) \neq f(X_q)$, for all $p \neq q$, then it is called Pi coloring of graph G [5].

Definition 2.7: (Incident Vertex Pi Coloring)

Let $G = (V, E)$ be a simple connected graph where V is vertex set, E is edge set, and X is a collection of order pair incident vertices of every single edge e in $E(G)$. Define a function $f : X \rightarrow P(C)$, where C is set of colors and $P(C)$ is its power set, such that $f(X_p) \neq f(X_q)$, for all $p \neq q$, then it is called Incident Vertex Pi Coloring of graph G [5].

III. Main Results

In this section, we discussed the Incident Vertex Pi coloring number results for Corona Product of graph namely P_2 with K_m, P_3 with K_m, C_3 with K_m, P_n with S_m, C_n with S_m, T_n with S_m, S_n with P_2.

3.1 Corona product graph of Path P_2 with Complete graph K_m and its Incident Vertex Pi Coloring

Theorem 3.1: The Incident Vertex Pi coloring of the Corona product graph of Path P_2 with Complete graph K_m is $2m+1$, for $m \geq 1$.

Proof: Consider Path $P_2 = u_1, u_2$ and Complete graph K_m with the vertex set $\{v_1, v_2, v_3, \ldots v_m\}$. Then, by the definition of the Corona product graph, we have a graph $P_2 \odot K_m$ with its vertex set is $V(P_2 \odot K_m) = \{u_1, u_2, v_{i1}, v_{i2}, for\ i:=1\ to \ldots m\}$. It has $2m+2$ vertices and $m^* m + m + 1$ edges. We observe that in this graph, there are two vertices of maximum degree $\Delta = d(u_2) = d(u_1) = m+1$.

Incident Vertex Pi coloring is defined as function $f : X \rightarrow P(C)$, such that it assigns distinct colors to every edge that is incident at its two end vertices. Here set X is a collection of the subset of order pairs of incident vertices of every edge of the Corona graph $P_2 \odot K_m$ and C is a set of distinct colors with its power set is $P(C)$. Now assign distinct colors to every edge that is incident at its two end vertices as below.

In this Corona product graph $P_2 \odot K_m$, there are two copies Complete graph on $m+1$ vertices each induced by vertex sets $\{u_1, v_{i1}\}$ and $\{u_2, v_{i2}\}$ for $i:1\ to \ldots m$. By the definition of Incident Vertex Pi coloring, the Complete graph on $m+1$ verities requires $m+1$ distinct colors for Incident Vertex Pi coloring [8]. Hence for first copy of Complete graph sets $\{u_1, v_{i1}\}$ for $i:1\ to \ldots m$ requires $m+1$ distinct colors. However, in the second copy of Complete graph $\{u_2, v_{i2}\}$, for $i:1\ to \ldots m$, we observe that it is possible to reuse only one color (say color 1 is used for vertex v_{12}). So, for second copy we needed m additional new colors. Thus, the Incident Vertex Pi coloring for vertices of the Corona product graph $P_2 \odot K_m$ is as below:

$v_{11} \rightarrow 1,\ v_{21} \rightarrow 2, \ldots\ v_{m1} \rightarrow m,\ u_1 \rightarrow m+1,\ v_{12} \rightarrow 1,\ v_{22} \rightarrow m+2,\ v_{32} \rightarrow m+3, \ldots\ v_{m2} \rightarrow 2m$ and $u_2 \rightarrow 2m+1$.

We notice that this coloring assigns distinct colors to every edge that is incident at its two end vertices in Corona product graph $P_2 \odot K_m$ as below.

- 1st copy of Complete graph: $f(X_p) = \{(i,j),\ for\ i = 1,2,\ldots m$ and $j = i+1, i+2, \ldots m+1\}$.
- 2ed copy of Complete graph: $f(X_p) = \{(i,j),\ for\ i = m+1, m+2, \ldots, 2m$ and $j = i+1, i+2, \ldots 2m+1$ and $(1,m+2), (1,m+3), \ldots (1,2m+1)\}$.
- Path $P_2 = u_1 u_2 : f(X_p) = \{(m+1, 2m+1)\}$.

Thus, for all distinct p and q, $f(X_p) \neq f(X_q)$.

Therefore, $IVPI(P_2 \odot K_m) = 2m+1$. Hence the proof.

The Incident Vertex Pi coloring of graph $P_2 \odot K_m$, for $m=4$ is shown in Figure 1.

3.2 Corona product graph of Path P_3 with Complete graph K_m and its Incident Vertex Pi Coloring

Theorem 3.2: The Incident Vertex Pi coloring of the Corona product graph of Path P_3 with Complete graph K_m is $3m$, for $m \geq 2$.

Proof: Consider Path graph $P_3 = \{u_1, u_2, u_3\}$ and Complete graph K_m with the vertex set $\{v_1, v_2, \ldots v_m\}$. By the definition of the Corona product graph, the resulting graph $P_3 \odot K_m$ has vertex set is $V(P_3 \odot K_m) = \{u_1, u_2, u_3, v_{i1}, v_{i2}, v_{i3}, for\ i:=1\ to \ldots m\}$. It has $3m+3$ vertices and $3m^*m+3m+4$ edges. We observe that in this graph, the maximum degree Δ is observed at u_2 given by $d(u_2) = m+2$.

Incident Vertex Pi coloring is defined as function $f : X \rightarrow P(C)$, such that it assigns distinct colors to every edge that is incident at its two end vertices. Here set X is a collection of the subset of order pairs of incident vertices of every edge of the Corona graph $P_3 \odot K_m$ and C is a set of distinct colors with its power set is $P(C)$. Now assign distinct colors to every edge that is incident at its two end vertices as below.

In this Corona product graph $P_3 \odot K_m$, there are three copies Complete graph on $m+1$ vertices each induced by vertex sets $\{u_1, v_{i1}\}$, $\{u_2, v_{i2}\}$ and $\{u_3, v_{i3}\}$ for $i:1\ to \ldots m$. By the definition of Incident Vertex Pi coloring, the Complete graph needed $m+1$ distinct colors for Incident Vertex Pi coloring [8]. Hence for first copy of Complete graph sets $\{u_1, v_{i1}\}$ for $i:1\ to \ldots m$ requires $m+1$ distinct colors.

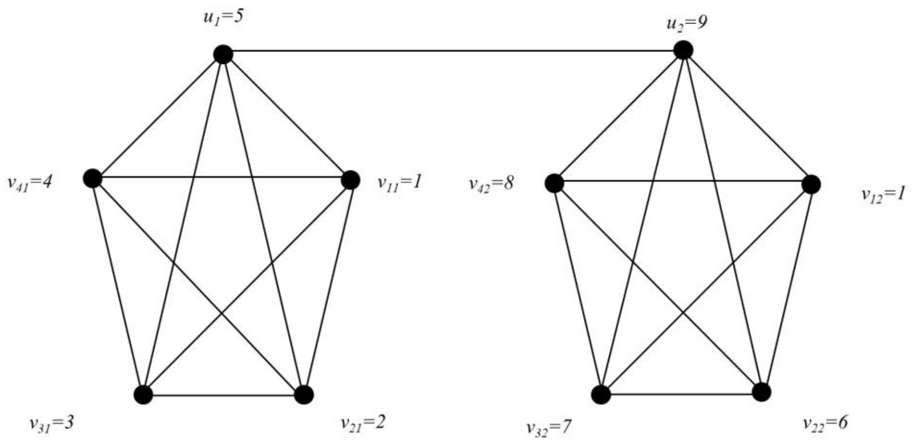

Figure 1. Incident Vertex Pi coloring of graph $P_2 \Theta K_m$, for m=4.

However, we observe that it is possible to reuse only one color from the color set of the first copy for the second copy of the Complete graph $\{u_2, v_{i2}\}$, because all vertices in Complete graph are adjacent to each other. Hence, we needed m more new colors for the second copy. Similarly, for the third copy of the Complete graph on $\{u_3, v_{i3}\}$, it's possible to reuse at most two colors (one from the first copy and another from the second copy), hence we required $m-1$ more new colors for third copy. Thus, we needed total $(m+1)+m+(m-1) = 3m$ colors for Incident Vertex Pi coloring to the graph $P_3 \Theta K_m$.

The Incident Vertex Pi coloring for vertices of the Corona product graph $P_3 \Theta K_m$ is as below:

First copy: $u_1 \rightarrow 1,\ v_{11} \rightarrow 2,\ v_{21} \rightarrow 3, \dots\ v_{m1} \rightarrow m+1,$

Second copy: $u_2 \rightarrow m+2,\ v_{12} \rightarrow m+3,\ v_{22} \rightarrow m+4,\ v_{32} \rightarrow m+5, \dots,\ v_{(m-1)2} \rightarrow 2m+1,\ v_{m2} \rightarrow 2$ *and*

Third copy: $u_3 \rightarrow 2m+2,\ v_{13} \rightarrow 2m+3,\ v_{23} \rightarrow 2m+4,\ v_{33} \rightarrow 2m+5, \dots,\ v_{(m-2)3} \rightarrow 3m,\ \ v_{(m-1)3} \rightarrow 3,\ v_{m3} \rightarrow m+3.$

We notice that this coloring assigns distinct colors to every edge that is incident at its two end vertices in Corona product graph $P_3 \Theta K_m$ as below.

- 1st copy of Complete graph: $f(X_p) = \{(i,j),$ *for* $i = 1,2,\dots m$ *and* $j = i+1, i+2, \dots m+1\}$.
- 2ed copy of Complete graph: $f(X_p) = \{(i,j),$ *for* $i = m+2, m+3, \dots 2m$ *and* $j = i+1, i+2, \dots 2m+1$ *and* $(2,m+2), (2,m+3), \dots (2,2m+1)\}$.
- 3rd copy of Complete graph: $f(X_p) = \{(i,j),$ *for* $i = 2m+2, 2m+3, \dots 3m-1$ *and* $j = i+1, i+2, \dots 3m;$ $(3,2m+2), (3,2m+3), \dots (3,3m)$ *and* $(m+3,2m+2), (m+3,2m+3), \dots (m+3, 3m), (m+3,3)\}$.
- Path $P_3 = u_1\, u_2\, u_3$: $f(X_p) = \{(1,m+2), (m+2,2m+2)\}$.

  Thus, for all distinct p and q, $f(X_p) \neq f(X_q)$.

  Therefore, $IVPI(P_3 \Theta K_m) = 3m$. Hence the proof.

The Incident Vertex Pi coloring of graph $P_3 \Theta K_m$, for m=4 is shown in Figure 2.

Theorem 3.3: The Incident Vertex Pi coloring of the Corona product graph of Cycle C_3 with Complete graph K_m is $3m$, for $m \geq 2$.

Proof: By using a similar logic of Theorem 3.2, we will get this result. We observe that, there is one additional edge from path P_3 to cycle C_3 in the Corona product graph $C_3 \Theta K_m$.

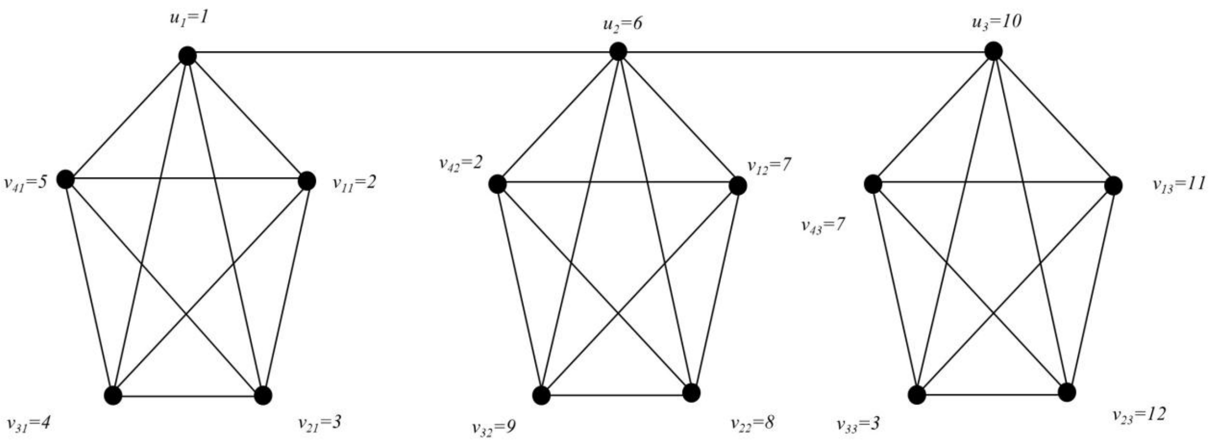

Figure 2. Incident Vertex Pi coloring of graph $P_3 \Theta K_m$, for m=4

3.3 Corona product graph of Path P_n, Cycle C_n and Tree T_n with Star graph S_m and its Incident Vertex Pi Coloring

Theorem 3.4: The Incident Vertex Pi coloring of the Corona product graph of Path P_n with Star graph S_m is $2n+m$, for $m \geq 3$.

Proof: Consider Path graph $P_n = \{v_1, v_2, \dots v_n\}$ and Star graph S_m with the vertex set $\{u, u_1, u_2, u_3, \dots u_m\}$, where u is central vertex and u_i are m pendent vertices. By definition of the Corona product graph, the resultant graph $P_n \odot S_m$ has vertex set is $V(P_n \odot S_m) = \{v_1, v_2, \dots v_n, u_i, u_{i1}, u_{i2}, u_{i3}, \dots u_{in}\}$, for $i = 1,2 \dots m$ and an edge set is $E(P_n \odot S_m) = \{(v_1, v_2), (v_2, v_3) \dots (v_{n-1}, v_n), (v_j, u_j), (v_j, u_{ij}), (u_j, u_{ij})$, for $j = 1,2 \dots n$ and $i = 1,2 \dots m\}$. It has $n(m+2)$ vertices and $n(2m+1)-1$ edges. We observe that in this graph, the maximum degree of the vertices $v_2, v_3, \dots v_{n-1}$ is $d(v_2) = \dots = d(v_{n-1}) = m+3$.

Incident Vertex Pi coloring is defined as function $f : X \rightarrow P(C)$, such that it assigns distinct colors to every edge that is incident at its two end vertices. Here set X is a collection of the subset of order pairs of incident vertices of every edge of the Corona graph $P_n \odot S_m$ and C is a set of distinct colors with its power set is $P(C)$. Now assign distinct colors to every edge that is incident at its two end vertices as below.

In $P_n \odot S_m$ Corona product graph, there are n copies of the Diamond graph on $(m+2)$ vertices each induced by vertex sets $\{(v_j, u_j, u_{ij}); i = 1,2, \dots m\}$, for $j = 1,2, \dots n$. By the definition of Incident Vertex Pi coloring, the Diamond graph requires $(m+2)$ distinct colors for Incident Vertex Pi coloring [8]. So, Incident Vertex Pi coloring for the first copy of the Diamond graph, we needed $(m+2)$ distinct colors. However, we notice that it is possible to reuse m colors assigned to the vertices $\{u_{i1}\}$ in first copy for corresponding m vertices in remaining all $n-1$ copies of the Diamond graph as well. So, let us we fix the colors for these m vertices across all n from copies in Diamond graph $\{u_{i1}\}, \{u_{i2}\}, \dots \{u_{in}\}$. Now we only need to color the remaining vertices $\{u_j, v_j\}$, for $j = 1,2, \dots n$. We observed that both vertices are adjacent to each other as well as adjacent to the set $\{u_{ij}\}$ of m vertices of n copies of the Diamond graph, hence we need $2n$ additional colors for Incident Vertex Pi coloring. Thus, we needed total $(2n+m)$ colors for Incident Vertex Pi coloring. The Incident Vertex Pi coloring for vertices of the Corona product graph $P_n \odot S_m$ is shown below:

First copy of Diamond graph: $u_{11} \rightarrow 1, u_{21} \rightarrow 2, u_{31} \rightarrow 3, \dots u_{m1} \rightarrow m, u_1 \rightarrow m+1, v_1 \rightarrow m+2$

Second copy of Diamond graph: $u_{12} \rightarrow 1, u_{22} \rightarrow 2, u_{32} \rightarrow 3, \dots, u_{m2} \rightarrow m, u_2 \rightarrow m+3, v \rightarrow m+4$, *and so on*

n^{th} copy of Diamond graph: $u_{1n} \rightarrow 1$, $u_{2n} \rightarrow 2$, $u_{3n} \rightarrow 3, \dots, v_{mn} \rightarrow m, v_n \rightarrow m+2n, u_n \rightarrow m+2n-1$.

We notice that this coloring assigns distinct colors to every edge that is incident at its two end vertices in Corona product graph $P_n \odot S_m$ as below.

- 1^{st} copy of Diamond graph: $f(X_p)=\{(i,m+1), (i,m+2)$, *for* $i=1,2, \dots m$ *and* $(m+1,m+2)\}$.
- 2^{ed} copy of Diamond graph: $f(X_p)=\{(i,m+3), (i,m+4)$, *for* $i=1,2, \dots m$ *and* $(m+3,m+4)\}$, and so on
- n^{th} copy of Diamond graph: $f(X_p)=\{(i,m+2n-1), (i,m+2n)$, *for* $i = 1,2, \dots m$ *and* $(m+2n-1,m+2n)\}$.
- Path $P_n = v_1, v_2 \dots v_n$: $f(X_p) = \{(m+2,m+4), (m+4,m+6) \dots (m+2n-2,m+2n)\}$.

Thus, for all distinct p and q, $f(X_p) \neq f(X_q)$.

Therefore, $IVPI(P_n \odot S_m)=2n+m$. Hence the proof.

The Incident Vertex Pi coloring of graph $P_n \odot S_m$, for $m=4$ and $n=3$ is shown in Figure 3.

Theorem 3.5: The Incident Vertex Pi coloring of the Corona product graph of Cycle C_n with Star graph S_m is $2n+m$, for $m \geq 3$.

Proof: By using a similar logic of Theorem 3.4, we will get this result. We observe that, there is one additional edge from path P_n to cycle C_n in the Corona product graph $C_n \odot S_m$.

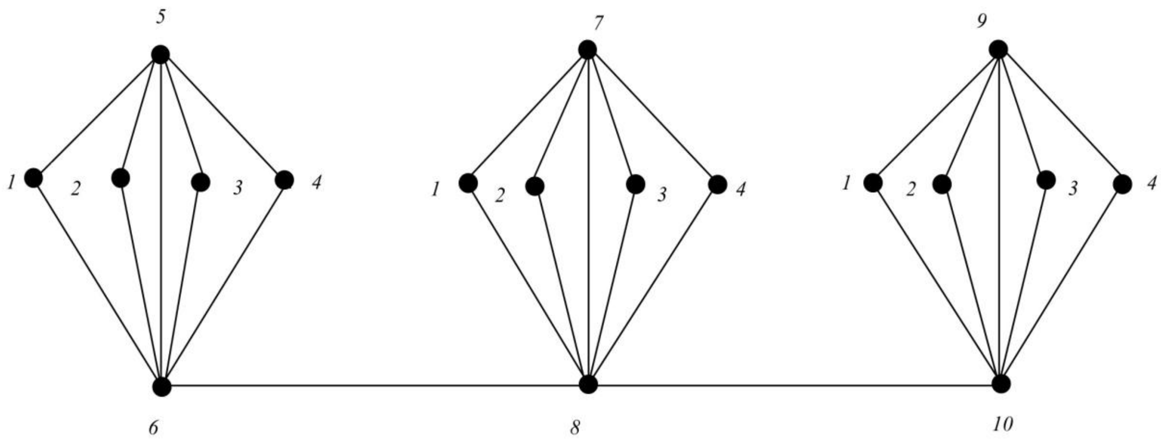

Figure 3. Incident Vertex Pi coloring of graph $P_n \odot S_m$, for m=4 and n=3.

Theorem 3.6: The Incident Vertex Pi coloring of the Corona product graph of Tree T_n with Star graph S_m is $2n+m$, for $m \geq 3$.

Proof: By using a similar logic of Theorem 3.4, we will get this result.

3.4 Corona product graph of Star graph S_n, Path P_2 and its Incident Vertex Pi Coloring

Theorem 3.7: The Incident Vertex Pi coloring of the Corona product graph of Star graph S_n with Path P_2 is $\Delta+1$, for $n \geq 4$.

Proof: Consider the Star graph S_n with the vertex set $\{v_1, v_2, ...v_{n+1}\}$, where v_1 is central vertex and u_2 to v_{n+1} are n pendent vertices. Additionally, $P_2 = \{u_1,u_2\}$ be the Path graph with two vertices. By definition of the Corona product graph, the resulting graph $S_n \odot P_2$ has vertex set is $V(S_n \odot P_2) = \{v_1, v_2, ... v_{n+1}; u_{1i},u_{2i}$, for $i = 1, 2...n + 1\}$ and an edge set is $E(S_n \odot P_2) = \{(v_1, v_i)$, for $i = 2,3...n + 1$; $(v_i, u_{1i}),(v_i, u_{2i}), (u_{1i},u_{2i})$, for $i = 1,2...n+1\}$. This graph consists of $3n+3$ vertices and $4n+3$ edges. Moreover, we observe that the maximum degree (Δ) of vertex v_1 is given by $d(v_1) = n+2$.

The Incident Vertex Pi coloring is defined as function $f: X \rightarrow P(C)$, such that this function assigns distinct colors to every edge based on its incident vertices, ensures that no two incident edges share the same color at their endpoints. Here set X is a collection of the subset of order pairs of incident vertices of every edge of the Corona graph $S_n \odot P_2$ and C is a set of distinct colors with its power set represented as $P(C)$. Now the incident vertex pi coloring is applied as follows:

In this Corona product graph, there are $n + 1$ copies of the cycle graph C_3 on $3n + 3$ vertices, each induced by vertex sets $\{(v_i,u_{1i},u_{2i})$, for $i = 1,2,...n+1\}$. Consider the set $X = \{X_1,X_2,...X_n,P_1,P_2,...P_{n+1},Q_1,Q_2,...Q_{n+1},R_1,R_2,...R_{n+1}\}$, where edges of Star graph S_n: $X_i = (v_1,v_{i+1})$, for $i = 1,2...n$; and edges of cycle graph $C_3 = P_iQ_iR_i$: $P_i = (v_i,u_{1i}),Q_i = (v_i,u_{2i}),R_i = (u_{1i},u_{2i})$, for $i = 1,2...n+1$ The Incident Vertex Pi coloring for vertices of the Corona product graph $S_n \odot P_2$ as below:

First copy of cycle graph $C_3 = P_1Q_1R_1$: $v_1 \rightarrow 1, u_{11} \rightarrow n+2, u_{21} \rightarrow n+3,$

Second copy of cycle graph $C_3 = P_2Q_2R_2$: $v_2 \rightarrow 2, u_{12} \rightarrow 3, u_{22} \rightarrow n+2,$

Third copy of cycle graph $C_3 = P_3Q_3R_3$: $v_3 \rightarrow 3, u_{13} \rightarrow 4, u_{23} \rightarrow n+3,$

Fourth copy of cycle graph $C_3 = P_4Q_4R_4$: $v_4 \rightarrow 4, u_{14} \rightarrow 5, u_{24} \rightarrow n+4,$ and so on

(n+1)th copy of cycle graph $C_3 = P_{n+1}Q_{n+1}R_{n+1}$: $v_{n+1} \rightarrow n+1, u_{1(n+1)} \rightarrow 2$ and $u_{2(n+1)} \rightarrow n+3$ *if n is even*, $u_{2(n+1)} \rightarrow 4$ *if n is odd.*

We notice that this coloring assigns distinct colors to every edge that is incident at its two end vertices as below.

- Star graph S_n with the vertex set $\{v_1, v_2, ...v_{n+1}\}$,

  $f(X_p) = \{(1,2),(1,3)...(1,n + 1)\}$.

- 1st copy of cycle graph $C_3 = P_1Q_1R_1$:

  $f(X_p) = \{(1,n+2),(1,n+3),(n+2,n+3)\}$.

- 2^{ed} copy of cycle graph $C_3 = P_2Q_2R_2$:

  $f(X_p) = \{(2,3),(2,n+2),(3,n+2)\}$.

- 3rd copy of cycle graph $C_3 = P_3Q_3R_3$:

  $f(X_p) = \{(3,4),(3,n+3),(4,n+3)\}$.

- (n+1)th copy of cycle graph $C_3 = P_{n+1}Q_{n+1}R_{n+1}$:

  $f(X_p) = \{(n+1,2),(n+1,n+3),(2,n+3)\}$ or $\{(n+1,2), (2,4), (n+1,4)\}$.

Thus, for all distinct p and q, $f(X_p) \neq f(X_q)$.

Therefore, IV PI$(S_n \odot P_2) = n + 3 = \Delta + 1$. Hence the proof.

The Incident Vertex Pi coloring of graph $S_n \odot P_2$ is shown in Figure 4.

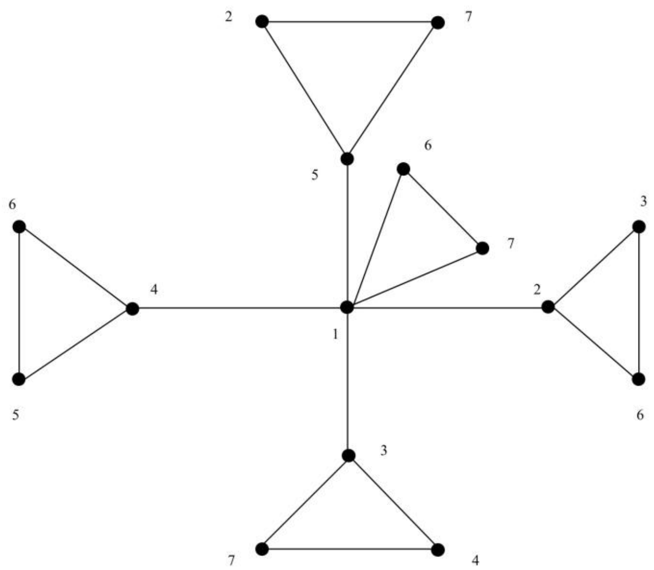

Figure 4. Incident Vertex Pi coloring of graph $S_n \odot P_2$

Corollary 3.8: The Incident Vertex Pi coloring of the Corona product graph of Star graph S_3 with Path P_2 is $\Delta+2$.

Proof: The Incident Vertex Pi coloring of the Corona product graph of Star graph S_3 with Path P_2 is shown in the Figure 5 below.

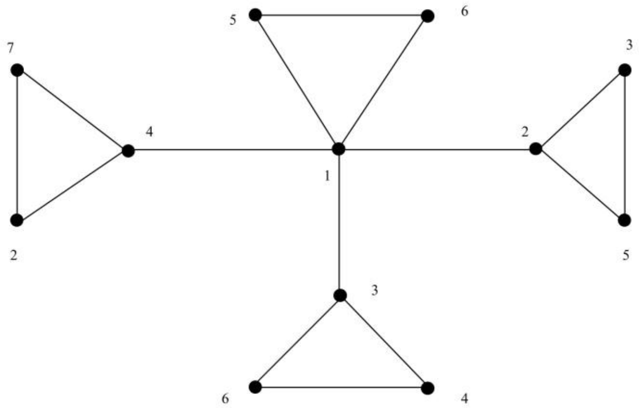

Figure 5. Incident Vertex Pi coloring of $S_3 \odot P_2$

IV. Conclusion

The concept of Incident Vertex Pi coloring for corona product graph has been explored along with its Chromatic number. We investigated the Incident Vertex Pi Chromatic of the Corona product of various graph families, including P_2 with K_m, P_3 with K_m, C_3 with K_m, P_n with S_m, C_n with S_m, T_n with S_m, and S_n with P_2. These results provide s new insights into the Incident Vertex Pi chromatic number of a graph. Based on our results, we determined the precise optimum Incident Vertex Pi chromatic numbers for these Corona products. Our finding indicates that the Incident Vertex Pi chromatic number of a graph is different than our traditional chromatic number.

TABLE I
Chromatic Number and Incident Vertex Pi Chromatic Number Comparison

Corona product graph	Chromatic No.	Incident Vertex Pi Chromatic No.
$P_2 \odot K_m$	$m+1$	$2m+1$
$P_3 \odot K_m$	$m+1$	$3m$
$P_n \odot S_m$	3	$2n+m$
$S_n \odot P_2$	3	$n+3$

Future study could explore to investigate Incident Vertex Pi coloring number for more families of corona product graph. Additionally, new researcher may explore Incident Vertex Pi coloring number as promising area of study in graph theory.

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