Defined-contribution Pension Reinsurance and Investment Game of Non-zero-sum Utilizing Jump-diffusion and Heston Models

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***Abstract*—In response to global aging challenges, establishing a sustainable pension investment management system has become crucial. This study proposes a defined-contribution (DC) pension risk mitigation framework that integrates investment and reinsurance strategies. Within this framework, we construct an investment game model of non-zero-sum for two pension investors with wealth maximization goals, allowing allocations to risk-free assets, risky assets, and reinsurance contracts. The higher-risk assets' time-varying market risk is modeled using the Heston Model which is characterized by stochastic volatility with mean-reversion. And the reinsurance surplus is denoted by a jump-diffusion model to capture the sudden financial shocks. Moreover, by utilizing standard dynamic programming and exponential utility preferences, we can derive a closed solution to the investment strategy throughout mathematical proofs. Our new model innovatively combines jump-diffusion processes with stochastic volatility from the Heston Model, expanding the theoretical foundation for pension investment optimization strategies. Practically, it offers pension managers a dynamic asset allocation tool which can increase portfolios' resistance to systemic risks.**

***Index Terms*—DC pension plans, reinsurance and investment, Non-zero-sum game, Jump-diffusion model, Heston stochastic volatility model**

I. Introduction

THE global aging population crisis is intensifying, and pension investment management has emerged as an important method to address this challenge. Defined Contribution (DC) pension plans are a cornerstone of modern pension systems, which are characterized by fixed contribution rates. But the retirement benefits available to participants remain subject to actuarial uncertainty, as the final amounts depend on both pre-retirement contributions and investment returns generated by fund managers' portfolio allocations. As markets grow in complexity, the efficacy of DC pension investment management directly impacts both the sustainability of pension systems and the investors' quality of life in retirement. Hence, developing pension investment optimization strategies has become a focus for academia, financial institutions, and pension trustees. To this end, some researchers have incorporated a variety of investment models into DC pension portfolios [1-5], and explored terminal wealth utility maximisation under the constraints of inflation risk [6], stochastic wage dynamics [7], and imperfect information [8-10].

With developments in research, dynamic portfolio optimization for defined-contribution pension systems has not only involved preliminary model uncertainty analysis, but also come to consider the competitive-cooperative behavior of multiple market participants, as well as the allocation of diverse risk-return assets. As a form of competition and cooperation, a non-zero-sum game differs from the common game of traditional financial markets. In this non-zero-sum game, the total benefits that all competitors can obtain are not fixed; that is to say, each participant can benefit, but this does not necessarily bring losses to other participants. There is the possibility of win-win cooperation between these participants [11-14]. Multiple investors can adopt competitive strategies in the market by using a model of non-zero-sum game to seek the optimal Nash equilibrium point for both themselves and their competitors, so that each investor cannot unilaterally change their strategy to affect returns [15]. Investors can also adopt a cooperative strategy, treating their goals as a whole [16]. Through cooperation, investors can obtain better trading opportunities and increase overall returns on investment.

In terms of applied research, Espinosa and Touzi (2015) made a breakthrough by integrating competitive game dynamics with methodological approaches in investment portfolio optimization [17]. They adopted the exponential utility function as an analytical tool and derived optimized strategies for maximizing relative wealth among multiple competing investors. Later, Zhu et al. (2019) proposed a dynamic optimization framework for reinsurance and investment leveraging the Heston model. This framework uses jump-diffusion processes to derive time-consistent equilibrium strategies for pension portfolios. It quantifies the nonlinear interactions parameters and risk aversion coefficients among stochastic volatility, and establishes a mechanism that hedges the discrete market shocks and optimizes long-term returns [18]. Within the framework of stochastic Malliavin calculus, He (2024) derived optimal strategies for players facing volatility-clustered inflation. Power and exponential utility functions were applied to

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determine strategies that maximize the terminal surplus relative to competing portfolios [19].

In financial practice, a large amount of the information that goes into investment decision-making processes has inherent uncertainties and is susceptible to exogenous perturbations. Asset prices often exhibit discontinuous dynamic jumps that deviate from the continuous Brownian motion assumptions. The kind of jump risk in investment decisions cannot in general be predicted by continuous price fluctuations [20][21]. In certain circumstances, the likelihood of investment decision failure greatly increases, leading to a sudden decrease in the overall value of the portfolio [22]. In this study we investigate and work towards prevention of the risks associated with jumping factors, which is of very great significance for investment decision-making [23] [24].

Recently, scholars and practitioners have increasingly concentrated on developing jump risk models, among which the jump-diffusion models is one of the important examples. These models assume that changes in asset prices conform to both jump and diffusion characteristics [20], and they can better capture the sudden changes in investment value. In the field of reinsurance-investment decision-making, proposed new models are increasingly accounting for the characteristics of jump diffusion. Chen et al. (2023) proposed that partial insider information regarding risky asset prices could be probabilistically forecasted. Moreover, they also suggested that a surplus process enables analysis of stochastic differential games in reinsurance-investment strategies [25]. Li et al. (2023) extended this methodology by investigating reinsurance and investment games with jump-diffusion characteristics under the Constant Elasticity model [26]. Additionally, based upon the default mechanism in Markovian models, Li et al. (2024) explored the non-cooperative equilibria through two jump-diffusion models which accounted for normal market shocks [27].

Inspired by these studies, we explored the optimal equilibrium combination of a DC pension system with discontinuous risk by using a stochastic fluctuation framework. Our work is partially similar to the concepts proposed by Chen et al. (2023) [25] and Li et al. (2023) [26]. However, Chen et al. (2023)[25] argued that insurance companies have internal investment information in advance, while Li et al. (2023) [26] analyzed jump-diffusion characteristics within the framework of Constant Elasticity of Variance (CEV), assuming that risk assets maintain a constant drift rate. However, financial markets often exhibit uncertainty, and empirical evidence also has consistently demonstrated the stochastic nature of the discontinuity of investment returns and risk. Notably, the existing literature lacks investigations of the randomness of the asset returns in DC pension plans which involve jump-conditioned reinsurance strategies.

In this study, we build upon the Heston stochastic volatility model, incorporating investment uncertainty and characterizing the reinsurance surplus process for DC pension systems through jump-diffusion risk modeling. A novel non-zero-sum game theoretical model integrating reinsurance and investment decisions is developed, with the aim of optimizing a pension portfolio. Next, by using dynamic programming and assuming exponential utility preferences, we perform the mathematical derivations to establish closed-form equilibrium investment strategies. This method quantifies the interaction between random fluctuation parameters. Within the proposed framework, the value function is $W^k(t,m(t),\varepsilon)=A(t)\tilde{m}_k+B(t)\ln\varepsilon+C_k(t)$. Compared with the models in the existing literature, this strategy $b_k^*(t)$ achieves enhanced parsimony, along with greatly lower computational complexity. This streamlined formulation can enhance operational feasibility for the practitioners implementing reinsurance-investment game theoretical strategies for DC pension schemes.

The other parts of this study are as follows: section II, we construct a non-zero-sum equilibrium model under the Heston model, in which the reinsurance surplus process of two DC pension investors follows a jump- diffusion model. Section III, the Nash equilibrium strategies are derived, assuming that the DC pension investment strategy adopts an exponential utility approach regarding reinsurance and capital allocation. Section IV discusses the method and offers conclusions.

II. Model construction

The choice of investment strategy has a significant impact on the final returns following accumulation of DC pension wealth. In order to investigate the mechanism of wealth accumulation and the laws of income changes, it is necessary to make reasonable abstractions of investment methods. Based on this, we first assume that DC pension investment consists of two investment methods: risk-free investment, and venture capital. Risk-free investment has stable returns but relatively low return rates, while venture capital has greater volatility risks but higher potential returns. Here, $\varepsilon_0(t)$ follows the dynamics:

$$\varepsilon_0(0)=1 \tag{1}$$

where the parameter $r>0$ denotes the instantaneous risk-free yield. The risky characteristics of the asset $\varepsilon(t)$ are modeled with the Heston framework:

$$\begin{aligned} &d\varepsilon(t)=\varepsilon(t)\left(r+\beta\Lambda(t)\right)dt+\sqrt{\Lambda(t)}\varepsilon(t)\,dW_1(t),\\ &\varepsilon(0)=\varepsilon_0>0 \end{aligned} \tag{2}$$

where $r+\beta\Lambda(t)$ represents the projected earnings, $\Lambda(t)$ is the square of the instantaneous volatility, $W_1(t)$ follows standard Brownian motion. The evolution process of volatility $\Lambda(t)$ according to:

$$\begin{aligned} &d\Lambda(t)=\upsilon\left(\vartheta-\Lambda(t)\right)dt+\sigma\sqrt{\Lambda(t)}dW_2(t),\\ &\Lambda_0=l_0>0 \end{aligned} \tag{3}$$

Here, $\upsilon>0$ represents the mean reversion speed; $\vartheta>0$ is the long-term equilibrium level; $\sigma>0$ represents the volatility of the fluctuation process, which satisfies the Feller condition $2\upsilon\vartheta\geq\sigma^2$; $W_2(t)$ also behaves as normal Brownian motion, and $W_1(t)$ correlates with $W_2(t)$ via the asset returns and volatility formation process; $\rho\in[-1,1]$ is the correlation coefficient between the two.

The DC pension system requires policyholders to pay pension fees at a fixed rate based on their salary income before retirement, establishing the basic framework for pension accumulation. But in practice, the policyholder's salary is not fixed. Wages are influenced by all kinds of factors such as macroeconomic fluctuations, industry cycles, and personal career development. These factors often exhibit random and unstable characteristics. In order to accurately simulate the accumulation process in DC pensions, we assume that salary distributions satisfy the following equation:

$$\frac{dY(t)}{Y(t)}=\left[r+\partial(t)+\omega\sigma_L\Lambda(t)\right]dt+\sigma_\Lambda\sqrt{\Lambda(t)}dW_1(t) \tag{4}$$

Here, the function t provides the structural basis for $\partial(t)$, whereas σ_L is the impact of risk on wage income. The Brownian motion $W_1(t)$ denotes exogenous economic shocks affecting income stability.

In addition, under the dual pressure of demographic transformation and policy adjustment, the decrease in population has led to a decline in pension bases. Delayed retirement policies implemented by numerous countries have heightened concerns among policyholders regarding long-term financial liquidity. These compounding factors have greatly amplified withdrawal risks for DC pension policyholders, which may destabilize the pension fund accumulation and potentially induce systemic financial strain. To address this challenge, DC pension managers could implement reinsurance strategies to transfer partial risks to some specialized reinsurance institutions, thereby establishing some risk diversification and loss-sharing mechanisms. Under this framework, we mathematically posit that the surplus process $\Re_k(t)$ of pension investors adopting reinsurance strategies follows the stochastic differential equation (SDE) below. This SDE quantifies the dynamic evolution process of pension surpluses under reinsurance coverage, thereby establishing an actuarial foundation for risk mitigation. Formally, the process is defined as:

$$d\Re_k(t)=\mathrm{I}_k dt+\sigma_k dB_k(t)-d\left(\sum_{i=1}^{\bar{N}_k(t)}Z_i^k\right) \tag{5}$$

where I_k is the risk premium income from reinsurance contracts; $\sigma_k>0$ represents the volatility parameter of the reinsurance strategy; $B_k(t)$ follows a standard Brownian motion, statistically independent of other factors; $\sum_{i=1}^{\bar{N}_k(t)}Z_i^k$ characterizes the pension surplus payout, modeled as a compound Poisson process.

Let $\alpha_k(t)$ characterize the reinsurance strategy, where the rate of return is $\mathrm{I}_k=(1+\theta_k)(\ell+\ell_k)\mu_k$, $\theta_k>0$ quantifies the risk margin factor in proportional reinsurance treaties. Let λ characterize the density of the Poisson $N(t)$. The reinsurance proportional rate paid by the pension fund is $(1-\alpha_k(t))(1+\Upsilon_k)(\ell+\ell_k)\mu_k$, and the reinsurance companies' safety loading coefficient is represented by $\Upsilon_k>\theta_k$. Upon surrender compensation, the pension investment manager pays out $100\alpha_k(t)\%$.

Let $C_k(t)=\sum_{i=1}^{\bar{N}_k(t)}Z_i^k$; then, the surplus process of the DC pension reinsurance strategy $a_k(t)$:

$$\begin{aligned}d\Re_k(t)&=\left[(\theta_k-\Upsilon_k)+\alpha_k(t)(1+\Upsilon_k)\right](\ell+\ell_k)\mu_k dt\\&\quad+\alpha_k(t)\delta_k dB_k(t)-\alpha_k(t)dC_k(t)\\&=(1+\theta_k)(\ell+\ell_k)\mu_k dt+\sigma_k dB_k(t)-dC_k(t)\end{aligned} \tag{6}$$

In the DC pension system, the accumulation of wealth in investment accounts is a dynamic process influenced by multiple factors. The wealth accumulation process, which is closely related to the strategic choices of investment managers, affects the future level of pension security for insured individuals. To construct an accurate model for wealth accumulation, it is necessary to identify the core driving factors of wealth changes. In order to accomplish this, we make a few more assumptions. $M_k^{\lambda_k}(t)$ is defined as the accumulated assets in the DC pension investment account, where $M_k^{\lambda_k}(0)=m_k$ represents the initial wealth value. Changes in pension wealth are sourced from return income from reinsurance business, income from portfolio management, and total amount of fixed contributions for pension. Let the reinsurance rate of return be $\Re_k(t)$, the random wage contribution rate be ξ, and the investment volume of risky assets be $b_k(t)$. We can then accordingly mathematically formulate wealth accumulation through a stochastic differential:

$$\begin{aligned}dM_k^{\lambda_k}&=d\Re_k(t)+b_k(t)\frac{d\mathcal{E}(t)}{\mathcal{E}(t)}+\left(M_k^{\lambda_k}(t)-b_k(t)\right)\frac{d\mathcal{E}_0(t)}{\mathcal{E}_0(t)}+\xi Y(t)dt\\&=\left[(1+\theta_k)(\ell+\ell_k)\mu_k dt+\sigma_k dB_k(t)-dC_k(t)\right]+b_k(t)\\&\quad\cdot\left[(r+\beta\Lambda(t))dt+\sqrt{\Lambda(t)}dW_1(t)\right]+\left(M_k^{\lambda_k}(t)-b_k(t)\right)rdt+\xi Y(t)dt\\&=\Big[M_k^{\lambda_k}(t)r+b_k(t)\beta\Lambda(t)dt+\left[(\theta_k-\Upsilon_k)+a_k(t)(1+\Upsilon_k)\right]\\&\quad+(\ell+\ell_k)\mu_k\Big]+\xi Y(t)\Big]dt+b_k(t)\sqrt{\Lambda(t)}dW_1(t)\\&\quad+\alpha_k(t)dB_k(t)-\alpha_k(t)dC_k(t)\end{aligned} \tag{7}$$

Definition 1 (Admissible Strategy). A DC pension reinsurance-investment strategy $\lambda_k(t)=(a_k,b_k)$ is deemed admissible if it satisfies:

1. Adaptiveness: let $\lambda_k(t)$ denote an adaptive controller related to $\{\Psi_t\}_{t\in[0,T]}$ and $E\left[\int_0^T\left(\alpha_k^2(t)+b_k^2(t)\right)dt\right]<\infty$;

2. Solvability: Eq.(7) is defined for $\forall(m_k,s)\in\Theta\times\Theta^+$, for which there exists a single solution, $\left\{M_k^{\lambda_k}(t)\right\}_{t\in[0,T]}$.

When both the DC pension reinsurance and investment strategies satisfy the aforementioned conditions, the strategy combination is deemed admissible. All qualifying strategies are collectively designated as the set M_k, which includes all theoretically sound and operationally viable viable strategic combinations available to DC investment managers pursuing dual objectives: risk mitigation and return maximization. This framework establishes a robust theoretical foundation for subsequent investigation into optimal strategy selection and Nash equilibrium solutions.

III. NASH EQUILIBRIUM

A. Nash equilibrium solution

Under conditions of constrained market resources and complex investment environments, the investment returns achieved by DC pension fund investors emerge not merely from their isolated portfolio strategies, but fundamentally through strategic interactions with other participants in the financial market. This study utilizes relative performance evaluation metrics to operationalize the game-theoretic interdependencies among DC pension investors [26]. We develop a stochastic differential game model to analyze the co-opetition dynamics in return maximization. The solution of the Nash equilibrium reveals a stable strategy profile and market equilibrium conditions for DC pension investment systems.

$$\Gamma_k^{\lambda_k,\lambda_j}\left(t,m_k,m_j,\varepsilon\right) := E_{t,m_k,m_j,\varepsilon}\left[\left(1-\bar{k}_k\right)M_k^{\lambda_k}(T)+\bar{k}_k\left(M_k^{\lambda_k}(T)-M_j^{\lambda_j}(T)\right)\right] -\frac{\wp_k}{2}Var_{t,m_k,m_j,\varepsilon}\left[\left(1-\bar{k}_k\right)M_k^{\lambda_k}(T)+\bar{k}_k\left(M_k^{\lambda_k}(T)-M_j^{\lambda_j}(T)\right)\right]$$

Here, $k,j\in\{1,2\},k\neq j$, γ_k quantifies the degree of risk avoidance of investors, where $\bar{k}_k$ quantifies the degree of attention of investors' comparative wealth, reflecting their behavioral focus on peer-comparative performance.

Let $M_k^{\varphi_k,\varphi_j}=M_k^{\varphi_k}(t)-\bar{k}_kM_j^{\varphi_j}(t)$ represent the accumulation process of comparative wealth of a DC pension investor.

$$\begin{aligned}
d\tilde{M}_k^{\lambda_k,\lambda_j}(t)&=dM_k^{\lambda_k}(t)-\bar{k}_kdM_j^{\lambda_j}(t)\\
&=\left[M_k^{\lambda_k}(t)r+b_k(t)\beta\Lambda(t)+\left[\left(\theta_k-\Upsilon_k\right)+\alpha_k(t)\left(1+\Upsilon_k\right)\right]\left(\ell+\ell_k\right)\mu_k+\xi Y(t)\right]dt\\
&+b_k(t)\sqrt{\Lambda(t)}dW_1(t)+\alpha_k(t)\sigma_kdB_k(t)-\alpha_k(t)dC_k(t)\\
&-\bar{k}_k\left[M_j^{\lambda_j}(t)r+b_j(t)\beta\Lambda(t)+\left[\left(\theta_j-\Upsilon_j\right)+\alpha_j(t)\left(1+\Upsilon_j\right)\right]\left(\ell+\ell_k\right)\mu_j+\xi Y(t)\right]dt\\
&-\bar{k}_kb_j(t)\sqrt{\Lambda(t)}dW_1(t)-\bar{k}_k\alpha_j(t)\sigma_jdB_j(t)+\bar{k}_k\alpha_j(t)dC_j(t)\\
&=\Big\{r\tilde{M}_k^{\lambda_k,\lambda_j}(t)+\beta\Lambda(t)\left(b_k(t)-\bar{k}_kb_j(t)\right)+\left[\left(\theta_k-\Upsilon_k\right)\right.\\
&+\alpha_k(t)(1+\Upsilon_k)\left.\right](\lambda+\lambda_k)\mu_k-\bar{k}_k\left[\left(\theta_j-\Upsilon_j\right)+\alpha_j(t)\left(1+\Upsilon_j\right)\right]\\
&\left(\lambda+\lambda_j\right)\mu_j+\xi Y(t)\left(1-\bar{k}_k\right)\Big]dt+\sqrt{\Lambda(t)}\left(b_k(t)-\bar{k}_kb_j(t)\right)dW_1(t)\\
&+\alpha_k(t)\sigma_kdB_k(t)-\bar{k}_k\alpha_j(t)\sigma_jdB_j(t)\\
&+\alpha_k(t)\sigma_kdB_k(t)-\bar{k}_k\alpha_j(t)\sigma_jdB_j(t)\\
&-\alpha_k(t)dC_k(t)+\bar{k}_k\alpha_j(t)dC_j(t)
\end{aligned}\tag{8}$$

And for an admissible strategy $(t,\tilde{m}_k,s)\in[0,T]\times\Theta\times\Theta^+$,

$$\hat{\Gamma}_k^{\lambda_k,\lambda_j^*}(t,\tilde{m}_k,\varepsilon):=E_{t,\tilde{m}_k,\varepsilon}\left[\tilde{M}_k^{\lambda_k,\lambda_j^*}(T)\right]-\frac{\wp_k}{2}Var_{t,\tilde{m}_k,\varepsilon}\left[\tilde{M}_k^{\lambda_k,\lambda_j^*}(T)\right]\tag{9}$$

In the field of DC type pension investment management, due to limited market resources and synergistic effects in investment strategies, the game between DC investment managers is essentially a non-zero-sum game, where one party's gains are not entirely based on the other party's losses; there is a possibility for a win-win scenario through strategic coordination. Meanwhile, due to the interference of random factors on pension investment, such as market fluctuations and economic cycles, this game also exhibits the structural properties of stochastic differential games. Accordingly, strategy selection and income changes for a pension investment game need to be characterized by dynamic differential equations. Therefore, the following Problem 1 is introduced to solve the Nash equilibrium. Essentially, the goal is to find a portfolio of investment strategies that follow the condition that neither investment manager can achieve higher pension accumulation returns or risk control effects by separately adjusting reinsurance and investment strategies. This should hold true across any time interval and for any potential strategy adjustments.

Problem 1: Obtaining the objective for the pension managers requires solving for the Nash equilibrium state, $\left(\lambda_1^*,\lambda_2^*\right)\in\Pi_1\times\Pi_2$, so that for any $(\lambda_1,\lambda_2)\in\Pi_1\times\Pi_2$, there is:

$$\begin{cases}\hat{\Gamma}_1^{\lambda_k,\lambda_j^*}(t,\tilde{m}_1,\varepsilon)\geq\hat{\Gamma}_1^{\lambda_k,\lambda_j^*}(t,\tilde{m}_1,\varepsilon),\\ \hat{\Gamma}_2^{\lambda_k,\lambda_j^*}(t,\tilde{m}_2,\varepsilon)\geq\hat{\Gamma}_2^{\lambda_k,\lambda_j^*}(t,\tilde{m}_2,\varepsilon).\end{cases}$$

B. Optimal solution of the model

In solving the Nash equilibrium, conventional static analysis methods are difficult to apply due to the dynamic decision and uncertain payoff mechanisms involved in stochastic differential games. To construct an analytical framework, it is necessary to introduce tools that can quantify the strategic value. Therefore, the value function is defined to measure the comprehensive expected value of investors, accounting for future stochastic returns across different strategy combinations:

$$W^k(t,\tilde{m}_k,\varepsilon)=\hat{J}_k^{\left(\lambda_k^*,\lambda_k^*\right)}(t,\tilde{m}_k,\varepsilon) = \sup_{\varphi_k\in\Pi_k}\hat{J}_k^{\left(\lambda_k,\lambda_j^*\right)}(t,\tilde{m}_k,\varepsilon)\tag{10}$$

Here, $\hat{\Gamma}_k^{\left(\lambda_k,\lambda_j^*\right)}(t,\tilde{m}_k,\varepsilon)$ is defined by Eq.(9), λ_k^* denotes the strategies of competing investors $k\in\{1,2\}$. The strategy is a stable solution derived from the optimal decision conditions of both parties in the game. Next, we need to introduce spatial definitions for further derivations and property analysis of value functions and equilibrium strategies:

$C^{1,2,2}\left([0,T]\times\Theta\times\Theta^+\right)=\left\{\varphi(t,\tilde{m}_k,s)\middle|\varphi(t,.,.)\right\}$ denotes first-order derivative of $[0,1]$, and $\varphi(;\tilde{m}_k,s)$ characterizes second-order derivative of $\tilde{m}_k$ concerning R and S on Θ^+, respectively.

The rigorous condition enables us to perform reasonable derivatives on the function of value when constructing the HJB equation using dynamic programming principles; it also enables us to solve for the equilibrium strategy via the optimization methods. Meanwhile, the continuous second-order differentiability is necessary to prove the existence, uniqueness, and stability of equilibrium strategies; the resulting theoretical deductions may enhance practical pension investment management in terms of analyzing different scenarios.

For a comprehensive evaluation of investment strategy shifts on the value function and equilibrium results, we need to introduce a mathematical tool that can quantify the adjustment effect of small investment strategies in DC pension stochastic differential investment games; that is to say, we must define variational operators. The variational operators can be used to reflect some changes in the accumulation process and the Nash equilibrium conditions of pension wealth, as a reaction to small changes in investors' reinsurance and investment strategies. The variational operator can be defined as:

$A^{(\lambda_k,\lambda_j)}$: $\forall \varphi^k(t,\tilde{m}_k,\varepsilon)\in C^{1,2,2}([0,\ T])\times\Theta\times\Theta^+$.

$$
\begin{aligned}
&A^{(\lambda_k,\lambda_j)}\varphi^k(t,\tilde{m}_k,\varepsilon)\\
&=\varphi^k(t,\tilde{m}_k,\varepsilon)+H_{k,j}^{(\lambda_k,\lambda_j)}(\tilde{m}_k,t)\varphi^k(t,\tilde{m}_k,\varepsilon)\\
&+\varepsilon(r+\beta\Lambda(t))\varphi_\varepsilon^k(t,\tilde{m}_k,\varepsilon)+\frac{1}{2}\Big[\Lambda(t)\big(b_k(t)-\bar{k}_k b_j(t)\big)^2\\
&+\alpha_k^2(t)\sigma_k^2+\bar{k}_k^2\alpha_j^2(t)\sigma_j^2-2\rho_0\sigma_k\alpha_j(t)\alpha_k(t)\bar{k}_k\sigma_j\\
&\cdot\varphi^k(t,\tilde{m}_k,\varepsilon)+\frac{1}{2}s^2\Lambda(t)\varphi_{\varepsilon\varepsilon}^k(t,\tilde{m}_k,\varepsilon)\\
&+\varepsilon\Lambda(t)\big(b_k(t)-\bar{k}_k b_j(t)\big)\varphi_{\varepsilon\varepsilon}^k(t,\tilde{m}_k,\varepsilon)\\
&+\lambda E\Big[\varphi^k\big(t,m_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j(t)Z^j,\varepsilon\big)\Big]-\varphi^k(t,\tilde{m}_k,\varepsilon)\\
&+\lambda E\Big[\varphi^k\big(t,\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j(t)Z^j,\varepsilon\big)\Big]\\
&-\varphi^k(t,\tilde{m}_k,\varepsilon)+\lambda_j E\Big[\varphi^k\big(t,\tilde{m}_k+\bar{k}_k\alpha_j(t)Z^j,\varepsilon\big)\Big]\\
&-\varphi^k(t,\tilde{m}_k,\varepsilon)+\lambda_k E\Big[\varphi^k\big(t,\tilde{m}_k-\alpha_k(t)Z^k,\varepsilon\big)-\varphi^k(t,\tilde{m}_k,\varepsilon)\Big]
\end{aligned}
$$

Here,

$$
\begin{aligned}
&H_{k,j}^{(\lambda_k,\lambda_j)}(\tilde{m}_k,t)=r\tilde{m}_k+\beta\Lambda(t)\big(b_k(t)-\bar{k}_k b_j(t)\big)\\
&+\big[(\theta_k-\Upsilon_k)+\alpha_k(t)(1+\Upsilon_k)\big]-\bar{k}_k\big[(\theta_j-\Upsilon_j)+\alpha_j(t)(1+\Upsilon_j)\big]\\
&\cdot(\ell+\ell_k)\mu_j+\xi Y(t)(1-\bar{k}_k)
\end{aligned}
$$

Variational operators establish the core link between the investment strategy optimization and equilibrium solution by quantifying changes in the value function under policy perturbations. By leveraging the definition of operator molecules formalized within the theoretical framework and the mathematical derivations of stochastic differential games for investment in the literature (Li et al., (2012) [28]; Li (2023) [26]), we can establish theoretical support for the methodology.

Theorem 1.

If $V^k(t,\tilde{m}_k,s)$ and $g^k(t,\tilde{m}_k,\varepsilon)\in C^{1,2,2}\big([0,T]\times\Theta\times\Theta^+\big)$, $k=1,2$ satisfy: $\forall(t,\tilde{m}_k,\varepsilon)\in[0,T]\times\Theta\times\Theta^+$,

$$
\begin{aligned}
&\sup_{\lambda_k\in\Pi_k}\Big\{A^{(\lambda_k,\lambda_j^*)}V^k(t,\tilde{m}_k,\varepsilon)-\frac{\wp_k}{2}A^{(\lambda_k,\lambda_j^*)}\big(g^k(t,\tilde{m}_k,\varepsilon)\big)^2\\
&+\wp_k g^k(t,\tilde{m}_k,\varepsilon)A^{(\lambda_k,\lambda_j^*)}g^k(t,\tilde{m}_k,\varepsilon)\Big\}=0 \qquad (11)
\end{aligned}
$$

$$V^k(T,\tilde{m}_k,\varepsilon)=\tilde{m}_k \qquad (12)$$

$$A^{(\lambda_k,\lambda_j^*)}g^k(t,\tilde{m}_k,\varepsilon)=0 \qquad (13)$$

$$g^k(T,\tilde{m}_k,\varepsilon)=\tilde{m}_k \qquad (14)$$

$$
\begin{aligned}
\Omega_k^*=\arg\sup_{\lambda_k\in\Pi_k}&\Big\{A^{(\lambda_k,\lambda_j^*)}V^k(t,\tilde{m}_k,\varepsilon)-\frac{\wp_k}{2}A^{(\lambda_k,\lambda_j^*)}\\
&\cdot\big(g^k(t,\tilde{m}_k,\varepsilon)\big)^2+\wp_k g^k(t,\tilde{m}_k,\varepsilon)A^{(\lambda_k,\lambda_j^*)}g^k(t,\tilde{m}_k,\varepsilon)\Big\} \qquad (15)
\end{aligned}
$$

And the equilibrium strategy λ_k^* follows the conditions:

$$W^k(t,\tilde{m}_k,\varepsilon)=V^k(t,\tilde{m}_k,\varepsilon),\ E_{t,\tilde{m}_k,\eta}\Big[\tilde{M}_k^{(\lambda_k^*,\lambda_j^*)}(T)\Big]=g^k(t,\tilde{m}_k,\varepsilon)$$

The strategy λ_k^* serves as the critical nexus between theoretical assumptions and practical investment strategy optimization, as it satisfies continuity and differentiability conditions for pension wealth trajectory projections while adhering to value-function-based optimization criteria for global optimality in stochastic market. These requirements provide DC pension managers with the unambiguous selection benchmarks. Theorem 2 formally establishes the Nash equilibrium representation of risk-sharing and asset allocation through stochastic differential game modeling. Its derivation is rooted in the foundations of Theorem 1 and the integration of complex market dynamics. The equilibrium solution explicitly resolves nonlinear strategic interdependencies among investors in the environment of constrained resources and competitive.

Theorem 2. The Nash equilibrium solution for risk sharing and asset allocation among DC pension investors can be as:

$$
\begin{cases}
\alpha_1^*(t)=\dfrac{\Gamma_1}{\Gamma_3},\\
b_1^*(t)=\dfrac{(\beta-C_B)(\wp_1+k_1\wp_1)}{(1-\bar{k}_1\bar{k}_2)\wp_1\wp_2 e^{r(T-t)}}
\end{cases}
$$

and

$$
\begin{cases}
\alpha_2^*(t)=\dfrac{\Gamma_2}{\Gamma_3},\\
b_1^*(t)=\dfrac{(\beta-C_B)(\wp_1+k_2\wp_2)}{(1-\bar{k}_1\bar{k}_2)\wp_1\wp_2 e^{r(T-t)}}
\end{cases}
$$

Here,

$$
\begin{aligned}
J_1&=\wp_2(\ell+\ell_1)\mu_1\Upsilon_1\big[\sigma_2^2+(\ell+\ell_2)\xi_2^2\big]\\
&+\wp_1\bar{k}_1(\ell+\ell_1)\mu_2\Upsilon_2\big[\rho_0\sigma_1\sigma_2+\lambda(\mu_1\mu_2+\rho)\big]\\
J_2&=\wp_1(\ell+\ell_1)\mu_2\Upsilon_2\big[\sigma_1^2+(\ell+\ell_2)\xi_1^2\big]\\
&+\wp_2\bar{k}_2(\ell+\ell_1)\mu_1\Upsilon_1\big[\rho_0\sigma_1\sigma_2+\lambda(\mu_1\mu_2+\rho)\big]
\end{aligned}
$$

$$J_3 = \wp_1\wp_2 e^{r(T-t)}\Big[\left(\sigma_1^2+(\ell+\ell_1)\zeta_1^2\right)\left(\sigma_2^2+(\ell+\ell_2)\xi_2^2\right) -\bar{k}_1\bar{k}_2\left(\rho_0\sigma_1\sigma_2+\lambda(\mu_1\mu_2+\rho)\right)^2\Big]$$

The value function is:

$$W^1(t,\tilde{m}_1,\varepsilon)=\tilde{m}_1 e^{r(T-t)}+\frac{B(t)}{\wp_1}\ln\varepsilon+\frac{C_1(t)}{\wp_1},$$

$$W^2(t,\tilde{m}_1,\varepsilon)=\tilde{m}_2 e^{r(T-t)}+\frac{B(t)}{\wp_2}\ln\varepsilon+\frac{C_2(t)}{\wp_2}.$$

where the value of $B(t)$ corresponds to the constant C_B Eqs.(24) and (26) yield the values of $C_1(t)$ and $C_2(t)$.

In a real financial market, asset price fluctuations often exhibit complex characteristics, showing both jumps and stochastic volatility. Theorem 2 is based on constructing a Heston stochastic volatility framework that includes jump-diffusion processes. It not only considers sudden changes in asset prices triggered by unexpected events, but also demonstrates the random evolution of volatility through the Heston model, thereby more closely aligning with the real market scenarios encountered in pension investment. Within this framework, Theorem 2 establishes a Nash equilibrium for risk sharing and asset allocation among DC pension investors. It integrates reinsurance strategies, the ratio of risky assets to risk-free assets, and the random wage fluctuations of contributors, clarifying the optimal strategy combinations for each investor in dynamic games. This model and framework can serve as a quantifiable and actionable reference for DC pension investment managers, so as to they can balance risk and return and achieve stable growth.

Proof: Derivation of the Nash equilibrium

When deriving the Nash equilibrium for DC pension stochastic differential games, it is necessary to define the variational operator that can directly quantify the influence of investment strategy changes on the value function. Eq.(11) can be transformed into a more tractable form for solving the Nash equilibrium by the variational operator:

$$\sup_{\lambda_k\in\Pi_k}\Bigg\{V_t^k(t,\tilde{m}_k,\varepsilon)+H_{k.j}^{(\lambda_k,\lambda_j^*)}V_{\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)+\varepsilon\left(r+\beta\Lambda(t)\right)$$
$$\cdot V_{\varepsilon}^k(t,\tilde{m}_k,\varepsilon)+\frac{1}{2}\Big[\Lambda(t)\left(b_k(t)-\bar{k}_k b_j^*(t)\right)^2+\sigma_k^2\alpha_k^2(t)$$
$$+\sigma_j^2\bar{k}_k^2\alpha_j^2(t)_k\alpha_k(t)\alpha_j^*(t)\sigma_k\sigma_j\Big]\cdot\Big[V_{\tilde{m}_k\tilde{m}_k}^k-\wp_k\left(g_{\tilde{m}_k}^k\right)^2\Big]$$
$$+\frac{1}{2}\varepsilon^2\Lambda(t)\Big[V_{\varepsilon\varepsilon}^k(t,\tilde{m}_k,\varepsilon)-\wp_k\left(g_{\varepsilon}^k(t,\tilde{m}_k,\Upsilon)\right)^2\Big]+\varepsilon\Lambda(t)$$
$$\cdot\left(b_k(t)-\bar{k}_k b_j^*(t)\right)\Big[V_{\tilde{m}_k\varepsilon}^k(t,\tilde{m}_k,\varepsilon)-\wp_k g_{\varepsilon}^k(t,\tilde{m}_k,\Upsilon)g_{\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)\Big]$$
$$+\ell E\Big[V^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)$$
$$-\frac{\wp_k}{2}\left(g^k\left(t,\tilde{m}_k-a_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)\right)^2$$
$$+\wp_k g^k(t,\tilde{m}_k,\varepsilon)g^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)\Big]$$
$$+\lambda_j E\Big[V^k\left(t,\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)-\frac{\wp_k}{2}\left(g^k\left(t,\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)\right)^2$$
$$+\wp_k g^k(t,\tilde{m}_k,\varepsilon)g^k\left(t,\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)\Big]$$
$$+\lambda_k E\Big[V^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k,\varepsilon\right)\varepsilon-\frac{\wp_k}{2}\left(g^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k,\varepsilon\right)\right)^2$$
$$+\wp_k g^k(t,\tilde{m}_k,\varepsilon)g^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k,\varepsilon\right)\Big]$$
$$-\left(\lambda+\lambda_j+\lambda_k\right)E\Big[V^k(t,\tilde{m}_k,\varepsilon)+\frac{\wp_k}{2}\left(g^k(t,\tilde{m}_k,\varepsilon)\right)^2\Big]\Bigg\}$$

$$=\sup_{\lambda_k\in\Pi_k}\left\{\sum_{i=1}^{10}F_i\right\}=0 \tag{16}$$

For Eqs.(12) and (16), considering the complexity of the model and the characteristics of the real financial market, we let the equations have the following formal solutions:

$$V^k(t,\tilde{m}_k,\varepsilon)=A(t)\tilde{m}_k+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k},$$

$$g^k(m,\tilde{m}_k,\varepsilon)=\bar{A}(t)\tilde{m}_k+\frac{\bar{B}(t)}{\wp_k}\ln s+\frac{\bar{C}_k(t)}{\wp_k},$$

$A(T)=\bar{A}(t)=1$, $B(T)=\bar{B}(t)=0$, and $C_k(T)=\bar{C}_k(T)=0$.

then: $V_t^k(t,\tilde{m}_k,\varepsilon)=A'(t)\tilde{m}_k+\frac{B'(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}$,

$V_{\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)=A(t)$, $V_{\varepsilon}^k(t,\tilde{m}_k,\varepsilon)=\frac{B(t)}{\wp_k}\cdot\frac{1}{\varepsilon}$, $V_{\tilde{m}_k\tilde{x}_k}^k(t,\tilde{m}_k,\varepsilon)=0$,

$V_{\tilde{m}_k\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)=0$, $V_{\tilde{m}_k\varepsilon}^k=0$, $V_{\varepsilon\varepsilon}^k(t,\tilde{m}_k,\varepsilon)=\frac{-B(t)}{\wp_k}\cdot\varepsilon^{-2}$,

$g_{\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)=\bar{A}(t)$, $g_t^k(m,\tilde{m}_k,\varepsilon)=\bar{A}'(t)\tilde{m}_k+\frac{\bar{B}'(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}$

$g_{\varepsilon}^k(t,\tilde{m}_k,\varepsilon)=\frac{\bar{B}(t)}{\wp_k}\cdot\varepsilon^{-1}$, $g_{\tilde{m}_k\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)=0$, $g_{\tilde{m}_k\varepsilon}^k(t,\tilde{m}_k,\eta)=0$,

$g_{\varepsilon\varepsilon}^k(t,\tilde{m}_k,\varepsilon)=-\frac{\bar{B}(t)}{\wp_k}\cdot\varepsilon^{-2}$.

One can substitute the above expressions into each item in Eq.(16) for detailed estimations. This approach needs to integrate an equation of pension wealth accumulation, a reinsurance return model, and a random wage payment mechanism. Through mathematical operations and logical deduction, the specific impact of each equilibrium strategy on dynamic changes in the investment pension account is quantified:

$$F_1=V_t^k(t,\tilde{m}_k,\varepsilon)=A'(t)\tilde{m}_k+\frac{B'(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k},$$

$$F_2=H_{k,j}^{(\lambda_k,\lambda_j^*)}V_{\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)=H_{k,j}^{(\lambda_k,\lambda_j^*)}A(t),$$

$$F_3=\varepsilon(r+\beta\Lambda(t))V_{\Upsilon}^k(t,\tilde{m}_k,\varepsilon)=(r+\beta\Lambda(t))\cdot\frac{B(t)}{\wp_k}$$

$$F_4=\frac{1}{2}\Big[\Lambda(t)\left(b_k(t)-b_j^*(t)\cdot\bar{k}_k\right)^2+\alpha_k^2(t)\sigma_k^2+\bar{k}_k^2\alpha_j^{*2}(t)\sigma_j^2$$
$$-2\rho_0\bar{k}_k a_k(t)\alpha_j^*(t)\sigma_k\sigma_j\Big]\Big[V_{\tilde{m}_k\tilde{m}_k}^k-\wp_k\left(g_{\tilde{m}_k}^k\right)^2\Big]$$
$$=\frac{1}{2}\Big[\Lambda(t)\left(b_k(t)-\bar{k}_k b_j^*(t)\right)^2+\alpha_k^2(t)\sigma_k^2+\bar{k}_k^2\alpha_j^{*2}(t)\sigma_j^2$$

$$-2\rho_0\bar{k}_k a_k(t)\alpha_j^*(t)\sigma_k\sigma_j\Big]\Big[-\wp_k\bar{A}^2(t)\Big]$$

$$F_5=\frac{1}{2}\varepsilon^2\Lambda(t)\left[V_{\varepsilon\varepsilon}^k(t,\tilde{m}_k,\varepsilon)-\wp_k\left(g_\varepsilon^k(t,\tilde{m}_k,\varepsilon)\right)^2\right]$$

$$=\frac{1}{2}\Lambda(t)\left[-\frac{B(t)}{\wp_k}-\frac{\bar{B}^2(t)}{\wp_k}\right],$$

$$F_6=\varepsilon\Lambda(t)\left(b_k(t)-\bar{k}_k b_j^*(t)\right)\left[V_{\tilde{m}_k\varepsilon}^k(t,\tilde{m}_k,\varepsilon)\right.$$

$$\left.-\wp_k g_\varepsilon^k(t,\tilde{m}_k,\varepsilon)\,g_{\tilde{m}_k}^k(t,\tilde{m}_k,\varepsilon)\right]$$

$$=\Lambda(t)\left(b_k(t)-\bar{k}_k b_j^*(t)\right)\left[-\bar{B}(t)\bar{A}(t)\right],$$

$$F_7=\ell E\Big\{V^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)$$

$$-\frac{\wp_k}{2}\left[g^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)\right]^2$$

$$+\wp_k g^k(t,\tilde{m}_k,\varepsilon)g^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)\Big\},$$

$$=\ell E\Big\{A(t)\left[\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\varepsilon_j^*(t)Z^j\right]+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}$$

$$-\frac{\wp_k}{2}\left[\bar{A}(t)\left(\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j\right)+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]^2$$

$$+\wp_k\left[\bar{A}(t)\tilde{m}_k+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]\Big[\bar{A}(t)\left(\tilde{m}_k-\alpha_k(t)Z^k\right.$$

$$\left.+\bar{k}_k a_j^*(t)Z^j\right)+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\Big]\Big\}$$

$$=\ell E\Big\{A(t)\left[\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j\right]+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}$$

$$-\frac{\wp_k}{2}\bar{A}^2(t)\left(\tilde{m}_k-\alpha_k(t)Z^k+\bar{k}_k\alpha_j^*(t)Z^j\right)\left(-\tilde{m}_k-\alpha_k(t)Z^k\right.$$

$$\left.+\bar{k}_k a_j^*(t)Z^j\right)+\wp_k\bar{A}(t)\tilde{m}_k\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]$$

$$+\frac{\wp_k}{2}\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]^2\Big\},$$

$$F_8=\ell_j E\Big\{V^k\left(t,\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)-\frac{\wp_k}{2}\left[g^k\left(t,\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j,\varepsilon\right)\right]^2$$

$$+\wp^k g^k(t,\tilde{m}_k,\varepsilon)g^k\left(t,\tilde{m}_k+\bar{k}_k\alpha_j^*Z^j,\varepsilon\right)\Big\}$$

$$=\ell_j E\Big\{A(t)\left(\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j\right)+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}$$

$$-\frac{\wp_k}{2}\left[\bar{A}(t)\left(\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j\right)+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]^2$$

$$+\wp_k\left[\bar{A}(t)\tilde{m}_k+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]$$

$$\cdot\left[\bar{A}(t)\left(\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j\right)+\frac{\bar{B}(t)}{\mathbb{Z}_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]$$

$$=\ell_j E\Big\{A(t)\left(\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j\right)+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}$$

$$-\frac{\wp_k}{2}\bar{A}^2(t)\left(\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j\right)\left(-\tilde{m}_k+\bar{k}_k\alpha_j^*(t)Z^j\right)$$

$$+\wp_k\bar{A}(t)\tilde{m}_k\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]+\frac{\wp_k}{2}\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]^2\Big\},$$

$$F_9=\ell_k E\Big\{V^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k,\varepsilon\right)-\frac{\wp_k}{2}\left[g\left(t,\tilde{m}_k-\alpha_k(t)Z^k,\varepsilon\right)\right]^2$$

$$+\wp_k g^k(t,\tilde{m}_k,\Upsilon)g^k\left(t,\tilde{m}_k-\alpha_k(t)Z^k,\varepsilon\right)\Big\}$$

$$=\ell_k E\Big\{A(t)\left(\tilde{m}_k-\alpha_k(t)Z^k\right)+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}$$

$$-\frac{\wp_k}{2}\left[\bar{A}(t)\left(\tilde{m}_k-\alpha_k(t)Z^k\right)+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]^2$$

$$+\wp_k\left[\bar{A}(t)\tilde{m}_k+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]$$

$$\cdot\left[\bar{A}(t)\left(\tilde{m}_k-\alpha_k(t)Z^k\right)+\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]\Big\}$$

$$=\ell_k E\Big\{A(t)\left(\tilde{m}_k-\alpha_k(t)Z^k\right)+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}$$

$$+\wp_k\bar{A}(t)\tilde{m}_k\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]+\frac{\wp_k}{2}\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]^2\Big\}$$

$$-\frac{\wp_k}{2}\bar{A}^2(t)\left(\tilde{m}_k-\alpha_k(t)Z^k\right)\left(-\tilde{m}_k-\alpha_k(t)Z^k\right),$$

$$F_{10}=-\left(\ell+\ell_j+\ell_k\right)E\left[V^k(t,\tilde{m}_k,\varepsilon)+\frac{\wp_k}{2}\left(g^k(t,\tilde{m}_k,\varepsilon)\right)^2\right]$$

$$=-\left(\ell+\ell_j+\ell_k\right)E\Big\{A(t)\tilde{m}_k+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}$$

$$+\frac{\wp_k}{2}\left[\bar{A}(t)\tilde{m}_k+\frac{\bar{B}(t)}{\wp_k}\ln s+\frac{\bar{C}_k(t)}{\wp_k}\right]^2\Big\}$$

$$=-\left(\ell+\ell_j+\ell_k\right)E\Big\{A(t)\tilde{m}_k+\frac{B(t)}{\wp_k}\ln\varepsilon+\frac{C_k(t)}{\wp_k}+\frac{\wp_k}{2}\bar{A}^2(t)\tilde{m}_k^2$$

$$+\wp_k\bar{A}(t)\tilde{m}_k\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]+\frac{\wp_k}{2}\left[\frac{\bar{B}(t)}{\wp_k}\ln\varepsilon+\frac{\bar{C}_k(t)}{\wp_k}\right]^2\Big\},$$

Based on the previous calculation, we can deduce:

$$F_7+F_8+F_9+F_{10}$$

$$=E\Big\{-(\ell+\ell_k)A(t)\alpha_k(t)Z^k+\left(\ell+\ell_j\right)\bar{k}_k\alpha_j^*(t)Z^jA(t)$$

$$-\frac{\wp_k}{2}\bar{A}^2(t)\left[\left(\ell+\ell_j\right)\left(\bar{k}_k\alpha_j^*(t)Z^j\right)^2+\left(\ell+\ell_k\right)\left(\alpha_k(t)Z^k\right)^2\right.$$

$$\left.-2\lambda\left(\alpha_k(t)Z^k\right)\left(\bar{k}_k\alpha_j^*(t)Z^j\right)\right]\Big\}$$

$$=\left[-(\ell+\ell_k)\alpha_k(t)\mu_k+\left(\lambda\!\!\!\!^-+\ell_j\right)\bar{k}_k\alpha_j^*(t)\mu_j\right]A(t)$$

$$-\frac{\wp_k}{2}\left(\ell+\ell_j\right)\bar{k}_k^{*2}\alpha_j^2(t)\xi_j^2\bar{A}^2(t)-\frac{\wp_k}{2}(\ell+\ell_k)\alpha_k^2(t)\xi_k^2\bar{A}^2(t)$$

$$+\gamma_k\lambda\alpha_k(t)\bar{k}_k\alpha_j^*(t)\left(\mu_i\mu_j+\rho\right)\bar{A}^2(t)$$

By substituting the estimates of the terms F_1-F_{10} into Eq.(16), the calculation yields:

$$\sup_{\lambda\!\!\!^-_k\in\Pi_k}\left\{A'(t)\tilde{m}_k+\frac{B'(t)}{\wp_k}\ln\Upsilon+\frac{C_k'(t)}{\wp_k}+\left[r\tilde{m}_k+\beta\Lambda(t)\left(b_k(t)-\bar{k}_k b_j^*(t)\right)\right.\right.$$

$$+\left[(\theta_k-\Upsilon_k)+\alpha_k(t)(1+\Upsilon_k)\right](\ell+\ell_k)\mu_k$$
$$-\left[\left(\theta_j-\Upsilon_j\right)+\alpha_j^*(t)\left(1+\Upsilon_j\right)-\alpha_j^*(t)\right]\left(\ell+\ell_j\right)\mu_j+\xi Y(t)\left(1-\bar{k}_k\right)\Big]A(t)$$
$$+\left(r+\beta\Lambda(t)\right)\cdot\frac{B(t)}{\wp_k}-\frac{1}{2}\Lambda(t)\cdot\frac{1}{\wp_k}\left[B(t)+\bar{B}^2(t)\right]$$
$$-\frac{1}{2}\Big[\Lambda(t)\left(b_k(t)-b_j^*(t)\bar{k}_k\right)^2+\alpha_k^2(t)\sigma_k^2+k_k^2\alpha_j^{*2}(t)\sigma_j^2$$
$$-2\rho_0\bar{k}_k\alpha_k(t)\alpha_j^*(t)\sigma_j\sigma_k\Big]\left[\wp_k\bar{A}^2(t)\right]-\Lambda(t)\left(b_k(t)-\bar{k}_k b_j^*(t)\right)$$
$$\cdot\bar{B}(t)\bar{A}(t)-\frac{\wp_k}{2}\left(\lambda+\lambda_j\right)\bar{k}_k^2\alpha_j^{*2}(t)\xi_j^2\bar{A}^2(t)-\frac{\wp_k}{2}\left(\ell+\ell_k\right)$$
$$\cdot\alpha_k^2(t)\xi_k^2\bar{A}^2(t)+\wp_k\lambda\alpha_k(t)\bar{k}_k\alpha_j^*(t)\left(\mu_k\mu_j+\rho\right)\bar{A}^2(t)\Big\}=0 \tag{17}$$

By computing the partial derivatives of $\alpha_k(t)$ and $b_k(t)$, the differential form of Eq.(17) can be reduced to the following form:

$$\wp_k\left[\sigma_k^2+(\ell+\ell_k)\xi_k^2\right]\bar{A}^2(t)\alpha_k(t)$$
$$=(\ell+\ell_k)\mu_k\Upsilon_k A(t)+\wp_k\left[\rho_0\bar{k}_k\sigma_j\sigma_k+\lambda\bar{k}_k\left(\mu_k\mu_j+\rho\right)\right]$$
$$\cdot\alpha_j^*(t)\bar{A}^2(t)\wp_k\Lambda(t)\bar{A}^2(t)b_k(t) \tag{18}$$
$$=\beta\Lambda(t)A(t)-\Lambda(t)\bar{A}(t)\bar{B}(t)+\wp_k\bar{k}_k b_j^*(t)\Lambda(t)\bar{A}^2(t) \tag{19}$$

Through analytical solution of the simultaneous Eqs.(18) - (19), we obtain:

$$\alpha_k^*(t)=\frac{(\ell+\ell_k)\mu_k\Upsilon_k A(t)\wp}{\wp_k\left[\sigma_k^2+(\ell+\ell_k)\xi_k^2\right]\bar{A}^2(t)}+\frac{\rho_0\sigma_k\sigma_j+\lambda\left(\mu_k\mu_j+\rho\right)}{\sigma_k^2+(\ell+\ell_k)\xi_k^2}\bar{k}_k\alpha_j^*(t) \tag{20}$$

$$b_k^*(t)=\frac{\beta A(t)}{\wp_k\bar{A}^2(t)}-\frac{\bar{B}(t)}{\wp_k\bar{A}(t)}+\bar{k}_k b_j^*(t) \tag{21}$$

When $\bar{k}_k$=0, the corresponding policy is the standard policy.

By substituting Eqs.(20) and (21) into Eqs.(17) and (18), one obtains:

$$A'(t)+\wp A(t)=0 \tag{22}$$

$$\frac{B'(t)}{\lambda_k}=0 \tag{23}$$

By applying the method of separation of variables under the constraint of initial conditions, we derive the solution:

$$A(t)=e^{r(T-t)} \tag{24}$$

$$B(t)=C_B=0 \tag{25}$$

Substituting $g^k(t,\tilde{m}_k,\varepsilon)$ into Eq.(13), then:

$$\left(\bar{A}'(t)+r\bar{A}(t)\right)\tilde{m}_k+\frac{\beta(t)}{\wp_k}\ln\varepsilon+\left(H_{k,j}^{\hat{\lambda}_k^*,\hat{\lambda}_j^*}-r\tilde{x}_k\right)\bar{A}(t)+\left(\wp+\beta\Lambda(t)\right)\frac{\bar{\beta}(t)}{\wp_k}$$
$$-\frac{1}{2\wp_k}\Lambda(t)\bar{B}(t)-(\ell+\ell_k)\bar{A}(t)\alpha_k^*(t)\mu_k+(\ell+\ell_k)\bar{A}(t)\bar{k}_k\alpha_j^*(t)\mu_j=0 \tag{26}$$

Assuming that:

$$\bar{A}'(t)+\wp\bar{A}(t)=0 \tag{27}$$

$$\frac{B'(t)}{\wp_k}=0 \tag{28}$$

The solution is:

$$\bar{A}(t)=e^{r(T-t)} \tag{29}$$

$$\bar{B}(t)=\mathrm{C}=0 \tag{30}$$

Substituting Eqs.(24)-(25) and Eqs.(29)-(30) into Eqs.(20)-(21), the following is obtained:

$$\begin{cases}\alpha_k^*(t)=\dfrac{(\ell+\ell_k)\mu_k\eta_k A(t)}{\wp_k\left[\sigma_k^2+(\ell+\ell_k)\xi_k^2\right]\bar{A}^2(t)}+\dfrac{\rho_0\sigma_k\sigma_j+\lambda\left(\mu_k\mu_j+\rho\right)}{\sigma_k^2+(\ell+\ell_k)\xi_k^2}\bar{k}_k\alpha_j^*(t)\\ b_1^*(t)=\dfrac{\beta A(t)}{\wp_k\bar{A}^2(t)}-\dfrac{\bar{B}(t)}{\wp_k\bar{A}(t)}+\bar{k}_k b_j^*(t)\end{cases}$$

Note that:

$$A(t)=\bar{A}(t)=e^{r(T-t)} \tag{31}$$

$$B(t)=\bar{B}(t)=C_B \tag{32}$$

where is the constant of the function. This is the basic wealth level of the pension account in a steady state. Substituting Eqs.(31)-(32) into Eqs.(20)-(21) and iterating the system yields:

$$\alpha_1^*(t)=\frac{(\ell+\ell_1)\mu_1\Upsilon_1}{\wp_1\left[\sigma_1^2+(\ell+\ell_1)\xi_1^2\right]e^{r(T-t)}}+\frac{\rho_0\sigma_1\sigma_2+\lambda(\mu_1\mu_2+\rho)}{\sigma_1^2+(\ell+\ell_1)\xi_1^2}\bar{k}_1\alpha_2^*(t) \tag{33}$$

$$\alpha_2^*(t)=\frac{(\ell+\ell_2)\mu_2\Upsilon_2}{\wp_2\left[\sigma_2^2+(\ell+\ell_2)\xi_2^2\right]e^{r(T-t)}}+\frac{\rho_0\sigma_1\sigma_2+\lambda(\mu_1\mu_2+\rho)}{\sigma_2^2+(\ell+\ell_2)\xi_2^2}\bar{k}_2\alpha_1^*(t) \tag{34}$$

$$b_1^*(t)=\frac{\beta}{\wp_1 e^{r(T-t)}}-\frac{C_B}{r_1 e^{r(T-t)}}+\bar{k}_1 b_2^*(t) \tag{35}$$

$$b_2^*(t)=\frac{\beta}{\wp_2 e^{r(T-t)}}-\frac{C_B}{r_2 e^{r(T-t)}}+\bar{k}_2 b_1^*(t) \tag{36}$$

In the solution for the DC pension model, Eqs.(33) and (34) are interrelated; Eq.(34) is the important intermediate expression. After substituting Eq.(34) into (33), algebraic operations are performed and stochastic processes are used to simplify and eliminate terms, ultimately obtaining a new expression to assist in solving for Nash equilibrium strategies and optimal investment solutions:

$$\alpha_1^*(t)=\frac{(\ell+\ell_1)\mu_1\Upsilon_1}{\wp_1\left[\sigma_1^2+(\ell+\ell_1)\xi_1^2\right]e^{r(T-t)}}+\frac{\rho_0\sigma_1\sigma_2+\ell(\mu_1\mu_2+\rho)}{\sigma_1^2+(\ell+\ell_1)\xi_1^2}\bar{k}_1$$
$$\cdot\left[\frac{(\ell+\ell_2)\mu_2\Upsilon_2}{\wp_2\left[\sigma_2^2+(\ell+\ell_2)\xi_2^2\right]e^{r(T-t)}}+\frac{\rho_0\sigma_1\sigma_2+\ell(\mu_1\mu_2+\rho)}{\sigma_2^2+(\ell+\ell_2)\xi_2^2}\bar{k}_2\alpha_1^*(t)\right]$$

Following the derivation steps, we arrive at the solution:

$$\alpha_1^*(t)=\frac{\Gamma_1}{\Gamma_3},$$

$$\alpha_2^*(t)=\frac{\Gamma_2}{\Gamma_3}$$

Here:

$$\Gamma_1=\wp_2(\ell+\ell_1)\mu_1\Upsilon_1\left[\sigma_2^2+(\ell+\ell_2)\xi_2^2\right]$$

$$+\wp_1\bar{k}_1(\ell+\ell_1)\mu_2\Upsilon_2\left[\rho_0\sigma_1\sigma_2+\ell(\mu_1\mu_2+\rho)\right]$$

$$\Gamma_2=\wp_1(\ell+\ell_2)\mu_2\Upsilon_2\left[\sigma_1^2+(\ell+\ell_1)\xi_1^2\right]$$

$$+\wp_2\bar{k}_2(\ell+\ell_1)\mu_1\Upsilon_1\left[\rho_0\sigma_1\sigma_2+\ell(\mu_1\mu_2+\rho)\right]$$

$$\Gamma_3=\wp_1\wp_2e^{r(T-t)}\left[\left(\sigma_1^2+(\ell+\ell_1)\zeta_1^2\right)\left(\sigma_2^2+(\ell+\ell_2)\xi_2^2\right)\right.$$

$$\left.-\bar{k}_1\bar{k}_2\left(\rho_0\sigma_1\sigma_2+\lambda(\mu_1\mu_2+\rho)\right)^2\right]$$

There is a tightly coupled iterative relationship between Eqs.(35) and (36). During each iteration cycle, intermediate results under the current parameters are computed throughout Eq.(36), which incorporates the dynamic valuations of key variables, the pension wealth accumulation and reinsurance returns. Propagate the calculation results of Eq.(36) as inputs into Eq.(35), then perform iterative recalibration to update the posterior parameters. Through successive iterations, upon convergence the algorithm yields:

$$b_1^*(t)=\frac{\beta-C_B}{\wp_1e^{r(T-t)}}+\bar{k}_1\left[\frac{\beta-C_B}{\wp_2e^{r(T-t)}}+k_2b_1^*(t)\right],$$

$$b_2^*(t)=\frac{\beta-C_B}{\wp_1e^{r(T-t)}}+\bar{k}_2b_1^*(t),$$

Where the solution is as follows:

$$b_1^*(t)=\frac{(\beta-C_B)(\wp_1+k_1\wp_1)}{(1-\bar{k}_1\bar{k}_2)\wp_1\wp_2e^{r(T-t)}}$$

$$b_2^*(t)=\frac{(\beta-C_B)(\wp_1+k_2\wp_2)}{(1-\bar{k}_1\bar{k}_2)\wp_1\wp_2e^{r(T-t)}}$$

IV. Discussion and Conclusions

This study comprehensively analyzes the reinsurance and investment strategies in DC pension systems, and proposes a methodology to develop robust operational strategies in stochastic financial market conditions with volatility clustering. The primary theoretical contribution of this work is to build the model of a Nash equilibrium optimization, in which the risk asset follows the Heston model, and the reinsurance surplus exhibits characteristics of jump-diffusion. By maximizing the utility of the index, an optimal strategy is derived to enhance the accumulation of terminal wealth and reduce discontinuous risks at the same time.

We have addressed limitations in previous research in terms of model construction, more rigorously considering the uncertainties of financial markets. Compared with models in earlier literature, our model has a few additional advantages: it is more concise in form, abandons complex and redundant structures, and is easier to understand and apply. Compared with traditional mean variance methods, this strategy has higher stability and can establish a formal link between the volatility smile effect and sustainability of DC pension plan, providing a computationally feasible solution for the jump-diffusion adaptation. In terms of generalizability, the model is not excessively limited by specific market environments or conditions, and can thus function in a wider range of financial market scenarios. In terms of practicality, the model strategy applies to realistic market situations, has stronger operability, can effectively guide investors' practical activities, helps investors make more reasonable investment decisions in complex markets, and aids asset preservation and appreciation for the DC pension systems.

The results of this study demonstrate strong practicality. As countries are confronted with aging populations and mounting pension payment pressures, pension investment management institutions can apply scientific investment and reinsurance strategies leveraging the game-theoretic models developed in this work. By optimizing the asset allocation, reducing the operational risks, and improving the pension stability, this approach can enhance the sustainability of the pension system and help ensure the quality of life for the retirees. Concurrently, given the globalization of financial markets and ongoing financial innovations, various financial institutions and investors might adopt these model's strategies when pursuing pension-related investments. These may enable them to better navigate volatile market conditions and strengthen competitiveness. For instance, insurance companies might utilize this model to determine the optimal reinsurance proportions and pricing for pension-linked reinsurance operations, achieving more efficient risk transfer and control. Moreover, investment institutions could apply the asset allocation strategies of the model to select superior investment targets and timing decisions, thereby boosting portfolio returns.

Although the investment decision model proposed in this study accounts for market uncertainty, real financial markets may exhibit even more complex behavior, such as varying market sentiment, policy uncertainties, and other uncertain factors. In future research, these topics might be explored to improve the present DC pension investment decision model, making it more applicable to complex market situations.

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