Expression of Constant Proportional Caputo Derivative in Terms of the Block Pulse Operational Matrix

Prabha R and Kiruthika S

Abstract—**In this paper, we introduce the constant proportional Caputo fractional derivative (CPC), which is defined for all differentiable functions that are locally integrable in the L^1 space of positive real numbers. The block pulse operational matrix is used to express the derivatives in differential equations that we solve using the block pulse method. Some initial value problems are tackled using the Shehu transform method. To model diseases more reliably and efficiently, we combine the Shehu transform method with the homotopy perturbation method for solving CPC differential equations.**

Index Terms—**CPC derivatives, block pulse method, orthogonal functions, Shehu transform.**

I. Introduction

THe history of fractional calculus goes back more than 3 decades, which came into existence as a result of a query about the value of the derivative of the non-integer order and has had significant developments both in theoretical and practical aspects. The classical derivatives (derivatives of integer order) can be interpreted geometrically; for instance, the first derivative is defined as the slope of the tangent, whereas the geometrical interpretation of fractional calculus has not gained much relevance. Due to this, the field of fractional calculus showed a slow development up to 1900 after which this field showed a rapid and expeditious development and shed its light in several applications not only in mathematics [18], [20], but also in science, engineering [15] and various medical fields [5]. The first application of fractional calculus was the Tautochrone problem proposed by Abel, which deals with finding the least time of descent by which a bead can move down a curve along the shortest path. Later on, differential equations of arbitrary order emerged as an important topic that is best suited for defining various dynamical systems more specifically and precisely since these operators are bestowed with some peculiar properties such as memory effect, hereditary property, etc. Hence, these operators are used in exploring the medical fields [7], [21], [32] by which the control strategies can be developed and modeled efficiently in such a way that the treatments can be made possible with minimum cost and maximum precision [19], [27]. Of the various models, the SIR model for childhood diseases [5] and the SIERD model have gained special attention in recent times because some pandemic diseases can affect the whole world economically and physically to a great extent [24]. Among the pandemic diseases such as Covid-19 modeling [21] past experiences such as social distancing, age, medical history and many other constraints are needed to describe treatment so that side effects are minimized [5], [26]. In addition to these properties, derivatives of arbitrary order satisfy the power law, exponential decay, and the generalized Mittag-Leffler function [17], [28] form the base for the definitions for various fractional operators [2], [4], [9], [25], [33].

The category of fractional operators is classified into different groups based on specific characteristics, such as whether they have a singular or non-singular kernel [16], [23]. Initially, the definitions that captured researchers' interest were proposed by Riemann-Liouville and Caputo. Among these, Caputo's definition was found to be more suitable for addressing problems involving fractional differential equations with initial conditions. Over time, several other definitions were introduced by Weyl, Marchaud, Hadamard, Caputo and Fabrizio, Atangana and Baleanu, among others.

Recently, a new fractional operator known as the proportional derivative operator has been defined; this operator emerges in control theory [11], [12], [27] and can be viewed as an extension of conformable derivatives. The introduction of this new derivative allows for a broader context in defining various processes and systems. Additionally, this operator can be applied to model various pandemic diseases [6], [21].

A.A. Kilbas et al. [2], H.J. Haubold et al. [16], and J. F. Gomez-Aguilar [17] have investigated the Mittag-Leffler function in the generalized form and its applications whereas T.A. Prabhakar [26], [33], [34] utilized generalized Mittag-Leffler memory to solve a singular integral problem.

To address fractional differential equations, A. Arikoglu and I. Ozkol utilized a transform technique [1]. The Shehu transform method was also employed in the article [3] to work with the Atangana-Baleanu derivative. For solving these fractional models, Maitama and Zhao investigated the combination of the Shehu transform technique and the homotopy perturbation method [30], while S. Momani and Z. Odibat numerically compared many approaches for solving fractional differential equations [31]. In addition, Akgul [11] used the Sumudu transform to solve differential equations that involve constant proportional Caputo operators, highlighting several applications.

Li Yuanlu, N. Sun [22], and C.H. Wang [10] used the

Manuscript received July 23, 2024; revised May 29, 2025.

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generalized block pulse matrix to solve fractional differential equations. Furthermore, A. Saadatmandi et al. [8] expanded the use of Legendre polynomials in differential equations by generalizing the Legendre operational matrix for fractional calculus. Baleanu et al. [11] introduced a new hybrid operator called the proportional Caputo derivative. E.K. Akgul et al. [14] applied the Laplace transform method to an economic model using this hybrid derivative

The SIR epidemic model for childhood diseases was examined by A.G. Selvam et al. [5], emphasizing its stability. T.A. Yildiz et al. [32] proposed optimal cancer treatments using fractional models, with and without singular kernels. Additionally, K. Rajagopal et al. [21] developed a fractional order model to address the novel coronavirus outbreak.

In this study, we analyze the block pulse method by applying it to the hybrid operator, presenting our findings as a sum of block pulse matrices. We also evaluate the Shehu transform of CPC derivatives and utilize these results to solve both linear and nonlinear CPC differential equations. The ability of the block pulse method to translate differential equations into algebraic equations represented by block pulse matrices is one of its main advantages, which greatly simplifies the problem-solving process. The CPC derivative is a newly developed operator, and ongoing research aims to find solutions to differential equations of this type. As part of this effort, we solve differential equations in both linear and nonlinear contexts using this method. Our goal is to produce results that are useful for practical applications involving CPC derivatives by converting them into block pulse matrices and subsequently solving the resulting algebraic equations.

In particular, when applying the CPC derivative, the results differ from those obtained using operators in other studies [8], [22], since we find sums of orthogonal matrices due to the hybrid nature of the operator. We also evaluate the CPC derivative of some standard functions, obtaining results expressed in terms of the Mittag-Leffler function, which can be utilized in application problems, such as solutions to nonlinear partial differential equations where periodic functions like trigonometric functions are involved. In this paper, we express the CPC differential equations in terms of block pulse matrices, allowing for an analysis of how the results differ when ordinary derivatives are replaced with CPC derivatives in disease modeling (e.g., the SIR model, SEIRD model). The numerical methods combined with transforms for solving differential equations are gaining much attention today. Furthermore, we evaluated the Shehu transform of the CPC derivative using the duality property referenced in earlier studies, where the authors cite AB, EA evaluated the Sumudu transform of the CPC derivative. The Shehu transform is subsequently applied to solve certain differential equations, with the results illustrated through examples. Finally, we present a method that combines the homotopy perturbation method with the Shehu transform, explaining how to solve a CPC differential equation using this approach [1], [12], [22], [29], [31].

The paper is organized as follows. Section II presents the study's concepts and techniques, and the CPC derivative is described using the operational matrix. The primary objective of Section III is to solve some linear CPC differential equations. In Section IV, some nonlinear CPC differential equations are solved using the block pulse approach. The evaluation of CPC derivatives for various standard functions is discussed in sections V, VI, and VII. In section VIII, the Shehu transform of this derivative is evaluated, and in section IX, some differential equations are solved using the Shehu transform method. The explanation of the homotopy perturbation Shehu transform method for solving a CPC differential equation is provided in section X. Finally, we conclude the paper in section XI.

II. Expression of CPC derivatives in terms of the block pulse operational matrix

In this section, we express the CPC derivatives in terms of an upper triangular matrix known as the block pulse operational matrix [8], [10], [22], for which we have used the concept of orthogonal functions [8].

The CPC derivative [11] of order $\alpha,\ 0<\alpha<1$ is given by

$$ {}_0^{CPC}D_t^{\wp} f(t) = K_1(\wp)\ {}_0^{RL}I_t^{1-\wp} f(t) + K_0(\wp)\, {}_0^{C}D_t^{\wp} f(t) \tag{1}$$

where K_1 and K_0 are functions of α.
The Riemann Liouville derivative of order $\wp$ was expressed using block pulse function as

$$ {}_0^{RL}I_t^{1-\wp} f(t) = \frac{1}{\Gamma(1-\wp)} \int_0^t (t-\tau)^{-\wp} f(\tau)d\tau, \tag{2}$$

where the integrand in this expression is the convolution of $t^{-\wp}$ and the function $f(t)$ which is absolutely integrable in [0,T) that can be represented through orthogonal(block pulse) functions $\eta_m(t), m=1,2,....$ The orthogonal functions are represented graphically in the figures below:
For the block pulse operational matrix,

$$ f(t) \approx f^T \eta_m(t), \text{ where T denotes the transpose of f(t)} \tag{3}$$

$$ f^T = [f_1\ f_2\ f_3\ ....f_m],\ \eta_m{}^T = [\eta_1(t)\ \eta_2(t)\ ....\eta_m(t)]$$

$$ f_i = \frac{m}{T}\int_0^t f(t)\eta_i(t)dt = \frac{m}{T}\int_{\frac{i-1}{m}T}^{\frac{i}{m}T} f(t)\eta_i(t)dt$$

$$ \eta_i(t) = \begin{cases} 1, & \frac{i-1}{m} \le t \le \frac{i}{m} \\ 0, & elsewhere \end{cases} \quad i=1,2,..$$

Therefore, the RL integral of the CPC derivative is represented in equation 2 as
$\frac{1}{\Gamma(1-\wp)}\int_0^t (t-\tau)^{-\wp} f(\tau)d\tau$

$$ \approx f^T \frac{1}{\Gamma(1-\wp)}\int_0^t (t-\tau)^{-\wp}\eta_i(\tau)d\tau$$

The RL integral of order $1-\wp$ is solved in terms of convolution of two functions $f_1 = t^{-\wp},\ f_2(t) = \eta_i(t),\ \wp > 0$. Then,

$$ \begin{aligned} & \frac{1}{\Gamma(1-\wp)}\int_0^t (t-\tau)^{-\wp}\eta_i(\tau)d\tau \\ = \quad & \frac{1}{\Gamma(1-\wp)}\int_0^t f_1(t-\tau)f_2(\tau)d\tau \end{aligned}$$

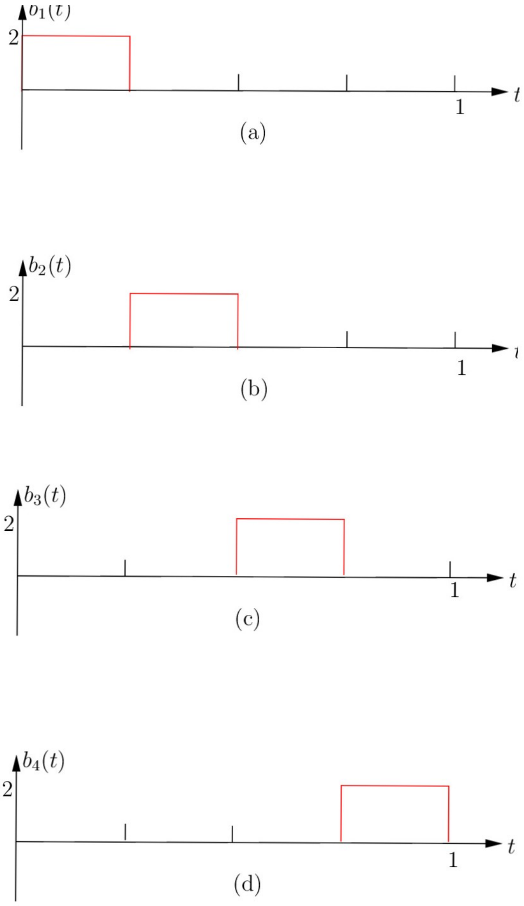

Fig. 1. Block pulse functions

Using the Laplace transform on the equation's two sides, we get,

$$L\left[\frac{1}{\Gamma(1-\wp)}(t-\tau)^{-\wp}\eta_i(\tau)d\tau\right]$$
$$=\frac{1}{\Gamma(1-\wp)}L\left[\int_0^t f_1(t-\tau)f_2(\tau)\right]$$
$$=\frac{1}{\Gamma(1-\wp)}F_1(s)F_2(s)$$

where $F_1(s)=L\left[t^{-\wp}\right]=\frac{\Gamma(1-\wp)}{s^{1-\wp}}$

$$F_2(s)=L\left[\eta_i(t)\right]=\frac{1}{s}\left\{e^{\frac{-(i-1)}{m}Ts}-e^{\frac{-i}{m}Ts}\right\}.$$

Thus, we have

$$L\left[\frac{1}{\Gamma(1-\wp)}\int_0^t(t-\tau)^{-\wp}\eta_i(\tau)d\tau\right]$$
$$=\frac{1}{\Gamma(1-\wp)}\frac{\Gamma(1-\wp)}{s^{1-\alpha}}\frac{1}{s}\left[e^{\frac{-(i-1)}{m}Ts}-e^{\frac{-i}{m}Ts}\right].$$

Multiplying and dividing by $1-\wp$ on the right-hand side of the above equation, we get

$$L\left[\frac{1}{\Gamma(1-\wp)}(t-\tau)^{-\wp}\eta_i(\tau)d\tau\right]$$
$$=\frac{1}{\Gamma(2-\wp)}\frac{\Gamma(2-\wp)}{s^{2-\wp}}\left[e^{\frac{-(i-1)}{m}Ts}-e^{\frac{-i}{m}Ts}\right]$$

When we apply the inverse Laplace transform to both sides of the equation above, we obtain

$$\frac{1}{\Gamma(1-\wp)}\int_0^t(t-\tau)^{-\wp}\eta_i(\tau)d\tau$$
$$=\frac{1}{\Gamma(2-\wp)}[(t-\frac{i-1}{m})^{1-\wp}u(t-\frac{i-1}{m}T)$$
$$-(t-\frac{i}{m})^{1-\wp}u(t-\frac{i}{m}T)] \quad (4)$$

where $u(t)$ is the unit step function, and we use the result of orthogonal function [22], so the equation 4 becomes

$$(t-\frac{i-1}{m}T)u(t-\frac{i-1}{m}T)\approx[0,0,...0,d_1,...d_{m-i+1}]\,\eta_m(t)$$
$$(t-\frac{i}{m}T)u(t-\frac{i}{m}T)\approx[0,0,...0,d_1,d_2,...d_{m-i}]\,\eta_m(t)$$

where

$$t^{1-\alpha}u(t)\approxeq[d_1\ d_2\ ...\ d_m]\,\eta_m(t)=C^T\eta_m(t)$$

and d_i's can be represented as

$$d_i=\frac{m}{T}\int_0^T f(t)\eta_i(t)dt$$
$$=\frac{m}{T}\int_{\frac{i-1}{m}T}^{\frac{i}{m}T}t^{1-\wp}u(t)dt$$
$$=\frac{m}{T}\int_{\frac{i-1}{m}T}^{\frac{i}{m}T}t^{1-\wp}dt$$
$$=\frac{m}{T}\frac{1}{2-\wp}\left[(\frac{i}{m})^{2-\wp}T^{2-\wp}-(\frac{i-1}{m})^{2-\wp}T^{2-\wp}\right]$$
$$=(\frac{T}{m})^{1-\wp}\left[\frac{(i)^{2-\alpha}-(i-1)^{2-\wp}}{2-\wp}\right]$$

Then equation 3 is expressed as

$$\frac{1}{\Gamma(1-\wp)}\int_0^t(t-\tau)^{-\wp}\eta_i(\tau)d\tau$$
$$\approx\frac{1}{\Gamma(2-\wp)}[0,0,...0,d_1,d_2-d_1,...d_{m-i+1}-d_{m-i}]\,\eta_m(t)$$
$$for\ \ i=1,2...m.$$

$$d_i-d_{i-1}=$$
$$(\frac{T}{m})^{1-\wp}\left[\frac{(i)^{2-\wp}-2(i-1)^{2-\wp}+(i-2)^{2-\wp}}{2-\wp}\right]$$

for

$$i=2,3,...,m-k$$

and $d_1=(\frac{T}{M})^{1-\wp}\frac{1}{2-\wp}$.

Therefore,

$$\frac{1}{\Gamma(1-\wp)}\int_0^t(t-\tau)^{-\wp}\eta_i(\tau)d\tau$$
$$\approx(\frac{T}{M})^{1-\wp}\frac{1}{\Gamma(2-\wp)}\frac{1}{2-\wp}[0,0...0,f_1,f_2...f_{m-i+1}]\,\eta_m(t)$$
$$\approx(\frac{T}{m})^{1-\wp}\frac{1}{\Gamma(3-\wp)}[0,0...0,f_1,\ f_2\ ...\ f_{m-i+1}]\,\eta_m(t)$$

where $f_1=1, f_k=k^{2-\wp}-2(k-1)^{2-\wp}+(k-2)^{2-\wp}$ $k=2,3,...m-i+1$.

The RL integral of CPC derivative in equation 4 is obtained in terms of block pulse function as

$$\frac{1}{\Gamma(1-\wp)}\int_0^t(t-\tau)^{-\wp}\eta_i(\tau)d\tau$$
$$=\frac{K_1(\wp)}{\Gamma(1-\wp)}\int_0^t(t-\tau)^{-\wp}\eta_i(\tau)d\tau$$
$$\approx(\frac{T}{m})^{1-\wp}\frac{K_1(\wp)}{\Gamma(3-\wp)}[0,0...0,f_1,f_2...f_{m-i+1}]\,\eta_m(t)$$
$$=\frac{K_1(\wp)}{\Gamma(1-\wp)}\int_0^t(t-\tau)^{-\wp}\eta_i(\tau)d\tau$$
$$\approx(\frac{T}{m})^{1-\wp}\frac{K_1(\wp)}{\Gamma(3-\wp)}.$$

$$\begin{bmatrix} f_1 & f_2 & . & . & f_m \\ 0 & f_1 & . & . & f_{m-1} \\ 0 & 0 & . & . & f_{m-2} \\ . & . & . & . & . \\ 0 & 0 & . & . & f_1 \end{bmatrix} \approx F_{1-\wp}\eta_m(t)$$

where

$$F_{1-\wp} = (\frac{T}{m})^{1-\wp}\frac{K_1(\wp)}{\Gamma(3-\wp)}\begin{bmatrix} f_1 & f_2 & . & . & f_m \\ 0 & f_1 & . & . & f_{m-1} \\ 0 & 0 & . & . & f_{m-2} \\ . & . & . & . & . \\ 0 & 0 & . & . & f_1 \end{bmatrix}$$

Thus, we have expressed the RL integral in the definition of CPC derivative in terms of a block pulse matrix which is an upper triangular matrix.

Proceeding as in the evaluation of RL integral, we have expressed the Caputo derivative of order α in the definition of CPC derivative regarding the operating matrix for block pulse. G_α as

$$G_\wp = K_0(\wp)(\frac{m}{T})^{\wp}\Gamma(\wp+2)\begin{bmatrix} g_1 & g_2 & . & . & g_m \\ 0 & g_1 & . & . & g_{m-1} \\ 0 & 0 & . & . & g_{m-2} \\ . & . & . & . & . \\ 0 & 0 & . & . & g_1 \end{bmatrix}$$

where
$g_1 = 1, g_2 = -h_2\ g_1, ..., g_m = -\sum_{i=2}^{m} g_i\ d_{m-i+1}$ and the h_i's are given by
$h_1 = 1, h_q = q^{\wp+1} - 2(q-1)^{\wp+1} + (q-2)^{\wp+1}$,
$q = 2, 3, ...m - i + 1$.
Thus, the CPC derivative of order $\wp$, $0 < \wp < 1$ can be expressed as a sum of two block pulse operational matrices.

$$^{CPC}_{0}D_t^{\wp} f(t) = F_{1-\wp}\eta_m(t) + G_\wp\eta_m(t) \tag{5}$$

Using the above result in equation (1), for $\wp = \frac{1}{2}$ by assuming the values of $K_1(\frac{1}{2}) = 1$ and $K_0(\frac{1}{2}) = 1$, $m = 4$, $T = 2$. Then

$$\begin{aligned} F_{1-\wp} &= F_{0.5} \\ &= (\frac{2}{4})^{0.5}\frac{1}{\Gamma(3-0.5)}\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ 0 & f_1 & f_2 & f_3 \\ 0 & 0 & f_1 & f_2 \\ 0 & 0 & 0 & f_1 \end{bmatrix} \end{aligned}$$

where $f_1 = 1,\ f_2 = 0.8284,\ f_3 = 0.5392,\ f_4 = 0.4361$.
Hence, we will get

$$F_{0.5} = \begin{bmatrix} .5318 & .4405 & .2867 & .2319 \\ 0 & .5318 & .4405 & .2867 \\ 0 & 0 & .5318 & .4405 \\ 0 & 0 & 0 & .5318 \end{bmatrix}$$

Also,

$$G_{0.5} = (\frac{4}{2})^{0.5}\Gamma(0.5)\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ 0 & g_1 & g_2 & g_3 \\ 0 & 0 & g_1 & g_2 \\ 0 & 0 & 0 & g_1 \end{bmatrix}$$

where

$$g_1 = 1, g_2 = -h_2 g_1,\ g_3 = -h_2 g_2\ - h_3 g_1,$$

$$g_4 = -h_2 g_3\ - h_3 g_2\ - h_4 g_1$$

The h_i 's are given by

$$h_1 = 1,\ h_2 = .8284,\ h_3 = -.4607,\ h_4 = .4364$$

Thus we get

$$g_1 = 1,\ g_2 = -.8284,\ g_3 = 1.1469,\ g_4 = -1.7681$$

After substituting these values in the matrix, we get

$$G_{0.5} = \begin{bmatrix} 1.8803 & 1.5576 & 2.1565 & 3.3246 \\ 0 & 1.8803 & 1.5576 & 2.1565 \\ 0 & 0 & 1.8803 & 1.5576 \\ 0 & 0 & 0 & 1.8803 \end{bmatrix}$$

Therefore,

$$^{CPC}_{0}D_t^{.5} f(t) = F_{.5}\eta_m(t) + G_{.5}\eta_m(t)$$

III. Solution of some linear CPC differential equations using operational matrix method

Example 1 Consider a linear CPC differential equation

$$^{CPC}_{0}D_t^{\beta} f(t) = \lambda g(t),\ \ where\ \ 0 < \beta < 1. \tag{6}$$

Using 3, 5 in the above equation 6, we get

$$\{F_{1-\beta} + H_\beta\}\,\eta_m(t) \approx \lambda G^T \eta_m(t)$$

Solving the algebraic equation, $G^T = \frac{1}{\lambda}\{F_{1-\beta} + H_\beta\}$ we will get the solution for 6.

Example 2 Consider the linear CPC differential equation of the form

$$^{CPC}_{0}D_t^{\beta} f(t) = \sum_{j=1}^{n} c_j(t)\ \ ^{CPC}_{0}D_t^{\beta_j} f(t) + c_0(t)y(t) + g(t)$$

where $0 < \beta < 1$ with initial conditions $y^m(0) = b_m$,
$m = 0, 1, 2, ...\lceil\beta\rceil - 1\ \beta_1 > \beta_2 > ... > \beta_n$, $^{CPC}_{0}D_t^{\beta}$ denotes the CPC operator of order β, $c_j(t)$ is a known function for j = 0, 1, 2,...n, the input and output functions being g(t) and y(t), respectively.

Before proceeding to find the solution, we write the solution in a modified form to reduce the given non-zero initial conditions to zero initial conditions. Hence we write the solution in the form $y(t) = y_\delta(t) + n(t)$ where $y_\delta(t)$ is a familiar function that meets the initial conditions

$$y^m(0) = b_m,\ m = 0, 1, 2, ...\lceil\beta\rceil - 1$$

and n(t) is a function whose value is not known.

$$^{CPC}_{0}D_t^{\beta} f(t) = \{F_{1-\beta} + G_\beta\}\,\eta_m(t)$$

Let us take $F_{1-\beta} + G_\beta = S_\beta$ and also using the modified initial conditions in the initial value problem, we get

$$^{CPC}_{0}D_t^{\beta} h(t) = \sum_{j=1}^{n} c_j(t)\ \ ^{CPC}_{0}D_t^{\beta_j} h(t) + c_0(t)h(t) + r(t)$$

with initial conditions

$$h^m(0) = 0, m = 0, 1, 2, ...\lceil\beta\rceil - 1$$

The input response n(t) and ${}_{0}^{CPC}D_t^{\beta}h(t)$ can be articulated using the block pulse functions as

$$\begin{aligned} r(t) &\approx R^T\eta_m(t) \\ {}_{0}^{CPC}D_t^{\beta}h(t) &\approx H^T\eta_m(t) \end{aligned}$$

where $R = [r_0, r_1, ...r_m]^T$ is a known but $H = [h_0, h_1, ...h_m]^T$ is an unknown column vector of order $m \times 1$

Similarly, $c_j(t)$ for j = 0,1,2,... n can be as well stated using the block pulse functions as

$$c_j(t) \approx {C_j}^T\eta_m(t)$$

where c_j is a known column vector of order $m \times 1$. Now,

$$\begin{aligned} {}_{0}^{CPC}D_t^{\beta_j}h(t) &= {}_{0}^{CPC}I_t^{\beta-\beta_j}\left[{}_{0}^{CPC}D_t^{\beta}h(t)\right] \\ &= {}_{0}^{CPC}I_t^{\beta-\beta_j}\left[H^T\eta_m(t)\right] \\ &= H^TN_{\beta-\beta_j}\eta_m(t) \end{aligned}$$

Substituting these equations in the modified CPC differential equation, we get

$$\begin{aligned} H^T\eta_m(t) = &\sum_{j=1}^{m}{C_j}^T\eta_m(t)[\eta_m(t)]^T\left[N_{\beta-\beta_j}\right]^TH \\ &+{C_0}^T\eta_m(t)[\eta_m(t)]^T\left[N_{\beta}\right]^TH + R^T\eta_m(t) \end{aligned}$$

Using the properties of orthogonal functions, we get

$$\eta_m(t)[\eta_m(t)]^T = \begin{bmatrix} \eta_1(t) & 0 & . & . & 0 \\ 0 & \eta_2(t) & . & . & 0 \\ 0 & 0 & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \eta_m(t) \end{bmatrix}$$

Set, $\left[N_{\beta-\beta_j}\right]^TH = Z_j = [z_{j1}, z_{j2}....z_{jm}]$. Then,

$$\begin{aligned} &\eta_m(t)[\eta_m(t)]^T\left[N_{\beta-\beta_j}\right]^TH \\ &= \begin{bmatrix} X_{j1} & 0 & . & . & 0 \\ 0 & X_{j2} & . & . & 0 \\ 0 & 0 & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & X_{jm} \end{bmatrix}\eta_m(t) \\ &= diag(X_j)\eta_m(t). \end{aligned}$$

Thus our modified CPC differential equation becomes

$$\begin{aligned} H^T\eta_m(t) = &\sum_{j=1}^{m}{C_j}^Tdiag(X_j)\eta_m(t) \\ &+{C_0}^T\left(diag\left[N_{\beta}\right]^TH\right) + R^T\eta_m(t) \end{aligned}$$

Hence we get

$$H^T = \sum_{j=1}^{m}{C_j}^Tdiag(X_j) + {C_0}^T\left(diag\left[N_{\beta}\right]^TH\right) + R^T$$

Taking transpose on both sides, we get

$$H = \sum_{j=1}^{m}C_jdiag(X_j) + \left(diag\left[N_{\beta}\right]^TH\right)C_0 + R$$

which stands for an algebraically solvable system of equations. Solving these equations, we will get the solution

$$h(t) = H^TN_{\beta}\eta_m(t)$$

IV. Solution of non-linear CPC differential equations using operational matrix method

Consider the non-linear CPC differential equation

$${}_{0}^{CPC}D_t^{\gamma}y(t) = \sum_{j=1}^{n}d_j\ {}_{0}^{CPC}D_t^{\gamma_j}y(t) + c_0[y(t)]^m + g(t)$$

subject to the initial conditions

$$y^k(0) = c_k, k = 0, 1, 2, ...\lceil\gamma\rceil - 1$$

where $\gamma > \gamma_1 > \gamma_1 > ....\gamma_n$, ${}_{0}^{CPC}D_t^{\gamma}$ denotes the CPC fractional derivative , d_j is a constant for j=0,1,2,...n. The computation of $[y(t)]^i$ is different from that of the linear case when we use this method.

Set

$${}_{0}^{CPC}D_t^{\gamma}y(t) \approx N^T\eta_m(t).$$

So $y(t) \approx N^TF_{\gamma}\eta_m(t)$.

Now,

$$\begin{aligned} N^TF_{\gamma} &= [X_1, X_2, ...X_m] \\ [y(t)]^i &= \left[{X_1}^i, {X_2}^i, ..., {X_m}^i\right]\eta_m(t) \end{aligned}$$

Substituting these into the above non-linear CPC differential equation and proceeding as in the linear case, we get the following.

$$\begin{aligned} N^T\eta_m(t) = &\sum_{j=1}^{n}d_jN^TF_{\gamma-\gamma_j}\eta_m(t) \\ &+c_0\left[{X_1}^i, {X_2}^i, ...{X_m}^i\right]\eta_m(t) + G^T\eta_m(t) \end{aligned}$$

V. CPC Derivatives of trigonometric functions

In this section, we study and evaluate the CPC derivatives of several functions like as trigonometric, exponential, and hyperbolic functions. In the derivation of CPC derivatives of the functions, we take $K_1(\frac{1}{2}) = K_0(\frac{1}{2}) = 1$.

A. CPC Derivative of sine function

$$\begin{aligned} {}_{0}^{CPC}D_t^{\frac{1}{2}}sint &= \frac{K_1(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t(t-\tau)^{-\frac{1}{2}}sin\tau\ d\tau \\ &+\frac{K_0(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t(t-\tau)^{-\frac{1}{2}}cos\tau\ d\tau \end{aligned}$$

Evaluating the first integral on the right hand side,

$$\begin{aligned} &\frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-\tau)^{-\frac{1}{2}}sin\tau\ d\tau \\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-\tau)^{-\frac{1}{2}}\tau\ d\tau - \frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-\tau)^{-\frac{1}{2}}\frac{\tau^3}{3!}\ d\tau \\ &+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t(t-\tau)^{-\frac{1}{2}}\frac{\tau^5}{5!}\tau\ d\tau - \cdots \end{aligned}$$

$$= \frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2}+1)}t^{1+\frac{1}{2}} - \frac{1}{3!}\frac{\Gamma(3+1)}{\Gamma(3+\frac{1}{2}+1)}t^{3+\frac{1}{2}}$$
$$+\frac{1}{5!}\frac{\Gamma(5+1)}{\Gamma(5+\frac{1}{2}+1)}t^{5+\frac{1}{2}} - \cdots$$
$$= -\sum_{k=1}^{\infty}\frac{(-t^2)^k}{\Gamma(2k+\frac{1}{2})}t^{-\frac{1}{2}}$$
$$= -t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(-t^2) - \frac{1}{\sqrt{\pi}}\right\}.$$

Evaluating the second term,

$$\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}cos\tau d\tau$$
$$= \frac{\Gamma(0+1)}{\Gamma(0+\frac{1}{2}+1)}t^{0+\frac{1}{2}} - \frac{1}{2!}\frac{\Gamma(2+1)}{\Gamma(2+\frac{1}{2}+1)}t^{2+\frac{1}{2}}$$
$$+\frac{1}{4!}\frac{\Gamma(4+1)}{\Gamma(4+\frac{1}{2}+1)}t^{4+\frac{1}{2}} - \cdots$$
$$= -\sum_{k=1}^{\infty}\frac{(-t^2)^k}{\Gamma(2k-\frac{1}{2})}t^{-\frac{3}{2}}$$
$$= -t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(-t^2) - \Gamma(-\frac{1}{2})\right\}.$$

Thus we get

$${}_0^{CPC}D_t^{\frac{1}{2}} sint$$
$$= -t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(-t^2) - \tfrac{1}{\sqrt{\pi}}\right\}$$
$$-t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(-t^2) - \Gamma(-\frac{1}{2})\right\}.$$

B. CPC Derivative of cosine function

Evaluating the first integral term,

$$\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}cos\tau \, d\tau$$
$$= \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} \, d\tau$$
$$-\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^2}{2!} \, d\tau$$
$$+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^4}{4!}\tau \, d\tau - ..$$
$$= \frac{\Gamma(1)}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}} - \frac{1}{2!}\frac{\Gamma(2+1)}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}$$
$$+\frac{1}{4!}\frac{\Gamma(4+1)}{\Gamma(\frac{11}{2})}t^{\frac{7}{2}} - \cdots$$
$$= -t^{-\frac{3}{2}}\sum_{k=1}^{\infty}\frac{(-t^2)^k}{\Gamma(2k-\frac{1}{2}}$$
$$-t^{-\frac{-3}{2}}\left\{E_{2,-\frac{1}{2}}(-t^2) - \Gamma(-\frac{1}{2})\right\}.$$

Now evaluating the second term,

$$\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}sin\tau d\tau$$
$$= \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\tau d\tau - \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^3}{3!}d\tau$$
$$+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^5}{5!}d\tau - \cdots$$
$$= \frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2}+1)}t^{1+\frac{1}{2}} - \frac{1}{3!}\frac{\Gamma(3+1)}{\Gamma(3+\frac{1}{2}+1)}t^{3+\frac{1}{2}}$$
$$+\frac{1}{5!}\frac{\Gamma(5+1)}{\Gamma(5+\frac{1}{2}+1)}t^{5+\frac{1}{2}} - \cdots$$
$$= -t^{-\frac{1}{2}}\sum_{k=1}^{\infty}\frac{(-t^2)^k}{\Gamma(2k+\frac{1}{2})}$$
$$= -t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(-t^2) - \frac{1}{\sqrt{\pi}})\right\}.$$

Thus we get

$${}_0^{CPC}D_t^{\frac{1}{2}} cost = -t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(-t^2) - \Gamma(-\tfrac{1}{2})\right\}$$
$$-t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(-t^2) - \tfrac{1}{\sqrt{\pi}}\right\}.$$

VI. CPC Derivative of the exponential function

$${}_0^{CPC}D_t^{\frac{1}{2}} e^t = \frac{K_1(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} \, e^\tau d\tau$$
$$+\frac{K_0(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} \, e^\tau d\tau$$

On evaluating the first integral term on the right hand side,

$$\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}e^\tau d\tau$$
$$= \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} \, d\tau + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau}{1!} \, d\tau$$
$$+\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^2}{2!}d\tau + \cdots$$
$$= \frac{\Gamma(0+1)}{\Gamma(0+\frac{1}{2}+1)}t^{0+\frac{1}{2}} + \frac{1}{1!}\frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2}+1)}t^{1+\frac{1}{2}}$$
$$+\frac{1}{2!}\frac{\Gamma(2+1)}{\Gamma(+2+\frac{1}{2}+1)}t^{5+\frac{1}{2}} + \cdots$$
$$= \sum_{k=1}^{\infty}\frac{t^{k-\frac{1}{2}}}{\Gamma(k+\frac{1}{2})}$$
$$= t^{-\frac{1}{2}}\left\{E_{1,\frac{1}{2}}(t) - \frac{1}{\sqrt{\pi}}\right\}.$$

The evaluation of the second integral gives the same expression as above. Hence we can conclude that

$${}_0^{CPC}D_t^{\frac{1}{2}} e^t = 2t^{-\frac{1}{2}}\left\{E_{1,\frac{1}{2}}(t) - \tfrac{1}{\sqrt{\pi}}\right\}.$$

VII. CPC Derivative of the hyperbolic functions

A. CPC Derivative of the hyperbolic sine function

$$ {}_0^{CPC}D_t^{\frac{1}{2}} sinht = \frac{K_1(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} sinh\tau\ d\tau + \frac{K_0(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} cosh\tau\ d\tau $$

$$ \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} sinh\tau\ d\tau = \frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2}+1)}t^{1+\frac{1}{2}} + \frac{1}{3!}\frac{\Gamma(3+1)}{\Gamma(3+\frac{1}{2}+1)}t^{3+\frac{1}{2}} + \frac{1}{5!}\frac{\Gamma(5+1)}{\Gamma(5+\frac{1}{2}+1)}t^{5+\frac{1}{2}} + \cdots = \sum_{k=1}^{\infty}\frac{(t^2)^k}{\Gamma(2k+\frac{1}{2})}t^{-\frac{1}{2}} = t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(t^2) - \frac{1}{\sqrt{\pi}}\right\}. $$

Evaluating the second term,

$$ \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} cosh\tau d\tau = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} d\tau + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^2}{2!} d\tau + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^4}{4!} d\tau + \cdots $$
$$ = \frac{\Gamma(0+1)}{\Gamma(0+\frac{1}{2}+1)}t^{0+\frac{1}{2}} + \frac{1}{2!}\frac{\Gamma(2+1)}{\Gamma(2+\frac{1}{2}+1)}t^{2+\frac{1}{2}} + \frac{1}{4!}\frac{\Gamma(4+1)}{\Gamma(4+\frac{1}{2}+1)}t^{4+\frac{1}{2}} + \cdots = t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(t^2) - \Gamma(-\frac{1}{2})\right\}. $$

Thus we have got

$$ {}_0^{CPC}D_t^{\frac{1}{2}} sinht = t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(t^2) - \frac{1}{\sqrt{\pi}}\right\} + t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(t^2) - \Gamma(-\frac{1}{2})\right\}. $$

B. CPC Derivative of the hyperbolic cosine function

$$ 0^{CPC}D_t^{\frac{1}{2}} cosht = \frac{K_1(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} cosh\tau\ d\tau + \frac{K_0(\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} sinh\tau\ d\tau $$

Evaluating the first integral term,

$$ \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} cosh\tau\ d\tau = \frac{1}{\Gamma(\frac{1}{2})}\left\{\int_0^t (t-\tau)^{-\frac{1}{2}}\ d\tau + \int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^2}{2!}\ d\tau\right\} + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^4}{4!}\tau\ d\tau + \cdots $$
$$ = \frac{\Gamma(0+1)}{\Gamma(0+\frac{1}{2}+1)}t^{0+\frac{1}{2}} + \frac{1}{2!}\frac{\Gamma(2+1)}{\Gamma(2+\frac{1}{2}+1)}t^{2+\frac{1}{2}} + \frac{1}{4!}\frac{\Gamma(4+1)}{\Gamma(4+\frac{1}{2}+1)}t^{5+\frac{1}{2}} + \cdots = t^{-\frac{3}{2}}\sum_{k=1}^{\infty}\frac{(t^2)^k}{\Gamma(2k-\frac{1}{2}} = t^{\frac{-3}{2}}\left\{E_{2,-\frac{1}{2}}(t^2) - \Gamma(-\frac{1}{2})\right\}. $$

Now evaluating the second term,

$$ \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}} sinh\tau d\tau = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\tau d\tau + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^3}{3!} d\tau + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\tau)^{-\frac{1}{2}}\frac{\tau^5}{5!} d\tau + \cdots $$
$$ = \frac{\Gamma(1+1)}{\Gamma(1+\frac{1}{2}+1)}t^{1+\frac{1}{2}} + \frac{1}{3!}\frac{\Gamma(3+1)}{\Gamma(3+\frac{1}{2}+1)}t^{3+\frac{1}{2}} + \frac{1}{5!}\frac{\Gamma(5+1)}{\Gamma(5+\frac{1}{2}+1)}t^{5+\frac{1}{2}} + \cdots = t^{-\frac{1}{2}}\sum_{k=1}^{\infty}\frac{(t^2)^k}{\Gamma(2k+\frac{1}{2})} = t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(t^2) - \frac{1}{\sqrt{\pi}})\right\}. $$

Thus we have got

$$ {}_0^{CPC}D_t^{\frac{1}{2}} cosht = t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(t^2) - \Gamma(-\frac{1}{2})\right\} + t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(t^2) - \frac{1}{\sqrt{\pi}}\right\}. $$

TABLE I
Table of the CPC derivatives of order $\frac{1}{2}$ for some special functions

f(t)	${}_0^{CPC}D_t^{\frac{1}{2}} f(t)$
sint	$-t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(-t^2) - \frac{1}{\sqrt{\pi}}\right\} - t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(-t^2) - \Gamma(-\frac{1}{2})\right\}.$
cos t	$-t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(-t^2) - \Gamma(-\frac{1}{2})\right\} - t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(-t^2) - \frac{1}{\sqrt{\pi}}\right\}.$
e^t	$2t^{-\frac{1}{2}}\left\{E_{1,\frac{1}{2}}(t) - \frac{1}{\sqrt{\pi}}\right\}.$
sinh t	$t^{-\frac{1}{2}}\left\{E_{2,\frac{1}{2}}(t^2) - \frac{1}{\sqrt{\pi}}\right\} + t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(t^2) - \Gamma(-\frac{1}{2})\right\}.$
cosh t	$t^{-\frac{3}{2}}\left\{E_{2,-\frac{1}{2}}(t^2) - \Gamma(-\frac{1}{2})\right\}$

VIII. Shehu transform of CPC Derivatives

The Shehu transform of a function h(t) is defined as

$$ H\left[h(t)\right] = V(s,u) = \int_0^{\infty} e^{\frac{-st}{u}} h(t) dt $$

whereas the Sumudu transform of CPC derivative [13] is given by

$$S\left[{}_{0}^{CPC}D_t^{\beta}f(t)\right] = K_1(\beta)G(u)s^{1-\beta} + K_0(\beta)(G(u)-f(0))s^{-\beta}$$

where G(u) is the Sumudu transform of f(t). Now the relationship between the Sumudu and Shehu transform is given by

$$V(s,r) = \frac{u}{s}G(\frac{u}{s})$$

Hence, Shehu transform of ${}_{0}^{CPC}D_t^{\beta}f(t)$ is given by

$$H\left[{}_{0}^{CPC}D_t^{\beta}f(t)\right] = \frac{u}{s}S\left[{}_{0}^{CPC}D_t^{\beta}f(t)\right]$$
$$= \frac{u}{s}\left[K_1(\beta)G(\frac{u}{s})(\frac{u}{s})^{1-\beta} + K_0(\beta)\left[G(\frac{u}{s})-h(0)\right](\frac{u}{s})^{-\beta}\right]$$
$$= K_1(\beta)(\frac{u}{s})^{1-\beta}V(s,u) + K_0(\beta)(\frac{u}{s})^{-\beta}V(s,u) - K_0(\beta)h(0)(\frac{u}{s})^{1-\beta}$$
$$= \left[K_1(\beta)\frac{u}{s} + K_0(\beta)\right](\frac{u}{s})^{-\beta}V(s,u) - K_0(\beta)h(0)(\frac{u}{s})^{1-\beta}$$

Thus, the Shehu transform of CPC derivative is given by

$$H\left[{}_{0}^{CPC}D_t^{\beta}f(t)\right] = \left[K_1(\beta)\frac{u}{s} + K_0(\beta)\right](\frac{u}{s})^{\beta}V(s,u) - K_0(\beta)h(0)(\frac{u}{s})^{1-\beta}$$

Here, we find out the Shehu transform of the function $t^{-\beta}$ which is required for solving the initial value problems.

$$H[f(t)] = V(s,u) = \int_0^{\infty} e^{\frac{-st}{u}}f(t)dt = \int_0^{\infty} e^{\frac{-st}{u}}t^{-\beta}dt$$

Now substituting $\frac{st}{u} = v$, we get

$$V(s,u) = \int_0^{\infty} e^{-v}(\frac{u}{s})^{1-\beta}v^{-\beta}dv = (\frac{u}{s})^{1-\beta}\int_0^{\infty} e^{-v}v^{-\beta}dv = (\frac{u}{s})^{1-\beta}\Gamma(1-\beta).$$

IX. Solution of some CPC differential equations using Shehu transforms

Example 1 Consider the problem that follows:

$${}_{0}^{CPC}D_t^{\eta}z(t) = j(t,z(t)), t>0, z(0)=c,\ c\in R$$

Let Z(s,m) and J(s,m) be the Shehu transforms of z(t) and j(t, y(t)) respectively. Taking Shehu transforms on both sides of the differential equation, we get

$$\left[K_1(\eta)\frac{m}{s} + K_0(\eta)\right](\frac{m}{s})^{-\eta}Z(s,m) - K_0(\eta)z(0)(\frac{m}{s})^{1-\eta} = J(s,m)$$
$$\left[K_1(\eta)\frac{m}{s} + K_0(\eta)\right](\frac{m}{s})^{-\eta}Z(s,m) = K_0(\eta)c(\frac{m}{s})^{1-\eta} + J(s,m)$$
$$Z(s,m) = \frac{J(s,m)}{\left[K_1(\eta)\frac{m}{s} + K_0(\eta)\right](\frac{m}{s})^{-\eta}} + \frac{K_0(\eta)c(\frac{m}{s})}{\left[K_1(\eta)\frac{m}{s} + K_0(\eta)\right]}$$

The solution can be obtained by applying the inverse transform on both sides.

Example 2 Examine the initial value problem that follows.

$${}_{0}^{CPC}D_t^{\lambda}l(t) = sint,\ t>0,\ l(0)=0$$

First, we find the Shehu transform of sint. By the definition of Shehu transform,

$$H[f(t)] = \int_0^{\infty} e^{\frac{-st}{w}}f(t)dt$$
$$H[sint] = \int_0^{\infty} e^{\frac{-st}{w}}sintdt = \frac{1}{(\frac{s}{w})^2+1}\left[\frac{-s}{w}sint - cos\frac{st}{w}\right]_0^{\infty}.$$

After applying the limits, we will get the Shehu transform of sint as

$$H[sint] = \frac{1}{(\frac{s}{w})^2+1}.$$

Let us consider the given differential equation. Taking Shehu transforms on both sides, we get

$$\left[K_1(\lambda)\frac{w}{s} + K_0(\lambda)\right](\frac{w}{s})^{-\lambda}L(s,w) - K_0(\lambda)l(0)(\frac{w}{s})^{1-\lambda} = \frac{1}{(\frac{s}{w})^2+1}\left[K_1(\lambda)(\frac{w}{s})^{1-\lambda} + K_0(\lambda)(\frac{w}{s})^{-\lambda}\right]$$

Hence we will get

$$L(s,w) = \frac{1}{(\frac{s}{w})^2+1\left[K_1(\lambda)(\frac{w}{s})^{1-\lambda} + K_0(\lambda)(\frac{w}{s})^{-\lambda}\right]}$$
$$= \left[1+(\frac{s}{w})^2\right]^{-1}\left[K_1(\lambda)(\frac{w}{s})^{1-\lambda} + K_0(\lambda)(\frac{w}{s})^{-\lambda}\right]^{-1}$$
$$= \left[1-(\frac{s}{w})^2+(\frac{s}{w})^4-...\right]K_1(\lambda)(\frac{w}{s})^{-\lambda}\left[\frac{K_0(\lambda)}{K_1(\lambda)}+\frac{w}{s}\right]^{-1}$$
$$= \left[1-(\frac{s}{w})^2+(\frac{s}{w})^4-...\right]K_0(\lambda)(\frac{w}{s})^{-\lambda}\left[1+\frac{K_1(\lambda)}{K_0(\lambda)}\frac{w}{s}\right]^{-1}$$
$$= K_0(\lambda)(\frac{w}{s})^{-\lambda}\left[1-(\frac{s}{w})^2+(\frac{s}{w})^4-...\right]\left[1-\frac{rw}{s}+...\right]$$

In the above equation we have let $\frac{K_1(\lambda)}{K_0(\lambda)} = r$. By using the term by term multiplication, we will get the above equation as

$$L(s,u) = K_0(\lambda)\left[(\frac{w}{s})^{-\lambda} - r(\frac{w}{s})^{1-\lambda} + (\frac{r^2}{s^2})(\frac{w}{s})^{2-\lambda} + \cdots\right] - \left[(\frac{w}{s})^{-2-\lambda} - r(\frac{w}{s})^{-1-\lambda} + r^2(\frac{w}{s})^{-\lambda}\right]$$

Taking the inverse Shehu transform on both sides, we will get

$$l(t) = K_0(\lambda)\left[\frac{t^{-\lambda-1}}{\Gamma(-\lambda)} - r\frac{t^{-\lambda}}{\Gamma(1-\lambda)} + \frac{r^2}{s^2}\frac{t^{1-\lambda}}{\Gamma(2-\lambda)}\cdots\right] - \left[\frac{t^{-3-\lambda}}{\Gamma(-2-\lambda)} - r\frac{t^{-2-\lambda}}{\Gamma(-1-\lambda)} + \cdots\right]$$

Thus the solution is obtained as the sum of powers of t of various orders. In the next section, the combination of homotopy perturbation with Shehu transform is applied to solve a CPC differential equation.

X. Explanation of homotopy perturbation Shehu transform method(HPSTM) with CPC fractional derivative

To clarify the specific steps involved in this approach, let's look at the following expression:

$$ {}_0^{CPC}D_t^{\beta}\psi(\delta,t) + R\psi(\delta,t) + N\psi(\delta,t) = \eta(\delta,t), \quad (7) $$

$0 < \beta \leq 1$ with initial condition

$$ \psi(\delta,0) = g(\delta) \quad (8) $$

where ${}_0^{CPC}D_t^{\beta}$ is the hybrid fractional operator, R,N are the differential operators of linear and nonlinear orders respectively, $\psi(\delta,t)$ is a familiar function and $\eta(\delta,t)$ is an function.

The HPSTM approach entails the following steps

Step 1: Implementing the Shehu transform on both sides of (7), we get

$$ H[{}_0^{CPC}D_t^{\beta}\psi(\delta,t)] + H[R\psi(\delta,t)] + H[N\psi(\delta,t)] $$
$$ = H[\eta(\delta,t)] $$
$$ i.e[K_1(\beta)\frac{u}{s} + K_0(\beta)]\frac{u}{s}^{\beta}V(s,u) - K_0(\beta)]\frac{u}{s}^{1-\beta}\psi(\delta,0) $$
$$ = H[\eta(\delta,t) - R\psi(\delta,t) + N\psi(\delta,t)] $$
$$ [K_1(\beta)\frac{u}{s} + K_0(\beta)]\frac{u}{s}^{\beta}V(s,u) $$
$$ = K_0(\beta)]g(x)\frac{u}{s}^{1-\beta} + H[\eta(\delta,t) - R\psi(\delta,t) + N\psi(\delta,t)] $$

Taking $K_0(\beta)$= $K_1(\beta)$ = 1, we get

$$ (\frac{u}{s}+1)\frac{u}{s}^{\beta}V(s,u) = g(\delta)\frac{u}{s}^{1-\beta} + H[\eta(\delta,t) $$
$$ -R\psi(\delta,t) + N\psi(\delta,t)H[\psi(\delta,t)] $$
$$ = \frac{g(\delta)\frac{u}{s}^{1-\beta}}{(\frac{u}{s}+1)\frac{u}{s}^{\beta}} + \frac{1}{(\frac{u}{s}+1)(\frac{u}{s})^{\beta}} $$
$$ \times H\left\{[\eta(\delta,t) - R\psi(\delta,t) + N\psi(\delta,t)]\right\} $$

Step 2: Taking the inverse Shehu transform on both sides of the above equation,

$$ \psi(\delta,t) = H(\delta,t) - H^{-1}\left\{\frac{1}{(\frac{u}{s}+1)(\frac{u}{s})^{\beta}}H[R\psi(\delta,t) + N\psi(\delta,t)]\right\}. \quad (9) $$

Step 3: To apply the homotopy perturbation Shehu transform method, we express the solution as a power series in terms of the homotopy p ϵ [0,1] as

$$ \psi(\delta,t) = \sum_{n=0}^{\infty} p^n \psi_n(\delta,t) \quad (10) $$

and the nonlinear term is decomposed as

$$ N\psi(\delta,t) = \sum_{n=0}^{\infty} p^n H_n(\psi) \quad (11) $$

where $H_n(\psi)$ are the He's polynomials and their values are computed by the following expression

$$ H_n(\psi_0,\psi_1,....\psi_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\sum_{j=o}^{n} p^j \beta_j\right]_{p=0}, n = 0,1,2.... \quad (12) $$

Now applying equations (10), (11), (12) in (9), we get

$$ \sum_{n=0}^{\infty} p^n \psi_n(\delta,t) = H(\delta,t) - pH^{-1}\frac{1}{(\frac{u}{s}+1)(\frac{u}{s})^{\beta}} $$
$$ H[R\sum_{n=0}^{\infty} p^n \psi_n(\delta,t) + \sum_{n=0}^{\infty} p^n H_n(\psi)]. $$

Comparing the coefficients of the powers of p, we get

$$ p^0 : \psi_0(\delta,t) = H(\delta,t) $$
$$ p^1 : \psi_1(\delta,t) = -H^{-1}\left[\frac{1}{(\frac{u}{s}+1)(\frac{u}{s})^{\beta}}H[R\psi_0(\delta,t) + H_0(\psi)]\right] $$
$$ p^2 : \psi_2(\delta,t) = -H^{-1}\left[\frac{1}{(\frac{u}{s}+1)(\frac{u}{s})^{\beta}}H[R\psi_1(\delta,t) + H_1(\psi)]\right] \quad (13) $$

Proceeding like this, we get the series $\psi(\delta,t) = \sum_{n=0}^{\infty}\psi_n(\delta,t)$ which gives the exact solution of the CPC differential equation.

XI. Conclusion

Fractional differential equations can be solved numerically using a variety of techniques, such as the power series approach, variational iteration method, and homotopy perturbation method. In this paper, we analyze a different numerical approach, called the operational matrix method, which is used to find the solution to a constant proportional Caputo differential equation. This method assists in the approximation of signals by transforming the given equations into integral equations. Additionally, we establish the Shehu transform of the CPC derivative to facilitate the application of this result in the solution of differential equations using the homotopy perturbation Shehu transform method. The solutions derived from this approach can be valuable in the design of models for various diseases.

References

[1] A. Arikoglu, Ozkol. I, "Solution of fractional differential equations by using differential transform method", Chaos, Solitons And Fractals, vol.34, no.5, pp.1473-1481, 2007.

[2] A.A. Kilbas, M. Saigo, R.K. Saxena, "Generalized Mittag-Leffler function and generalized fractional calculus operators", Integral Transforms and Special Functions, vol.15, no.1, pp.31–49, 2004.

[3] A. Bokhari, "Application of Shehu transform to Atangana-Baleanu derivatives", Journal of Mathematics and Computer Science, vol.20, no.2, pp.101-107, 2019.

[4] B. Thakur, S. Gupta, "An Iterative Algorithm for Numerical Solution of Nonlinear Fractional Differential Equation using Legendre Wavelet Method", IAENG International Journal of Applied Mathematics, vol.54, no.3, pp.380-389, 2024.

[5] A.G. Selvam, M. Vianny, D.A. and M.Jacintha, "Stability in a fractional order SIR epidemic model of childhood diseases with discretization", Journal of Physics: Conference Series, IOP Publishing, Vol. 1139, no.1, p.012009, 2018.

[6] A. El-Ajou, Z. Odibat, S. Momani, and A. Alawneh, "Construction of Analytical Solutions to Fractional Differential Equations Using Homotopy Analysis Method", IAENG International Journal of Applied Mathematics, vol.40, no.2 pp.43-51, 2010.

[7] A. Jajarmi , A. Yusuf, D. Baleanu, M. Inc, "A new fractional HRSV model and its optimal control: a non-singular operator approach", Physica A, vol.547, no.6, p.123860, 2020.

[8] A. Saadatmandi , M. Dehghan, "A new operational matrix for solving fractional-order differential equations", Computers and Mathematics with Applications, vol.59, no.3, pp.1326-36, 2010.

[9] A. Vinodkumar, K. Malar, M. Gowrisankar, P. Mohankumar, ”Existence, uniqueness and stability of random impulsive fractional differential equations”, Acta Mathematica Scientia, vol.36, no.2, pp.428-442, 2016.

[10] C.H. Wang, ”On the generalization of block pulse operational matrices for fractional and operational calculus”, Journal of the Franklin Institute, vol.315, no.2, pp.91-102, 1983.

[11] D. Baleanu, A. Fernandez, A. Akgul, ”On a fractional operator combining proportional and classical differintegrals”, Mathematics, vol.8, no.3, p.360, 2020.

[12] E.K. Akgül, ”Solutions of the linear and nonlinear differential equations within the generalized fractional derivatives”, Chaos, vol.29, 023108, 2019.

[13] E.K.Akgül, A.Akgul, ”Sumudu transform for some applications with constant proportional Caputo derivative”, Authorea Preprints .

[14] E.K. Akgül, A. Akgul, D. Baleanu, ”Laplace transform method for economic models with constant proportional Caputo derivative”,Fractal and Fractional, vol.4,no.3,p.30, 2020.

[15] H.G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y.Q. Chen, ”A new collection of real world applications of fractional calculus in science and engineering”, Communications in Nonlinear Science and Numerical Simulation, vol.64, pp. 213–231, 2018.

[16] H.J. Haubold, A.M. Mathai, and R.K. Saxena, ”Mittag-leffler functions and their applications”, Journal of Applied Mathematics and Computing, vol.2011, p.298628, 2011.

[17] J.F. Gómez-Aguilar, A. Atangana, ”New insight in fractional differentiation: power, exponential decay and Mittag-Leffler laws and applications”, The European Physical Journal Plus, vol.132, no.1, pp.1-21, 2017.

[18] K. Chen and W. Zhang, ”Study on a New Fractional Logistic Model via Deformable Fractional Derivative”, IAENG International Journal of Applied Mathematics, vol.54, no.8, pp.1540-1544, 2024.

[19] K. Balachandran, S. Kiruthika, ”Existence of solutions of abstract fractional impulsive semilinear evolution equations”, The Electronic Journal of Qualitative Theory of Differential Equations, no.4, pp 1-12, 2010.

[20] K. Diethelm, N.J. Ford, ”Analysis of fractional differential equations”, Journal of Mathematical Analysis and Applications, vol.265, no.2, pp.229-248, 2002.

[21] K. Rajagopal, N. Hasanzadeh, F. Parastesh, Hamarash II, S.Jafari, I.Hussain, ”A fractional-order model for the novel coronavirus (COVID-19) outbreak”, Nonlinear Dynamics, vol.101, no.1, pp.711-8, 2020.

[22] Li. Yuanlu, N. Sun, ”Numerical solution of fractional differential equations using the generalized block pulse operational matrix”, Computers and Mathematics with Applications, vol.62, no.3, pp.1046-1054, 2011.

[23] M. Caputo, M. Fabrizio, ”A new definition of fractional derivative without singular kernel”, Progress in Fractional Differentiation and Applications, vol.1, no.2, pp.1-13, 2015.

[24] M. Berir, R. Saadeh, M. Abdoon, A. Qazza, and D. Almutairi, ”A fractional study for solving the sir model and chaotic system”, IAENG International Journal of Applied Mathematics, vol.54, no.2, pp.212-217, 2024.

[25] P. Prakash, S. Choudhary, V. Daftardar-Gejji, ”Exact solutions of generalized nonlinear time-fractional reaction–diffusion equations with time delay”, The European Physical Journal Plus, vol.135, no.6, pp.1-24, 2020.

[26] P.Veeresha, D. G. Prakasha, and Devendra Kumar, ”Fractional SIR epidemic model of childhood disease with Mittag-Leffler memory”, Fractional Calculus in Medical and Health Science, pp.229-248, 2020.

[27] R. Karthikeyan, S. Pasam, S. Sudheer, V. Teja, S. Tripathi, ”Fuzzy Fractional Order PID Based Parallel Cascade Control System”, Recent Advances in Intelligent Informatics, Springer, vol.235, pp. 293-302, 2014.

[28] R. Prabha, S.Kiruthika, ”Fractional derivatives of some special functions using ABR and ABC derivatives”, Journal of Physics: Conference Series, IOP Publishing, Vol. 1850, no. 1, p.012040, 2021.

[29] S. Das, ”Analytical solution of a fractional diffusion equation by variational iteration method”, Computers and Mathematics with Applications, vol.57, no.3, pp.483-487, 2009.

[30] S. Maitama, W. Zhao, ”Homotopy perturbation Shehu transform method for solving fractional models arising in applied sciences”, Journal of Applied Mathematics and Computational Mechanics, vol.20, no.1, pp.71-82, 2021.

[31] S. Momani, Z.Odibat, ”Numerical comparison of methods for solving linear differential equations of fractional order”, Chaos, Solitons And Fractals, vol.31, no.5, pp.1248-1255, 2007.

[32] T.A. Yıldız, S.Arshad , D. Baleanu, ”New observations on optimal cancer treatments for a fractional tumor growth model with and without singular kernel”, Chaos, Solitons and Fractals, 117, pp.226-39, 2018.

[33] T.R. Prabhakar, ”A singular integral equation with a generalized Mittag Leffler function in the kernel”, Yokohama Mathematical Journal, vol.19, pp.7-15, 1971 .

[34] Z. Yuan, T. Du, ”The Multipoint-based Hermite–Hadamard Inequalities for Fractional Integrals with Exponential Kernels”, IAENG International Journal of Applied Mathematics, vol.54 no.6,pp.1013-1025, 2024.