Multigrid Domain Decomposition Methods for Solving Stokes-Darcy Coupled Problems

Qingbo Xu, Liyun Zuo

Abstract—**In the present work, we address the coupling problem between fluid flow and porous media flow. A hybrid numerical method is proposed to solve the coupled Stokes-Darcy problem with Beavers-Joseph-Saffman (BJS) interface conditions. The method is presented in detail, rigorously tested, and its error estimates are thoroughly analyzed, demonstrating its robustness and effectiveness in handling the coupled system.**

Index Terms—**Stokes equations, Darcy's law, Multigrid method, Domain decomposition method.**

I. Introduction

In recent years, In practical applications, scenarios involving fluid dynamics across multiple regions with coupled physical phenomena are common. A typical instance is the interaction between free-flowing fluids and porous media flows. The free-flow region, described by the Stokes equations, and the porous media domain, characterized by Darcy's law, are interconnected through specific boundary conditions. This coupled approach provides an effective methodology for addressing numerous engineering challenges. For instance, this coupling model is closely related to applications in oil exploration and extraction, groundwater contamination contral, agricultural irrigation, and the transport of drugs in blood flow in medicine. Therefore, studying and addressing this coupling problem is of great significance.

Due to the importance of such coupling problems, many researchers have focused their studies on numerical methods for solving them, including stabilized finite element methods[1], [2], [3], [4], [5], domain decomposition methods[6], [7], [8], [9], [10], [11], [12], two-gril and multigrid methods[13], [14], [15], [16], local and parallel methods[17], [18], [19], discontinuous finite element[20], [21], [22], [23], Lagrange multiplier methods[24], [25], and several other approaches[26], [27], [28], [29], [30]. Among these methods, we are particularly interested in domain decomposition methods and multigrid methods. Multigrid methods stand out for their high efficiency, strong generality, and low computational cost, but their implementation is complex, and handing boundary conditions can be challenging. On the other hand, domain decomposition methods offer efficient parallelism and can reduce the complexity of global problem-solving.

In 2024, Sun et al. proposed a decoupling method combining two-grid and domain decomposition in [12]. They first used Robin-type domain decomposition to obtain an approximate solution on a corase grid and then applied a two-grid framework to construct a domain decomposition method on a fine grid, significantly improving the algorithms efficiency. In 2024,Zheng et al. extended the work in [31] from the Stokes-Darcy equations in [12]. The paper was published under the title: *"Two-Grid Domain Decomposition Method for Coupling Fluid Flow with Porous Media Flow"* in the *IAENG International Journal of Applied Mathematics*. The two-grid domain decomposition method still exhibited excellent convergence for the extended equations.

The convergence performance of the two-grid approach is constrained by the precision of its coarse-grid discretization. Therefore, we extend the two-grid domain decomposition method in [12] to multigrid domain decomposition method. This approach first obtains an approximate solution on the coarse grid using a Robin-Robin domain decomposotion method and updates the interface conditions based on the corase grid solution. The Stokes and Darcy subsystems are subsequently computed across progressively refined grids. Theoretical error examination demonstrates the algorithm's convergence.

This manuscript proceeds in the following manner: Section II presents the coupled Stokes-Darcy system incorporating Beavers-Joseph-Saffman (BJS) boundary conditions at the interface. Section III outlines the fundamental principles of the domain decomposition approach. Section IV presents the multigrid domain decomposition method. Section V conducts an in-depth error analysis of the multigrid domain decomposition method, focusing on its convergence. Section VI offers a comprehensive summary of the study's outcomes.

II. Coupled Stokes-Darcy Problem

This section investigates a Stokes-Darcy coupled system defined on a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$. The computational domain consists of two distinct subdomains: the free-flow subdomain Ω_e and the porous medium Ω_c, which intersect along their common boundary $\Gamma = \partial\Omega_e \cap \partial\Omega_c$. It is important to note that $\Omega_e \cap \Omega_c = \emptyset$ and $\overline{\Omega}_e \cup \overline{\Omega}_c = \overline{\Omega}$. Let $\Gamma_e = \Omega_e \setminus \Gamma$ and $\Gamma_c = \Omega_c \setminus \Gamma$.

Within the free-flow subdomain Ω_e, the fluid dynamics are described by the Stokes system:

$$\begin{cases} -\nabla \cdot (\boldsymbol{T}(u_e, p_e)) = f_e, \\ \nabla \cdot u_e = 0, \end{cases} \tag{1}$$

where

$$\boldsymbol{T}(u_e, p_e) = -p_e\boldsymbol{I} + 2\nu\boldsymbol{D}(u_e),$$
$$\boldsymbol{D}(u_e) = \frac{1}{2}(\nabla u_e + \nabla^T u_e),$$

u_e and p_e are construed respectively as the fluid velocity and the kinematic pressure in Ω_e. Besides, f_e is the external body

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force, and $\boldsymbol{T}(u_e, p_e)$ is the stress tensor with the identity matrix $\boldsymbol{I}$ and kinematic viscosity of the fluid $\nu > 0$.

In the porous medium Ω_c, the fluid flow is governed by the Darcy equation:

$$\begin{cases} \nabla \cdot u_c = f_c, \\ u_c = -\boldsymbol{K}\nabla\phi_c, \end{cases} \tag{2}$$

Let u_c denote the fluid flow rate within the porous medium Ω_c. For simplicity, we consider the hydraulic conductivity tensor to be isotropic, denoted by $\boldsymbol{K}$. The piezometric head, ϕ_p, is defined as the sum of the elevation z and the ratio of the dynamic pressure p_c to the product of the fluid density ρ and gravitational acceleration g, i.e., $\phi_p = z + \frac{p_c}{\rho g}$. Furthermore, the source term f_c is chosen to fulfill the necessary conditions for the problem's well-posedness.

$$\int_{\Omega_c} f_c = 0.$$

The coupling of Darcy's law and the continuity condition (2) generates an elliptic partial differential equation:

$$-\nabla \cdot (\boldsymbol{K}\nabla\phi_c) = f_c. \tag{3}$$

Assume the u_e and ϕ_c satisfying homogeneous Dirichlet boundary conditions:

$$u_e = 0 \ \text{ on } \Gamma_e, \qquad \phi_c = 0 \ \text{ on } \Gamma_c.$$

Three coupling conditions are enforced at the interface Γ:

$$\begin{cases} u_e \cdot n_c + u_c \cdot n_c = 0, \\ -n_e \cdot (\boldsymbol{T}(u_e, p_e) \cdot n_e) = g\phi_c, \\ -\tau_i \cdot (\boldsymbol{T}(u_e, p_e) \cdot n_e) \\ \quad = \dfrac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}} \cdot \tau_i \cdot u_e, \ i = 1, \cdots, d-1, \end{cases} \tag{4}$$

Let n_e and n_c be the unit outward normals to the fluid and porous medium at the common boundary Γ, respectively. The vectors τ_i represent the set of orthogonal unit tangent vectors to the interface Γ. The constant α is a parameter, and $\prod$ is defined as the ratio of the product of the hydraulic conductivity tensor $\boldsymbol{K}$ and the kinematic viscosity ν to the acceleration due to gravity g, i.e., $\prod = \frac{\boldsymbol{K}\nu}{g}$.

To derive the weak form of the mixed formulation, we define

$$\begin{aligned} H_e &= \{v_e \in (H^1(\Omega_e))^d : v_e = 0 \ \text{ on } \Gamma_e\}, \\ H_c &= \{\psi_c \in H^1(\Omega_c) : \psi_c = 0 \ \text{ on } \Gamma_c\}, \\ W &= H_e \times H_c, \\ Q_e &= L^2(\Omega_e). \end{aligned}$$

We use $(\cdot,\cdot)_{\Omega_X}$ and $\|\cdot\|_{L^2(\Omega_X)}$ to denote the standard L^2-scalar product of the spaces $L^2(\Omega_X)(X = e, c)$ and the associated L^2-norms of the space $L^2(\Omega_X)$, respectively.

III. Domain Decomposition Method

This section provides an overview of the domain decomposition method as described in [6]. The coupled Stokes-Darcy equations are decoupled into two distinct subproblems through this approach, with solutions being computed concurrently in both the free-flow subdomain (Ω_e) and porous medium domain (Ω_c). By employing domain decomposition, the computational scale is significantly reduced, enabling the use of standard software tools to solve each subproblem independently.

We now introduce the essential Robin-type boundary conditions at the interface. Given two positive constants λ_e and λ_c, there are associated functions g_e and g_c defined on the interface Γ that adhere to the following equation:

$$n_e \cdot (\boldsymbol{T}(u_e, p_e) \cdot n_e) + \lambda_e u_e \cdot n_e = g_e, \tag{5}$$
$$\lambda_c \boldsymbol{K}\nabla\phi_c \cdot n_c + g\phi_c = g_c. \tag{6}$$

By (4), we can get

$$g_e = \lambda_e u_e \cdot n_e - g\phi_c \qquad \text{on } \Gamma, \tag{7}$$
$$g_c = \lambda_c u_e \cdot n_e + g\phi_c \qquad \text{on } \Gamma. \tag{8}$$

It can be readily confirmed that the interface conditions (4) are tantamount to the previously stated Robin-type conditions (5)-(6) on the condition that the functions g_e and g_c meet the necessary compatibility criteria at the interface Γ.

Thus we obtain the variational formulation for the stationary Stokes-Darcy problem: for two given functions g_e, g_c and two normal numbers λ_e, λ_c, find $(u_e, p_e) \in H_e \times Q_e$, $\phi_c \in H_c$ such that

$$\lambda_c a_c(\phi_c, \psi) + \langle g\phi_c, \psi\rangle = \langle g_c, \psi\rangle + \lambda_c (f_c, \psi), \qquad \forall\ \psi \in H_c, \tag{9}$$

$$\begin{aligned} & a_e(u_e, v_e) - b_e(v_e, p_e) + \lambda_e \langle u_e \cdot n_e, v_e \cdot n_e\rangle \\ & + \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}} \langle u_e \cdot \tau_i, v_e \cdot \tau_i\rangle \\ & = \langle g_e, v_e \cdot n_e\rangle + (f_e, v_e), \qquad \forall\ v_e \in H_e, \end{aligned} \tag{10}$$

$$b_e(u_e, q) = 0, \qquad \forall\ q \in Q_e, \tag{11}$$

The bilinear forms are

$$\begin{aligned} a_e(u_e, v_e) &= 2\nu(\boldsymbol{D}(u_e), \boldsymbol{D}(v_e)), \\ a_c(\phi_c, \psi) &= (\boldsymbol{K}\nabla\phi_c, \nabla\psi), \\ b_e(u_e, q) &= (\nabla \cdot v_e, q), \end{aligned}$$

Chen and colleagues have established the well-posedness of the weak formulation (9)-(12) in [6].

For the Robin-Robin domain decomposition method, it is crucial to analyze the subsequent finite element discretization process. Let $\mathcal{T}_h$ represent a standard quasi-uniform triangulation of $\overline{\Omega}$ with a mesh parameter $h > 0$. Additionally, we denote the partition of Γ induced by $\mathcal{T}_h$ as $\mathcal{B}_h$.

Let $H_{e,h} \subset H_e$, $Q_{e,h} \subset Q_e$, and $H_{c,h} \subset H_c$ be the finite element subspaces defined on the partition $\mathcal{T}_h$. The P2-P1 finite element pair is used for the NS problem, while the P2 finite element is employed for the Darcy problem to ensure compatibility.

$$\begin{aligned} H_{e,h} &= \{v_{e,h} \in (H^1(\Omega_e))^d : v_{e,h}\,|_T \in (\mathbb{P}_2(T))^d \\ &\qquad \forall T \in \mathcal{T}_{e,h}, v_{e,h}\,|_{\Gamma_e} = 0\}, \\ Q_{e,h} &= \{q_{e,h} \in L^2(\Omega_e) : q_{e,h}\,|_T \in \mathbb{P}_1(T) \\ &\qquad \forall T \in \mathcal{T}_{e,h}\}, \\ H_{c,h} &= \{\psi_{c,h} \in H^1(\Omega_c) : \psi_{c,h}\,|_T \in \mathbb{P}_2(T) \\ &\qquad \forall T \in \mathcal{T}_{c,h}, \psi_{c,h}\,|_{\Gamma_c} = 0\}, \end{aligned}$$

the Spaces $H_{e,h}$ and $Q_{e,h}$ satisfy the inf-sup condition.

The discrete trace space over the interface Γ is defined as:

$$X_h = \{g_h \in L^2(\Gamma): \; g_h \mid_\tau \in \mathbb{P}_2(\tau) \;\; \forall \tau \in \mathcal{B}_h, \; g_h \mid_{\partial\Gamma} = 0\} \\ = H_{e,h} \mid_\Gamma \cdot n_f = H_{c,h} \mid_\Gamma .$$

Drawing upon the Robin conditions for the Stokes-Darcy equation and the compatibility conditions (7)-(8), we can outline the Robin-Robin domain decomposition method, as described in [6]:

1) Initial values of g_e^0 and g_c^0 are guessed.

2) For n=0,1,2, find $\phi_{c,h}^n \in H_{c,h}$ satisfy

$$\lambda_c a_c(\phi_{c,h}^n, \psi) + \langle g\phi_{c,h}^n, \psi\rangle = \langle g_{c,h}^n, \psi\rangle + \lambda_c(f_c, \psi), \quad \forall \; \psi \in H_{c,h}, \quad (12)$$

and $(u_{e,h}^n, p_{e,h}^n) \in H_{e,h} \times Q_{e,h}$ satisfy

$$a_e(u_{e,h}^n, v_e) - b_e(v_e, p_{e,h}^n) + \lambda_e\langle u_e e, h^n \cdot n_e, v_e \cdot n_e\rangle + \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle u_{e,h}^n \cdot \tau_i, v_e \cdot \tau_i\rangle = \langle g_{e,h}^n, v_e \cdot n_e\rangle + (f_e, v_e), \quad \forall \; (v_e, q_e) \in H_{e,h} \times Q_{e,h}, \quad (13)$$

$$b_s(u_{e,h}^n, q) = 0, \quad \forall \;\; q \in Q_e, \quad (14)$$

respectively.

3) Update $g_{e,h}^{n+1}, g_{c,h}^{n+1}$ by the following way:

$$g_{e,h}^{n+1} = \frac{\lambda_e}{\lambda_c} g_{c,h}^n - (1 + \frac{\lambda_e}{\lambda_c}) g\phi_{c,h}^n,$$
$$g_{c,h}^{n+1} = -g_{e,h}^n + (\lambda_e + \lambda_c) u_{e,h}^n \cdot n_e.$$

Theoretical convergence of the domain decomposition scheme is guaranteed, as proved in [6]. Moreover, the method guarantees an error bound that is independent of the mesh parameter h, provided that $\lambda_e < \lambda_c$ and the parameters λ_e and λ_c are carefully selected to meet specified control criteria. This Robin-Robin technique provides FEM solutions for decoupled Stokes-Darcy systems with Robin BCs. as described in equations (5)-(6). Specifically, for given functions $g_{e,h}$, $g_{c,h}$ and scalar parameters λ_e, λ_c, the method aims to find the triplet $(u_{e,h}, p_{e,h}) \in H_{e,h} \times Q_{e,h}$ and $\phi_{c,h} \in H_{c,h}$ that satisfy the required conditions.

$$\lambda_c a_c(\phi_{c,h}, \psi) + \langle g\phi_{c,h}, \psi\rangle = \langle g_{c,h}, \psi\rangle + \lambda_c(f_c, \psi), \quad \forall \; \psi \in H_{c,h}, \quad (15)$$

$$a_e(u_{e,h}, v_e) - b_e(v_e, p_{e,h}) + \lambda_e\langle u_{e,h} \cdot n_e, v_e \cdot n_e\rangle + \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle u_{e,h} \cdot \tau_i, v_e \cdot \tau_i\rangle = \langle g_{e,h}, v_e \cdot n_e\rangle + (f_e, v_e), \quad \forall \; (v_e, q_e) \in H_{e,h} \times Q_{e,h}, \quad (16)$$

$$b_e(u_{e,h}^n, q) = 0, \quad \forall \;\; q \in Q_e, \quad (17)$$

with the compatibility conditions:

$$g_{e,h} = \lambda_e u_{e,h} \cdot n_e - g\phi_{c,h} \quad \text{on } \Gamma, \quad (18)$$
$$g_{c,h}\lambda_c u_{e,h} \cdot n_e + g\phi_{c,h} \quad \text{on } \Gamma. \quad (19)$$

IV. THE MULITIGRID DOMAIN DECOMPOSITION METHOD

This section focuses on a decoupling strategy for the Stokes-Darcy system. Drawing inspiration from the Robin-Robin domain decomposition method [6] and the Multigrid domain decomposition approach [11], we develop a Multigrid-based domain decomposition scheme for the Stokes-Darcy problem with BJS interface conditions.

The Multigrid domain decomposition method for addressing the coupled Stokes-Darcy problem entails a two-phase process, which is detailed below.

1) On a coarse grid with mesh size H and $H = h_0$, we recall domain decomposition method to solve problems (15)-(17). Then we obtain the coarse grid result g_{e,h_0}, g_{c,h_0}.

2) An modified fine grid problem is constructed and solved by finding $(u_e^{h_i+1}, p_e^{h_i+1}) \in H_{e,h} \times Q_{e,h}$, $\phi_c^{h_i+1} \in H_{c,h}$, such that

$$\lambda_c a_c(\phi_c^{h_i+1}, \psi) + \langle g\phi_c^{h_i+1}, \psi\rangle = \langle g_{c,h_i}, \psi\rangle + \lambda_c(f_c, \psi), \quad \forall \; \psi \in H_{c,h}, \quad (20)$$

$$a_e(u_e^{h_i+1}, v_e) - b_e(v_e, p_e^{h_i+1}) + \lambda_e\langle u_e^{h_i+1} \cdot n_e, v_e \cdot n_e\rangle + \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle u_e^{h_i+1} \cdot \tau_i, v_e \cdot \tau_i\rangle = \langle g_{e,h_i}, v_e \cdot n_e\rangle + (f_e, v_e), \quad \forall \; (v_e, q_e) \in H_{e,h} \times Q_{e,h}. \quad (21)$$

$$b_e(u_e^{h_i+1}, q) = 0, \quad \forall \;\; q \in Q_e, \quad (22)$$

The multigrid domain decomposition method combines the strengths of the multigrid approach with the domain decomposition framework, offering a powerful tool for solving complex multi-domain and multi-physics problems. By leveraging the hierarchical structure of multigrid methods and the localized treatment of domain decomposition techniques, it efficiently manages strong interactions between different models across distinct domains. This hybrid approach not only enhances numerical stability but also improves convergence rates, making it particularly effective for addressing challenging coupling phenomena. Furthermore, for decoupled solutions, this method can significantly improve computational efficiency by reducing both memory requirements and processing time while maintaining high accuracy. Its versatility and robustness make it a preferred choice for large-scale simulations in scientific and engineering applications.

V. ERROR ANALYSIS

In this section, we reiterate the reasoning from [6] to illustrate the convergence of the introduced Multigrid Domain Decomposition method. To concise the presentation, we adopt the notation $m \lesssim n$ signifying that m is bounded above by Cm for some constant C, which may vary with the specific context. We now review the error bounds for the decoupled algorithm as detailed in [6]:

$$\|u_e - u_{e,h}\|_1 \lesssim h^2, \qquad \|u_e - u_{e,h}\| \lesssim h^3,$$
$$\|\phi_c - \phi_{c,h}\|_1 \lesssim h^2, \qquad \|\phi_c - \phi_{c,h}\| \lesssim h^3,$$
$$\|p_c - p_{c,h}\| \lesssim h^2.$$

For the finite element approximation given by equations (15)-(17), we express the error functions, which are related to the discrepancies between the solution components on the coarse and fine meshes, as follows:

$$\begin{aligned} \varrho_{e,H} &= g_{e,h} - g_{e,h_i}, & \varrho_{c,H} &= g_{c,h} - g_{p,h_i}, \\ \delta_{e,H} &= u_{e,h} - u_{e,h_i}, & \delta_{c,H} &= \phi_{p,h} - \phi_{p,h_i}, \\ \varsigma_{e,H} &= p_{e,h} - p_{e,h_i}. \end{aligned}$$

Then, by means of the triangle inequality, we can easily obtain several basic error estimates about the numerical solution of (15)-(17) on coarse and fine meshes

$$\begin{aligned} \|\delta_{e,H}\|_1 &\lesssim H^2, & \|\delta_{e,H}\| &\lesssim H^3, \\ \|\delta_{c,H}\|_1 &\lesssim H^2, & \|\delta_{c,H}\| &\lesssim H^3, \\ \|\varsigma_{e,H}\| &\lesssim H^2. \end{aligned} \tag{23}$$

To implement the error estimation, the following lemma is essential:

Lemma 1: Along the interface Γ, two error estimates on $\varrho_{e,H}$ and $\varrho_{c,H}$ related with the interface conditions have the forms of

$$\|\varrho_{e,H}\|_\Gamma \lesssim (\lambda_s + g)H^{\frac{5}{2}}, \tag{24}$$

$$\|\varrho_{c,H}\|_\Gamma \lesssim (\lambda_c + g)H^{\frac{5}{2}}. \tag{25}$$

Proof: According to the definition of $\varrho_{e,H}$, $\varrho_{p,H}$ and (18)-(19), we can derive the following formula

$$\begin{aligned} \varrho_{e,H} &= \lambda_e \delta_{e,H} \cdot n_e - g\lambda_{c,H}, \\ \varrho_{c,H} &= \lambda_c \delta_{e,H} \cdot n_e + g\lambda_{c,H}. \end{aligned}$$

Using the Young inequality we can launch

$$\begin{aligned} \|\varrho_{e,H}\|_\Gamma &= \|\lambda_e \delta_{e,H} \cdot n_e - g\delta_{e,H}\|_\Gamma \\ &\leq \lambda_e \|\delta_{e,H} \cdot n_e\|_\Gamma + g\|\delta_{c,H}\|_\Gamma. \end{aligned}$$

Based on the trace inequality, we are aware that there exists a constant C such that

$$\begin{aligned} \|\delta_{e,H} \cdot n_e\|_\Gamma &\leq C\|\delta_{e,H}\|^{\frac{1}{2}} \|\delta_{e,H}\|_1^{\frac{1}{2}}, \\ \|\delta_{c,H}\|_\Gamma &\leq C\|\delta_{c,H}\|^{\frac{1}{2}} \|\delta_{c,H}\|_1^{\frac{1}{2}}, \end{aligned}$$

then we can conclude that

$$\begin{aligned} \lambda_f \|\delta_{e,H} \cdot n_e\|_\Gamma + g\|\delta_{c,H}\|_\Gamma &\leq \lambda_e C\|\delta_{e,H}\|^{\frac{1}{2}} \|\delta_{e,H}\|_1^{\frac{1}{2}} \\ &\quad + gC\|\delta_{c,H}\|^{\frac{1}{2}} \|\delta_{c,H}\|_1^{\frac{1}{2}} \\ &\leq \lambda_e C H^{\frac{3}{2}} H + gCH^{\frac{3}{2}} H \\ &\lesssim (\lambda_e + g)H^{\frac{5}{2}}. \end{aligned}$$

The error estimate of $\|\varrho_{p,H}\|_\Gamma$ can be obtained in the same way. ■

Building on the aforementioned groundwork, we can derive the error estimate for the Multigrid domain decomposition method as follows:

Theorem 1: Let $(u_{e,h}, p_{e,h}, \phi_{c,h})$ be the solution derived from domain decomposition method, and assume that $(u_e^{h_i+1}, p_e^{h_i+1}, \phi_c^{h_i+1})$ is the solution derived from Multigrid domain decomposition method, the following error estimates hold:

$$\|\phi_{p,h} - \phi_c^{h_i+1}\|_1 \lesssim \frac{\lambda_c + g}{\boldsymbol{J}\lambda_c} H^{\frac{5}{2}}, \tag{26}$$

$$\|u_{e,h} - u_e^{h_i+1}\|_1 \lesssim (\lambda_s + g)H^{\frac{5}{2}}, \tag{27}$$

$$\begin{aligned} &\|p_{e,h} - p_e^{h_i+1}\| \\ &\lesssim (2\nu + \lambda_e C_e^2 + \frac{\nu\alpha\sqrt{d}C_e^2}{\sqrt{trace(\prod)}} + c_e)(\lambda_e + g)H^{\frac{5}{2}}, \end{aligned} \tag{28}$$

Proof: On the fine grid,let $h = h_{i+1}$ in (20)-(22),subtracting (20)-(22) from (15)-(17) yields

$$\begin{aligned} &\lambda_c a_c(\phi_{c,h_{i+1}} - \phi_c^{h_i+1}, \psi) + \langle g(\phi_{c,h_{i+1}} - \phi_c^{h_i+1}), \psi\rangle \\ &\qquad = \langle g_{c,h} - g_{c,h_i}, \psi\rangle, \quad \forall\ \psi \in H_{c,h}. \end{aligned} \tag{29}$$

$$\begin{aligned} &a_e(u_{e,h_{i+1}} - u_e^{h_i+1}, v_e) - b_e(v_e, p_{e,h_{i+1}} - p_e^{h_i+1}) \\ &+ \lambda_e\langle(u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot n_e, v_e \cdot n_e)\rangle \\ &+ \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle u_{e,h_{i+1}} - u_e^{h_i+1} \cdot \tau_i, v_e \cdot \tau_i\rangle \\ &\qquad = \langle (g_{e,h} - g_{e,h_i}), v_e \cdot n_e\rangle, \quad \forall\ v_e \in H_{e,h}. \end{aligned} \tag{30}$$

$$b_e(u_{e,h_{i+1}} - u_e^{h_i+1}, q) = 0, \qquad \forall\ q \in Q_e. \tag{31}$$

Let $\psi = \phi_{c,h_{i+1}} - \phi_c^{h_{i+1}} \in H_{c,h}$ in (29)

$$\begin{aligned} &\lambda_c a_c(\phi_{c,h_{i+1}} - \phi_c^{h_i+1}, \phi_{c,h_{i+1}} - \phi_c^{h_i+1}) \\ &+ \langle g(\phi_{c,h_{i+1}} - \phi_c^{h_i+1}), \phi_{c,h_{i+1}} - \phi_c^{h_i+1}\rangle \\ &\qquad = \langle g_{c,h} - g_{c,h_i}, \phi_{c,h_{i+1}} - \phi_c^{h_i+1}\rangle. \end{aligned} \tag{32}$$

Utilizing the Cauchy-Schwarz inequality in conjunction with the trace inequality, we can conclude that

$$\begin{aligned} &\|\phi_{p,h} - \phi_p^h\|_1^2 \leq \frac{1}{\boldsymbol{K}} a_p(\phi_{p,h} - \phi_p^h, \phi_{p,h} - \phi_p^h), \\ &a_p(\phi_{p,h} - \phi_p^h, \phi_{p,h} - \phi_p^h) \\ &\qquad \leq a_p(\phi_{p,h} - \phi_p^h, \phi_{p,h} - \phi_p^h) + \frac{g}{\xi_p}\|\phi_{p,h} - \phi_p^h\|_\Gamma^2, \end{aligned}$$

then, from Lemma 1 , we get the following inequality,

$$\begin{aligned} \|\phi_{c,h_{i+1}} - \phi_c^{h_i+1}\|_1^2 &\leq \frac{1}{\boldsymbol{J}}[a_c(\phi_{c,h_{i+1}} - \phi_c^{h_i+1}, \phi_{c,h_{i+1}} - \phi_c^{h_i+1}) \\ &\leq \frac{1}{\boldsymbol{J}}[\lambda_c a_c(\phi_{c,h_{i+1}} - \phi_c^{h_i+1}, \phi_{c,h_{i+1}} - \phi_c^{h_i+1}) \\ &\quad + g\|\phi_{c,h_{i+1}} - \phi_c^{h_i+1}\|^2] \\ &\leq \frac{1}{\boldsymbol{J}\lambda_c} < g_{c,h} - g_{c,h_i}, \phi_{c,h_{i+1}} - \phi_c^{h_i+1} > \\ &\lesssim \frac{\lambda_c + g}{\boldsymbol{J}\lambda_c} H^{\frac{5}{2}} \|\phi_{c,h_{i+1}} - \phi_c^{h_i+1}\|_1. \end{aligned} \tag{33}$$

We can get (26) by eliminating $\|\phi_{c,h_{i+1}} - \phi_c^{h_i+1}\|_1$ from (33).

Setting $v_e = u_{e,h_{i+1}} - u_e^{h_i+1} \in H_{e,h}, q = p_{e,h_{i+1}} - p_e^{h_i+1} \in Q_{e,h}$ and substituting into (29), we have

$$\begin{aligned} &a_e(u_{e,h_{i+1}} - u_e^{h_i+1}, u_{e,h_{i+1}} - u_e^{h_i+1}) \\ &- b_e(u_{e,h_{i+1}} - u_e^{h_i+1}, p_{e,h_{i+1}} - p_e^{h_i+1}) \\ &+ \lambda_e\langle(u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot n_e, (u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot n_e\rangle \\ &+ \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle u_{e,h_{i+1}} - u_e^{h_i+1} \cdot \tau_i, u_{e,h_{i+1}} - u_e^{h_i+1} \cdot \tau_i\rangle \\ &= \langle g_{e,h} - g_{e,h_i}, (u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot n_e\rangle, \end{aligned} \tag{34}$$

by the $Korn's$ inequality, there exists C_e that makes

$$\|u_{e,h_{i+1}}-u_e^{h_i+1}\|_1^2 \leq \frac{C_e^2}{2\nu}a_e(u_{e,h_{i+1}}-u_e^{h_i+1},u_{e,h_{i+1}}-u_e^{h_i+1}),$$

then we can get

$$\begin{aligned}
&\|u_{e,h_{i+1}} - u_e^{h_i+1}\|_1^2 \\
&\leq \frac{C_e^2}{2\nu}[a_e(u_{e,h_{i+1}} - u_e^{h_i+1}, u_{e,h_{i+1}} - u_e^{h_i+1}) \\
&+ \lambda_e\|(u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot n_e\| \\
&+ \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\|u_{e,h_{i+1}} - u_e^{h_i+1} \cdot \tau_i\|] \\
&\leq \frac{C_e^2}{2\nu}\langle g_{e,h} - g_{e,h_i}, (u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot n_e\rangle \\
&\lesssim \frac{C_e^2}{2\nu}\|g_{e,h} - g_{e,h_i}\|\|u_{e,h_{i+1}} - u_e^{h_i+1}\|_1 \qquad (35)
\end{aligned}$$

Let $q = p_{e,h_{i+1}} - p_e^{h_i+1} \in Q_{e,h}$, there exist $v_e \in H_{e,h}$ such that

$$\|p_{e,h_{i+1}} - p_e^{h_i+1}\| \leq \frac{-b_e(v_e, p_{e,h_{i+1}} - p_e^{h_i+1})}{\|v_e\|_1}.$$

It can also be inferred from (33) that

$$\begin{aligned}
&\|p_{e,h_{i+1}} - p_e^{h_i+1}\| \\
&\leq \frac{1}{\|v_e\|_1}\Big[|a_e(u_{e,h_{i+1}} - u_e^{h_i+1}, v_e)| \\
&+ \lambda_e|\langle(u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot n_e, v_e \cdot n_e\rangle| \\
&+ \sum \frac{\nu\alpha\sqrt{d}}{\sqrt{trace(\prod)}}|\langle(u_{e,h_{i+1}} - u_e^{h_i+1}) \cdot \tau_i, v_e \cdot \tau_i\rangle|\Big] \\
&\leq (2\nu + \lambda_e C_{tr}^2 + \frac{\nu\alpha\sqrt{d}C_{tr}^2}{\sqrt{trace(\prod)}})\|u_{e,h_{i+1}} - u_{e^{h_i+1}}\|_1 \\
&+ C_{tr}\|g_{e,h} - g_{e,h_i}\|_\Gamma \\
&\lesssim (2\nu + \lambda_e C_{tr}^2 + C_{tr} + \frac{\nu\alpha\sqrt{d}C_{tr}^2}{\sqrt{trace(\prod)}})(\lambda_e + g)H^{\frac{5}{2}}.
\end{aligned}$$

which completes the proof of (28). ■

Drawing from Theorem 1 and the triangle inequality, we can establish the error estimate for the solution obtained by the Multigrid domain decomposition method in relation to the exact solution as detailed below.

Corollary 1: Let $(u_e^{h_{i+1}}, p_e^{h_i+1}, \phi_c^{h_i+1}) \in (H_{e,h} \times Q_{e,h} \times H_{c,h})$, and $(u_{e,h}, p_{e,h}, \phi_{c,h}) \in (H_e \times Q_e \times H_c)$ be the solution of Mulitigrid domain decomposition method and (9)-(10), respectively. Choosing $H = h^{\frac{2}{5}}$, we have

$$\begin{aligned}
&\|\phi_{c,h_{i+1}} - \phi_c^{h_i+1}\|_1 \lesssim h, \\
&\|u_{e,h_{i+1}} - u_e^{h_i+1}\|_1 + \|p_{e,h_{i+1}} - p_e^{h_i+1}\| \lesssim h.
\end{aligned}$$

The accuracy of the Multigrid Domain Decomposition algorithm has been established. Additionally, the theoretical framework can be broadened to encompass higher-order elements, provided the continuous solution exhibits sufficient regularity. In conclusion, future studies may enhance the algorithm's accuracy through more rigorous analytical investigations.

VI. Conclusion

This manuscript presents a multigrid domain decomposition algorithm specifically designed for solving coupled Stokes–Darcy equations. We conduct a comprehensive convergence analysis that demonstrates the algorithm's robust convergence rates and numerical stability. The results show that our proposed approach achieves high convergence accuracy while maintaining computational efficiency, making it particularly suitable for large-scale simulations. The method's scalability further enhances its practical value for complex coupled problems.

The algorithm's modular design suggests significant potential for extension to other multi-domain, multi-physics systems. Future work will focus on applications to more sophisticated interface conditions, including the Stokes–Darcy system with Beavers–Joseph interface conditions. Such extensions could substantially broaden the method's applicability across computational science and engineering domains, particularly for problems requiring accurate modeling of fluid–porous media interactions.

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