Application of Second Order Coupled Lucas Sequence

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***Abstract* - Fibonacci numbers and polynomials have been widely studied due to their importance in mathematics, physics, and business. The Coupled Fibonacci Sequence (CFS) and Multiplicative Coupled Fibonacci Sequence (MCFS) contain useful identities but depend on previous terms for computation. The Lucas Sequence (LS) also displays notable properties in number theory. This study investigates the second-order Coupled Lucas Sequence (CLS), in which two interdependent sequences evolve in tandem. Through mathematical analysis and simulations, we uncover patterns, periodicities, and structural relationships within the sequence. Additionally, the research explores its promising applications in cryptography, optimization, and algorithm design. A deeper understanding of CLS enhances number theory and offers insights into broader mathematical systems. This study contributes to mathematical research by revealing intricate connections between sequences and emphasizing the elegance and utility of coupled sequences across disciplines.**

***Index Terms*- LS, FS, CFS, MCFS, CLS.**

I. INTRODUCTION

Numerous fields, including algebra, combinatorics, approximation theory, geometry, graph theory, and number theory itself, have benefited from it., the Fibonacci numbers and polynomials play a crucial role. Perhaps the most well-known application of the Fibonacci numbers is in the rabbit breeding puzzle, which Leonardo de Pisa first presented in his book "Liber-Abaci" in 1202. Numerous authors have explored their various characteristics and broadened usefulness. The Fibonacci and Lucas numbers are undoubtedly two of the most fascinating mathematical sequences, as illustrated in Koshy's book [1]. A long list of identities can be found in Vajda's book [2] and includes numerous identities. There is a long form of unity matrices and determinants to study Fibonacci numbers. A.K. Awasthi, Vikas Ranga, and Kamal Dutt [14] discuss the extension of Fibonacci sequences using specific multiplicative schemes.

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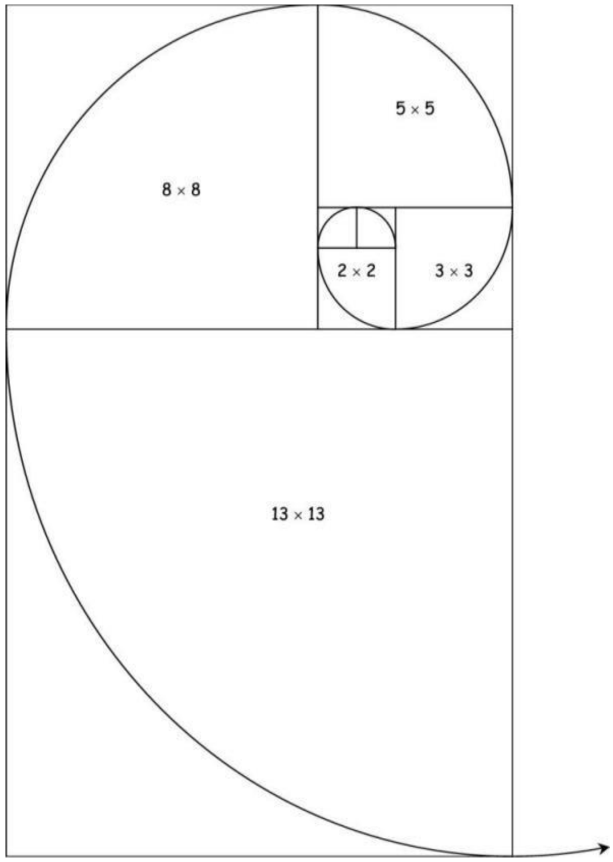

Fig 1. Fibonacci Spiral with golden ratio

Fig 1, represents the Fibonacci Spiral, formed using squares with side lengths following the Fibonacci sequence (2 × 2, 3 × 3, 5 × 5, 8 × 8,…). A quarter-circle arc inside each square creates a spiral-like curve, approximating the Golden Spiral, seen in nature, art, and architecture. It visually demonstrates the connection between the Fibonacci sequence and the Golden Ratio, showcasing proportional and symmetrical growth patterns.

The "Coupled Lucas Sequence of Second Order" emerges as a captivating exploration within the domain of number theory, building upon the foundations laid by the classical LS. This innovative extension introduces a dynamic interplay between two distinct second-order LS, weaving a tapestry of numerical relationships that transcend the conventional boundaries of sequence theory. As a testament to the continuous evolution of mathematical inquiry, this study delves into the intricacies of the coupled sequences, unraveling a myriad of patterns, properties, and applications. By introducing coupling mechanisms between two such sequences, a new and intriguing mathematical entity emerges. This coupled relationship manifests as a simultaneous evolution of two interconnected sequences, influencing each other's progression in a harmonious dance of numerical dynamics.

The origins of the Fibonacci and Lucas numbers as determinants of some tridiagonal matrices were investigated by Cahill and Narayan [3]. K.T. Atanassov [4] and Suman, Amitava, K. Sisodiya introduce respectively the interlinked second order recurrence relation and interlinked Jacobsthal Sequence by constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ naming them as 2F Sequences.

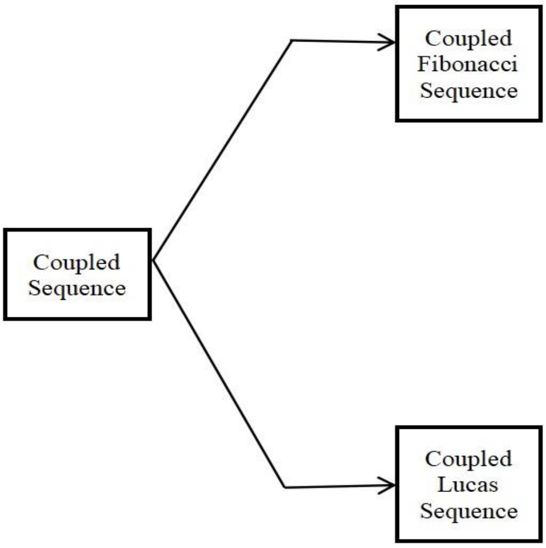

Fig 2. Structure of Coupled Sequence

Fig. 2 illustrates a hierarchical relationship between sequences. At the top level is the "Coupled Sequence", which branches into two distinct types: Coupled Fibonacci Sequence. One branch leads to the "Coupled Fibonacci Sequence," suggesting it is a variant or extension of the traditional Fibonacci sequence, possibly modified by a coupling rule or relationship. Coupled Lucas Sequence has the branch leads to the "Coupled Lucas Sequence," indicating a similar variant or extension of the Lucas sequence, also with some form of coupling rule. This structure shows that the "Coupled Sequence" serves as a foundational concept that can lead to either a coupled version of the Fibonacci or Lucas sequences, depending on the branching path.

According to the scheme

$\alpha_{n+2} = \beta_{n+1} + \beta_n,\ \ n \geq 0$

$\beta_{n+2} = \alpha_{n+1} + \alpha_n,\ \ n \geq 0$

Taking $\alpha_0 = a, \beta_0 = b, \alpha_1 = c, \beta_1 = d$ where a, b, c and d are integers, As seen in [4, 6, 7], he expanded his work in the same approach. Hirschhorn provides explicit solutions in [5, 8] to Atanassov [7] long-standing difficulties on second and third order recurrence relations. Recently, order five coupled recurrence relations of this type were found by Singh, Sikhwal, and Jain [9]. Moreover, Carlitz et al. [10] provided a description for a unique sequence.

II. CLS OF SECOND ORDER

The sequences $\{L_i\}_{i=0}^{\infty}$ and $\{\ell_i\}_{i=0}^{\infty}$ will coincide and the sequence $\{L_i\}_{i=0}^{\infty}$ will turn into a generalized Lucas sequence if we set $a = b$ and $c = d$.

Where,

$L_0(a,c) = a, L_1(a,c) = c$

$L_{n+2}(a,c) = \ell_{n+1}(a,c) + 2\ell_n(a,c),$

$L_n = a, b, d + 2c, b + 2a + 2d$

$\ell_n = c, d, b + 2a, d + 2c + 2b$

Following are the first few terms.

Table I
First few terms of second order CLS

n	L_n	ℓ_n
0	a	c
1	b	d
2	$d + 2c$	$b + 2a$
3	$b + 2a + 2d$	$d + 2c + 2b$
4	$d + 2c + 4b + 4a$	$b + 2a + 4d + 4c$
5	$6d + 8c + 5b + 2a$	$5d + 2c + 6b + 8a$

Taking Lucas sequence

$L_{n+2} = L_{n+1} + 2L_n,\ n \geq 0$

$\ell_{n+2} = \ell_{n+1} + 2\ell_n,\ n \geq 0$

We defined 2-L Sequences as coupled order recurrence relations for Lucas numbers and Lucas sequences.

$L_{n+2} = \ell_{n+1} + 2\ell_n,\ n \geq 0$

$\ell_{n+2} = L_{n+1} + 2L_n,\ n \geq 0$

$L_0 = a, L_1 = b, \ell_0 = c, \ell_1 = d$

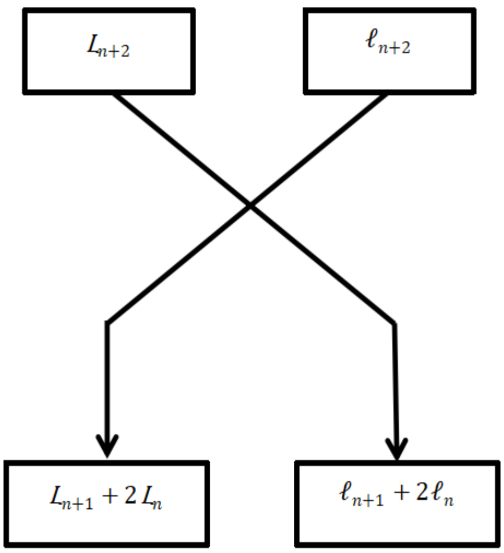

Fig 3. Structure of Scheme of CLS

Fig 3 illustrates the hierarchical structure of the scheme of CLS under addition. 2nd order CLS represents the basic CLS with one scheme, where the terms are derided by adding the last term and twice the second to last term of the sequence.

III. MAIN IDENTITIES

We can derive the following properties from the above terms:

Theorem 1: For every odd number $n \geq 3$.

$$\frac{L_n - L_1}{2} = (L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2})$$

OR

$$L_n - L_1 = 2(L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2})$$

Proof: We will use a mathematical induction method to demonstrate this conclusion.

For $n = 3$,

$$\frac{L_3 - L_1}{2} = \frac{\ell_2 + 2\ell_1 - L_1}{2}$$

$$= \frac{L_1 + 2L_0 + 2\ell_1 - L_1}{2}$$

$$= \frac{2L_0 + 2\ell_1}{2}$$

$$= L_0 + \ell_1$$

or

$$L_3 - L_1 = 2(L_0 + \ell_1)$$

The result is accurate for $n = 2$

therefore we suppose the same for n.

We will now demonstrate that for $n + 2$.

$$\frac{L_{n+2} - L_1}{2} = \frac{\ell_{n+1} + 2\ell_n - L_1}{2}$$

$$= \frac{L_n + 2L_{n-1} + 2\ell_n - L_1}{2}$$

$$= \frac{L_n - L_1}{2} + \frac{2L_{n-1} + 2\ell_n}{2}$$

$$= (L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2}) + (L_{n-1} + \ell_n)$$

$$= (L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots L_{n-1} + \ell_n)$$

Therefore, the statement holds true for the case of $n + 2$, completing the inductive step and proving the result.

Example-1 based on Theorem 1

Let $\{L_n\}_{n=0}^{\infty}$ and $\{\ell_n\}_{n=0}^{\infty}$ be two infinite sequences.

$L_{n+2} = \ell_{n+1} + 2\ell_n,\ n \geq 0$

$\ell_{n+2} = L_{n+1} + 2L_n,\ n \geq 0$

We are given a sequence L_n with initial terms 1 and 3, and a sequence ℓ_n with initial terms 2 and 4.
The Initial terms of the sequence under CLS of second order will provide proving pattern as expressed in further proving.

Table II
Initial terms of the sequence under CLS of second order

n	ℓ_n	L_n
0	2	1
1	4	3
2	5	8
3	14	13
4	29	24
5	50	57
6	105	108
7	222	205
8	421	432
9	842	865
10	1729	1684

Now we will apply the theorem on this example

$$\frac{L_n - L_1}{2} = (L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2})$$

For $n = 3$ in L.H.S.

$$\Rightarrow \quad \frac{L_3 - L_1}{2} = \frac{13 - 3}{2}$$

$$\Rightarrow \quad = 5$$

Now $n = 3$ in R.H.S

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2} = L_0 + \ell_1$$

$$= 1 + 4$$

$$= 5$$

=L.H. S

For $n = 5$ in L.H.S

$$\Rightarrow \quad \frac{L_5 - L_1}{2} = \frac{57 - 3}{2}$$

$$\Rightarrow \quad = 27$$

Now $n = 5$ in R.H.S.

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2}$$
$$= L_0 + \ell_1 + L_2 + \ell_3$$

$$= 1 + 4 + 8 + 14$$

$$= 27$$

=L.H. S

Hence the conclusion is valid.

For $n = 7$ in L.H.S

$$\Rightarrow \quad \frac{L_7 - L_1}{2} = \frac{205 - 3}{2}$$

$$\Rightarrow \quad = 101$$

Now $n = 7$ in R.H.S.

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots L_{n-3} + \ell_{n-2}$$

$$= L_0 + \ell_1 + L_2 + \ell_3 + L_4 + \ell_5$$

$$= 1 + 4 + 8 + 14 + 24 + 50$$

$$= 101$$

=L.H.S

For $n = 9$ in L.H.S

$$\Rightarrow \quad \frac{L_7 - L_1}{2} = \frac{865 - 3}{2}$$

$$\Rightarrow \quad = 431$$

Now $n = 9$ in R.H.S.

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots L_{n-3} + \ell_{n-2}$$

$$= L_0 + \ell_1 + L_2 + \ell_3 + L_4 + \ell_5 + L_6 + \ell_7$$

$$= 1 + 4 + 8 + 14 + 24 + 50 + 108 + 222$$

$$= 431 \text{ =L.H. S}$$

Example-2 based on Theorem 1

Table III
Initial terms of the Generalized CLS of second order

n	ℓ_n	L_n
0	a	b
1	c	d
2	$2b + d$	$2a + c$
3	$2a + c + 2d$	$2b + 2c + d$
4	$4a + 2b + 4c + d$	$2a + 4b + c + 4d$
5	$2a + 8b + 5c + 6d$	$8a + 2b + 6c + 5d$

Now we will apply the theorem on this example

$$\frac{L_n - L_1}{2} = (L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots L_{n-3} + \ell_{n-2})$$

For $n = 3$ in L.H.S.

$$\Rightarrow \quad \frac{L_3 - L_1}{2} = \frac{2b + 2c + d - d}{2}$$

$$\Rightarrow \quad = b + c$$

Now $n = 3$ in R.H.S

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots L_{n-3} + \ell_{n-2} = L_0 + \ell_1$$

$$\Rightarrow \quad = b + c$$

=L.H.S

For $n = 5$ in L.H.S

$$\Rightarrow \quad \frac{L_5 - L_1}{2} = \frac{8a + 2b + 6c + 5d - d}{2}$$

$$\Rightarrow \quad = 4a + b + 3c + 2d$$

Now $n = 5$ in R.H.S.

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots L_{n-3} + \ell_{n-2}$$

$$= L_0 + \ell_1 + L_2 + \ell_3$$

$$= b + c + 2a + c + 2a + c + 2d$$

$$\Rightarrow \quad = 4a + b + 3c + 2d$$

=L.H. S

Hence, the conclusion is valid on both the examples for every odd number $n \geq 3$

Theorem 2: For every even number $n \geq 2$.

$$\frac{L_n - \ell_1}{2} = (\ell_0 + L_1 + \ell_2 + L_3 + \cdots \ldots \ldots \ldots L_{n-3} + \ell_{n-2})$$

OR

$$L_n - \ell_1 = 2(\ell_0 + L_1 + \ell_2 + L_3 + \cdots \ldots \ldots \ldots L_{n-3} + \ell_{n-2})$$

Proof: We will use a mathematical induction method to demonstrate this conclusion.

For $n = 2$,

$$\frac{L_2 - \ell_1}{2} = \frac{\ell_1 + 2\ell_0 - \ell_1}{2}$$

$$= \ell_0$$

The result is accurate for $n = 2$, therefore we suppose the same for n.

We will now demonstrate that for $n = 2$,

$$\frac{L_{n+2} - \ell_1}{2} = \frac{\ell_{n+1} + 2\ell_n - \ell_1}{2}$$

$$= \frac{L_n + 2L_{n-1} + 2\ell_n - \ell_1}{2}$$

$$= \frac{L_n - \ell_1}{2} + \frac{2L_{n-1} + 2\ell_n}{2}$$

$$= (\ell_0 + L_1 + \ell_2 + L_3 + \cdots \ldots \ldots \ldots + L_{n-3} + \ell_{n-2}) + (L_{n-1} + \ell_n)$$

$$= (\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots L_{n-1} + \ell_n)$$

Thus, the outcome is accurate for $n + 2$.

Example based on Theorem 2

Let $\{L_n\}_{n=0}^{\infty}$ and $\{\ell_n\}_{n=0}^{\infty}$ be two sequences.

$L_{n+2} = \ell_{n+1} + 2\ell_n,\ n \geq 0$

$\ell_{n+2} = L_{n+1} + 2L_n,\ n \geq 0$

We are given a sequence L_n with initial terms 2 and 4, and a sequence ℓ_n with initial terms 1 and 3.

Table IV
Second-order CLS's initial few terms

n	ℓ_n	L_n
0	1	2
1	3	4
2	8	5
3	13	14
4	24	29
5	57	50
6	108	105
7	205	222
8	432	421
9	865	842
10	1684	1729

Now we will apply the theorem on this example

$$\frac{L_n - \ell_1}{2} = (\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2})$$

For $n = 4$ in L.H.S.

$$\Rightarrow \quad \frac{L_4 - \ell_1}{2} = \frac{29 - 3}{2}$$

$$\Rightarrow \quad = 13$$

Now $n = 4$ in R.H.S

$$\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2} = \ell_0 + L_1 + \ell_2$$

$$= 1 + 4 + 8$$

$$= 13$$

=L.H. S

For $n = 6$ in L.H.S.

We have the result as per requirement.

$$\Rightarrow \quad \frac{L_6 - \ell_1}{2} = \frac{105 - 3}{2}$$

$$\Rightarrow \quad = 51$$

Now $n = 6$ in R.H.S

$$\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2}$$

$$= \ell_0 + L_1 + \ell_2 + L_3 + \ell_4$$

$$= 1 + 4 + 8 + 14 + 24$$

$$= 51$$

=L.H.S

For $n = 8$ in L.H.S

$$\Rightarrow \quad \frac{L_8 - \ell_1}{2} = \frac{421 - 3}{2}$$

$$\Rightarrow \quad = 209$$

Now $n = 8$ in R.H.S

$$\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2}$$

$$= \ell_0 + L_1 + \ell_2 + L_3 + \ell_4 + L_5 + \ell_6$$

$$= 1 + 4 + 8 + 14 + 24 + 50 + 108$$

$$= 209$$

=L.H.S

For $n = 10$ in L.H.S

$$\Rightarrow \quad \frac{L_{10} - \ell_1}{2} = \frac{1729 - 3}{2}$$

$$\Rightarrow \quad = 863$$

Now $n = 10$ in R.H.S

$$\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots L_{n-3} + \ell_{n-2}$$

$$= \ell_0 + L_1 + \ell_2 + L_3 + \ell_4 + L_5 + \ell_{6+} + L_7 + \ell_8$$

$$= 1 + 4 + 8 + 14 + 24 + 50 + 108 + 222 + 432$$

$$= 863$$

=L.H. S

Hence the conclusion is valid.

Theorem-2 can also be proved using the values of Table-III.

Theorem 3: For every odd number $n \geq 3$.

$$\frac{\ell_n - \ell_1}{2} = (\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2})$$

OR

$$\ell_n - \ell_1 = 2(\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2})$$

Example based on Theorem 3

Let $\{L_n\}_{n=0}^{\infty}$ and $\{\ell_n\}_{n=0}^{\infty}$ be two sequences.

$L_{n+2} = \ell_{n+1} + 2\ell_n,\ n \geq 0$

$\ell_{n+2} = L_{n+1} + 2L_n,\ n \geq 0$

We are given a sequence L_n with initial terms 2 and 3, and a sequence ℓ_n with initial terms 1 and 2.

Table V
Initial terms of the second-order CLS

n	ℓ_n	L_n
0	1	2
1	2	3
2	7	4
3	10	11
4	19	24
5	46	39
6	87	84
7	162	179
8	347	336
9	694	671
10	1343	1388

Now we will apply the theorem on this example

$$\frac{\ell_n - \ell_1}{2} = (\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2})$$

For $n = 5$ in R.H.S

$$\Rightarrow \quad \frac{\ell_5 - \ell_1}{2} = \frac{46 - 2}{2}$$

$= 22$

=L.H.S

Now $n = 5$ in L.H.S

$\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2}$

$= \ell_0 + L_1 + \ell_2 + L_3$

$= 1 + 3 + 7 + 11$

$= 22$

=R.H. S

For $n = 7$ in R.H.S

We have the result as per requirement.

$$\Rightarrow \quad \frac{\ell_7 - \ell_1}{2} = \frac{162 - 2}{2}$$

$= 80$

=L.H. S

Now $n = 7$ in L.H.S

$\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2}$

$= \ell_0 + L_1 + \ell_2 + L_3 + \ell_4 + L_5$

$= 1 + 3 + 7 + 11 + 19 + 39$

$= 80$

=R.H. S

For $n = 9$ in R.H.S

$$\Rightarrow \quad \frac{\ell_9 - \ell_1}{2} = \frac{694 - 2}{2}$$

$\Rightarrow \quad = 346$

=L.H.S

Now $n = 9$ in L.H.S

$\ell_0 + L_1 + \ell_2 + L_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2}$

$= \ell_0 + L_1 + \ell_2 + L_3 + \ell_4 + L_5 + \ell_6 + L_7$

$= 1 + 3 + 7 + 11 + 19 + 39 + 87 + 179$

$= 346$

=R.H. S

Hence the conclusion is valid.

Theorem-3 can also be proved using the values of Table-III

Theorem 4: For every even number $n \geq 2$.

$$\frac{\ell_n - L_1}{2} = (L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2})$$

Or

$$\ell_n - L_1 = 2(L_0 + \ell_1 + L_2 + \ell_3 + \cdots \dots \dots \dots \ell_{n-3} + L_{n-2})$$

Example based on Theorem 4

Let $\{L_n\}_{n=0}^{\infty}$ and $\{\ell_n\}_{n=0}^{\infty}$ be two sequences.

$L_{n+2} = \ell_{n+1} + 2\ell_n,\ n \geq 0$

$\ell_{n+2} = L_{n+1} + 2L_n,\ n \geq 0$

We are given a sequence L_n with initial terms 2 and 2, and a sequence ℓ_n with initial terms 1 and 1.
First few terms of second order CLS will provide proving pattern as expressed in further proving.
The table number VI also provide required results of ℓ_n and L_n.

Table VI
First few terms of second order CLS

n	ℓ_n	L_n
0	1	2
1	1	2
2	6	3
3	7	8
4	14	19
5	35	28
6	66	63
7	119	136
8	262	251
9	523	500
10	1002	1047

Now we will apply the theorem on this example

$$\frac{\ell_n - L_1}{2} = (L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots \ell_{n-3} + L_{n-2})$$

For n =6 in L.H.S

$$\Rightarrow \qquad \frac{\ell_6 - L_1}{2} = \frac{66-2}{2}$$

$$\Rightarrow \qquad = 32$$

=R.H.S

Now $n = 6$ in R.H.S

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots \ell_{n-3} + L_{n-2}$$
$$= L_0 + \ell_1 + L_2 + \ell_3 + L_4$$
$$= 2 + 1 + 3 + 7 + 19$$

=32

=L.H.S

For n =8 in L.H.S

$$\Rightarrow \qquad \frac{\ell_8 - L_1}{2} = \frac{262-2}{2}$$

$$\Rightarrow \qquad = 130$$

=R.H.S

Now $n = 8$ in R.H.S

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots \ell_{n-3} + L_{n-2}$$
$$= L_0 + \ell_1 + L_2 + \ell_3 + L_4 + + \ell_5 + L_6$$
$$= 2 + 1 + 3 + 7 + 19 + 35 + 63$$

=130

=L.H. S

For n =10 in L.H.S

$$\Rightarrow \qquad \frac{\ell_{10} - L_1}{2} = \frac{1002-2}{2}$$

$$\Rightarrow \qquad = 500$$

=R.H.S

Now $n = 10$ in R.H.S

$$L_0 + \ell_1 + L_2 + \ell_3 + \cdots \ldots \ldots \ldots \ell_{n-3} + L_{n-2}$$
$$= L_0 + \ell_1 + L_2 + \ell_3 + L_4 + + \ell_5 + L_6 + \ell_7 + L_8$$
$$= 2 + 1 + 3 + 7 + 19 + 35 + 63 + 119 + 251$$

=500

=L.H. S

Hence the conclusion is valid.

Theorem-4 can also be proved using the values of Table-III

Theorem 5: For every positive integer n.

$$\frac{L_{n+2}L_{n+1} - \ell_{n+2}\ell_{n+1}}{\ell_{n+2}\ell_n - L_{n+2}L_n} = 2$$

or

$$L_{n+2}L_{n+1} - \ell_{n+2}\ell_{n+1} = 2(\ell_{n+2}\ell_n - L_{n+2}L_n)$$

Proof: We will prove this result by method of mathematical induction

For $n = 1$,

$$\frac{L_3L_2 - \ell_3\ell_2}{\ell_3\ell_1 - L_3L_1} = \frac{(\ell_2 + 2\ell_1)L_2 - (L_2 + 2L_1)\ell_2}{(L_2 + 2L_1)\ell_1 - (\ell_2 + 2\ell_1)L_1}$$

$$= \frac{\ell_2L_2 + 2\ell_1L_2 - L_2\ell_2 - 2L_1\ell_2}{L_2\ell_1 + 2L_1\ell_1 - \ell_2L_1 - 2\ell_1L_1}$$

$$= \frac{2\ell_1L_2 - 2L_1\ell_2}{L_2\ell_1 - \ell_2L_1}$$

$$= 2\left[\frac{\ell_1L_2 - L_1\ell_2}{L_2\ell_1 - \ell_2L_1}\right]$$

$$= 2$$

The result is accurate for $n = 1$,

Therefore we suppose the same for n .We will now demonstrate that for $n + 1$,

$$\frac{L_{n+3}L_{n+2} - \ell_{n+3}\ell_{n+2}}{\ell_{n+3}\ell_{n+1} - L_{n+3}L_{n+1}}$$

$$\frac{L_{n+3}L_{n+2}-\ell_{n+3}\ell_{n+2}}{\ell_{n+3}\ell_{n+1}-L_{n+3}L_{n+1}}$$
$$=\frac{(\ell_{n+2}+2\ell_{n+1})L_{n+2}-(L_{n+2}+2L_{n+1})\ell_{n+2}}{(L_{n+2}+2L_{n+1})\ell_{n+1}-(\ell_{n+2}+2\ell_{n+1})L_{n+1}}$$
$$=\frac{\ell_{n+2}L_{n+2}+2\ell_{n+1}L_{n+2}-L_{n+2}\ell_{n+2}-2L_{n+1}\ell_{n+2}}{L_{n+2}\ell_{n+1}+2L_{n+1}\ell_{n+1}-\ell_{n+2}L_{n+1}-2\ell_{n+1}L_{n+1}}$$
$$=\frac{2\ell_{n+1}L_{n+2}-2L_{n+1}\ell_{n+2}}{L_{n+2}\ell_{n+1}-\ell_{n+2}L_{n+1}}$$
$$=2\left[\frac{\ell_{n+1}L_{n+2}-L_{n+1}\ell_{n+2}}{L_{n+2}\ell_{n+1}-\ell_{n+2}L_{n+1}}\right]$$
$$=2$$

Hence the result is true for $n+1$.

Example-1 based on Theorem 5

Let $\{L_n\}_{n=0}^{\infty}$ and $\{\ell_n\}_{n=0}^{\infty}$ be two sequences.

$L_{n+2}=\ell_{n+1}+2\ell_n,\ n\geq 0$

$\ell_{n+2}=L_{n+1}+2L_n,\ n\geq 0$

We are given a sequence L_n with initial terms 1 and 1, and a sequence ℓ_n with initial terms 2 and 2.

Table VII
Introductory terms of the second-order CLS

n	ℓ_n	L_n
0	2	1
1	2	1
2	3	6
3	8	7
4	19	14
5	28	35
6	63	66
7	136	119
8	251	262
9	500	523
10	1047	1002

Now we will apply the theorem on this example

$$\frac{L_{n+2}L_{n+1}-\ell_{n+2}\ell_{n+1}}{\ell_{n+2}\ell_n-L_{n+2}L_n}=2$$

Put $n=1$,

$$\frac{L_3L_2-\ell_3\ell_2}{\ell_3\ell_1-L_3L_1}=\frac{(11\times 4)-(10\times 7)}{(10\times 2)-(11\times 3)}$$
$$=\frac{44-70}{20-33}$$
$$=2$$

Put $n=2$,

$$\frac{L_4L_3-\ell_4\ell_3}{\ell_4\ell_2-L_4L_2}=\frac{(24\times 11)-(19\times 10)}{(19\times 7)-(24\times 4)}$$
$$=\frac{(264)-(190)}{(133)-(96)}$$
$$=2$$

Put $n=3$,

$$\frac{L_5L_4-\ell_5\ell_4}{\ell_5\ell_3-L_5L_3}=\frac{(39\times 24)-(46\times 19)}{(46\times 10)-(39\times 11)}$$
$$=\frac{(936)-(874)}{(460)-(429)}$$
$$=2$$

Put $n=5$,

$$\frac{L_7L_6-\ell_7\ell_6}{\ell_7\ell_5-L_7L_5}=\frac{(119\times 66)-(136\times 63)}{(136\times 28)-(119\times 35)}$$
$$=\frac{7854-8568}{3808-4165}$$
$$=\frac{714}{357}$$
$$=2$$

Put $n=6$,

$$\frac{L_8L_7-\ell_8\ell_7}{\ell_8\ell_6-L_8L_6}=\frac{(262\times 119)-(251\times 136)}{(251\times 63)-(262\times 66)}$$
$$=\frac{31178-34136}{15813-17292}$$
$$=\frac{2958}{1479}$$
$$=2$$

Put $n=7$,

$$\frac{L_9L_8-\ell_9\ell_8}{\ell_9\ell_7-L_9L_7}=\frac{(523\times 262)-(500\times 251)}{(500\times 136)-(523\times 119)}$$
$$=\frac{137026-125500}{68000-62237}$$
$$=\frac{11526}{5763}$$
$$=2$$

Put $n = 8$,

$$\frac{L_{10}L_9 - \ell_{10}\ell_9}{\ell_{10}\ell_8 - L_{10}L_8} = \frac{(1002 \times 523) - (1047 \times 500)}{(1047 \times 251) - (1002 \times 262)}$$

$$= \frac{524046 - 523500}{262797 - 262524}$$

$$= \frac{546}{273}$$

$$= 2$$

Hence, the conclusion holds true for all positive integers n, confirming its validity through mathematical induction and consistent logical reasoning throughout.

Example-2 based on Theorem 5

Table VIII
Initial terms of the Generalized CLS of second order

n	ℓ_n	L_n
0	b	a
1	d	c
2	$2a + c$	$2b + d$
3	$2b + 2c + d$	$2a + c + 2d$
4	$2a + 4b + c + 4d$	$4a + 2b + 4c + d$
5	$8a + 2b + 6c + 5d$	$2a + 8b + 5c + 6d$

Now we will apply the theorem on this example

$$\frac{L_{n+2}L_{n+1} - \ell_{n+2}\ell_{n+1}}{\ell_{n+2}\ell_n - L_{n+2}L_n} = 2$$

Put $n = 2$,

To perform this calculation, we need to do it separately.

$$L_4L_3 = (4a + 2b + 4c + d)(2a + c + 2d)$$

$$= 8a^2 + 4ab + 12ac + 10ad + 2bc + 4c^2 + 9cd + 4bd + 2d^2$$

$$\ell_4\ell_3 = (2a + 4b + c + 4d)(2b + 2c + d)$$

$$= 4ab + 8b^2 + 10bc + 12bd + 4ac + 2c^2 + 9cd + 2ad + 4d^2$$

$$\ell_4\ell_2 = (2a + 4b + c + 4d)(2a + c)$$

$$= 4a^2 + 8ab + 4ac + 8ad + 4bc + c^2 + 4cd$$

$$L_4L_2 = (4a + 2b + 4c + d)(2b + d)$$

$$8ab + 4b^2 + 8bc + 4bd + 4ad + 4cd + d^2$$

Subtract $\ell_4\ell_3$ from L_4L_3

$$L_4L_3 - \ell_4\ell_3 = (8a^2 - 8b^2 + 2c^2 - 2d^2 + 8ac + 8ad - 8bc - 8bd)$$

$$= 2(4a^2 - 4b^2 + c^2 - d^2 + 4ac + 4ad - 4bc - 4bd)$$

Subtract $\ell_4\ell_2$ from L_4L_2

$$\ell_4\ell_2 - L_4L_2 = (4a^2 - 4b^2 + c^2 - d^2 + 4ac + 4ad - 4bc - 4bd)$$

$$\frac{L_4L_3 - \ell_4\ell_3}{\ell_4\ell_2 - L_4L_2}$$

$$= (4a^2 - 4b^2 + c^2 - d^2 + 4ac + 4ad - 4bc - 4bd)$$

$$= \frac{2(4a^2 - 4b^2 + c^2 - d^2 + 4ac + 4ad - 4bc - 4bd)}{(4a^2 - 4b^2 + c^2 - d^2 + 4ac + 4ad - 4bc - 4bd)}$$

$$= 2$$

Hence the conclusion is valid for all positive integers n.

IV. CONCLUSION

The exploration of the second-order Coupled Lucas Sequence provides deep insights into the broader domain of sequence theory, particularly in relation to classical sequences such as the Fibonacci and Generalized Fibonacci–Lucas sequences. Historically, these sequences have been celebrated for their intriguing properties and wide-ranging applications across number theory, computer science, cryptography, and even natural phenomena. This study continues that tradition, advancing our understanding of recursive sequences and expanding the scope of their practical relevance.

The Lucas sequences, much like the Fibonacci sequences, are defined by recurrence relations. However, the defining characteristic of the Second-Order Coupled Lucas Sequence lies in its coupling mechanism, which interweaves two independent sequences into a unified structure. This coupling introduces both complexity and elegance, as each term in one sequence depends not only on the preceding terms of its own sequence but also on the corresponding terms of the other. This interdependence creates intricate patterns and dependencies, resulting in behaviors significantly more complex than those found in traditional Fibonacci or Lucas sequences.

Through theoretical analysis, it has been demonstrated that these coupled sequences exhibit unique identities and properties that set them apart from other well-known sequences. By employing a combination of inductive reasoning and computational methods, researchers can uncover new identities and relationships within the Coupled Lucas Sequence. Inductive reasoning, in particular, is instrumental in predicting novel outcomes, as it enables the extrapolation of known patterns to reveal previously unrecognized properties of the sequence.

The initial values of the two sequences in the Coupled Lucas Sequence of Second Order play a significant role in determining their behavior. These initial values act as seeds that define the growth and evolution of the sequences over time. Small changes in these initial conditions can lead to vastly different outcomes, revealing the sensitivity and complexity of the system. The recurrence relations, which govern the progression of the sequences, ensure that each term is calculated based on a fixed formula, but the interaction between the two sequences adds an additional layer of unpredictability and complexity to the system.

One of the most fascinating aspects of this research is the way the simultaneous evolution of the two sequences creates a harmonious relationship between them. Each term is intricately linked not only to the preceding terms of its own sequence but also to the corresponding terms in the coupled sequence. This interconnectedness suggests that the sequences are working together in tandem, with each one influencing the other's progression in a delicate balance. This relationship adds a deeper level of structure to the sequences, offering potential insights for other areas of mathematics, particularly in the study of dynamical systems and complexity theory.

The investigation has also uncovered practical applications of the Coupled Lucas Sequence of Second Order. Beyond its theoretical significance, this sequence can be applied in fields such as cryptography, where the complex relationships between terms can be utilized to generate secure encryption keys. Furthermore, the sequence's intricate patterns and behaviors hold potential in computer science, particularly in algorithms related to recursive functions and optimization problems.

In conclusion, the study of the coupled Lucas sequence of the second order has revealed a rich mathematical structure that seamlessly combines theoretical elegance with practical applications. The interplay of recurrence relations and coupling mechanisms introduces new complexities, challenging our understanding of traditional sequences and opening up fresh avenues for research and discovery. By further exploring the properties of these sequences, mathematicians can gain deeper insights into recursion, interdependence, and complexity, enriching the broader field of sequence theory. The potential for uncovering new identities and applications within this framework is vast, promising exciting developments in both theoretical and applied mathematics.

Furthermore, the practical applications of the coupled sequence span various domains. Although inherently mathematical, the coupling mechanism shows promise for use in cryptography, optimization, and other fields where the dynamic interplay of numerical relationships can be leveraged for practical purposes. This highlights the importance of pure mathematical exploration, demonstrating that abstract concepts can lead to meaningful real-world applications.

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