# Total Coloring for Several Families of 4-Regular Circulant Graphs

Wenzheng An, Chunling Tong, Senyuan Su, Xingli Xu

***Abstract*—4-Regular circulant graphs are among the most important classes of graphs, and their total coloring has attracted much attention in recent years. Nevertheless, determining the total chromatic numbers for most 4-regular circulant graphs remains an open problem, despite considerable efforts. In this paper, we address this issue and determine the total chromatic numbers for several families of 4-regular circulant graphs.**

***Index Terms*—total coloring, total chromatic number, regular graph, circulant graph**

## I. INTRODUCTION

LET $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A total k-coloring of a graph $G$ is a mapping $\sigma: V(G) \cup E(G) \rightarrow \{1,2,\cdots,k\}$, such that no two adjacent or incident elements of $V(G) \cup E(G)$ are assigned the same color. The smallest number of colors needed for such a coloring of $G$ is called the total chromatic number, denoted as $\chi''(G)$. Determining total chromatic number is known to be NP-complete [1], and remains NP-hard even for r-regular bipartite graphs with $r \geq 3$ [2].

There is a long-standing total coloring conjecture independently proposed by Behzad [3] and Vizing [4]. It states that the total chromatic number $\chi''(G)$ satisfies $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$ for a simple graph G, where $\Delta(G)$ is the maximum degree of G. The conjecture implies that for any simple graph G, $\chi''(G)$ can only take one of two possible values: $\Delta(G)+1$ or $\Delta(G)+2$. Usually, a graph with $\chi''(G)=\Delta(G)+1$ is referred to as Type I, while a graph with $\chi''(G)=\Delta(G)+2$ is referred to as Type II.

The conjecture has been verified by a number of graphs, and the exact values of total chromatic number for certain classes of graphs have been determined in previous studies [5]–[9]. However, the total chromatic numbers for most 4-regular circulant graphs remain unresolved even after many efforts. In this paper, we investigate the total coloring of 4-regular circulant graphs, aiming to determine their total chromatic numbers as comprehensively as possible. The paper is organized as follows. Section II provides a brief overview of 4-regular circulant graphs and their total coloring. Section III presents our main findings, where the total chromatic numbers for several families of 4-regular circulant graphs are given. Finally, Section IV is the conclusion.

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## II. A REVIEW OF 4-REGULAR CIRCULANT GRAPHS AND THEIR TOTAL COLORING

A 4-regular circulant graph $C_n(d_1,d_2)$ is the graph that has vertex set $V=\{v_0,v_1,\cdots,v_{n-1}\}$ and edge set $E=\bigcup_{i=1}^{2} E_i$ with $E_i=e_0^i,e_1^i,\cdots,e_{n-1}^i$ and $e_j^i=v_j v_{(j+d_i)\bmod n}$, where $1 \leq d_1 < d_2 \leq \lfloor\frac{n-1}{2}\rfloor$ and indices of the vertices are considered modulo n.

4-regular circulant graphs are among the most important classes of graphs, and many articles have been devoted to the study of their coloring, especially total coloring [10]–[17]. Studies in this area have thrown up some interesting results. According to [5], a 4-regular circulant graph must belong to either Type I or Type II, i.e. its total chromatic number is 5 or 6. Campos and de Mello proved that $C_n(1,2)$ is Type I, except for $C_7(1,2)$ which is Type II [12]. Tong et al. demonstrated similar results, proving that $C_n(1,4)$ is Type I, except for $C_{13}(1,4)$ which falls into Type II [13]. Khennoufa and Togni refined these findings by showing that $C_{5p}(1,k)$ for k (mod 5) = 2,3 with $p \geq 1$ and $k < \frac{5p}{2}$, and $C_{6p}(1,k)$ for k (mod 3) $\neq 0$ with $p \geq 3$ and $k < 3p$ are Type I [14]. Nigro et al. extended these results, proving that $C_{3p}(1,3)$ with $p \geq 3$ but $p \neq 4$, $C_{3tp}(1,p)$ with $t \geq 1$ and p (mod 3) = 0, and $C_{(8p+6q)k}(3,2k)$ with $k \geq 1$, $p,q \geq 0$ and $p+q \neq 0$ are Type I [15], [16]. Navaneeth et al. further expanded the classification, identifying that $C_{5p}(1,k)$ for k (mod 5) = 1,4 with $p \geq 1$ and $k < \frac{5p}{2}$, $C_{9p}(1,k)$ for $\frac{9p}{\gcd(9p,k)} = 3s$ ($s \in N$) with $2 \leq k < \frac{9p}{2}$, $C_{3p}(a,b)$ for $\frac{3p}{\gcd(3p,b)} = 3s$ ($s \in N$) with p (mod 2) = 1 and $\gcd(a,b)=1$, and $C_{6p}(a,b)$ for a, b(mod 3) $\neq 0$ with p(mod 2) = 0 or with p (mod 2) = 1 and $\gcd(a,b)=1$ are Type I [17].

Table I presents all the families of Type I 4-regular circulant graphs identified in previous studies. By contrast, Type II 4-regular circulant graphs are much less common, and only a few instances have been confirmed so far, including $C_7(1,2)$, $C_{12}(1,3)$, and $C_{13}(1,4)$.

TABLE I
FAMILIES OF TYPE I 4-REGULAR CIRCULANT GRAPHS OBTAINED IN PREVIOUS STUDIES

| | |
|---|---|
| $C_n(1,2)$ | $n \geq 5$ but $n \neq 7$ |
| $C_n(1,4)$ | $n \geq 7$ but $n \neq 13$ |
| $C_{3p}(1,3)$ | $p \geq 3$ but $p \neq 4$ |
| $C_{3tp}(1,p)$ | $t \geq 1,\ p\,(\text{mod } 3) = 0$ |
| $C_{5p}(1,k)$ | $p \geq 1,\ k\,(\text{mod } 5) \neq 0$ |
| $C_{6p}(1,k)$ | $p \geq 3,\ k\,(\text{mod } 3) \neq 0$ |
| $C_{9p}(1,k)$ | $p \geq 1,\ \frac{9p}{gcd(9p,\ k)} = 3s(s \in N)$ |
| $C_{(8p+6q)k}(3,2k)$ | $k \geq 1, p, q \geq 0$ and $p + q \neq 0$ |
| $C_{3p}(a,b)$ | $\frac{3p}{gcd(3p,\ b)} = 3s(s \in N,$ $p\,(\text{mod } 2) = 1 \wedge \gcd(a,b) = 1)$ |
| $C_{6p}(a,b)$ | $a, b\ (\text{mod } 3) \neq 0$ ($p$ (mod 2) $= 0$ or $p$ (mod 2) $= 1 \wedge \gcd(a,b) = 1)$ |

## III. Total Coloring for Several Families of 4-regular Circulant Graphs

As stated above, researchers have determined the total chromatic numbers for some special families of 4-regular circulant graphs. We now try to find the total chromatic numbers for a wider range of 4-regular circulant graphs. Specifically, our study focuses on the graphs $C_{tp}(1, t\mu + \lambda)$ with $t,\ p \geq 1$ and $\mu \geq 0$. To achieve this, we employ a computational approach to systematically search for valid total coloring schemes. Subsequently, we rigorously prove the correctness of the identified coloring schemes, ensuring their validity and reliability.

For simplicity, we use $(i_1 i_2 \cdots i_t)^p$ to represent $\underbrace{i_1 i_2 \ldots i_t \cdots i_1 i_2 \ldots i_t}_{p}$, where $i_1, i_2, \cdots, i_t \in \{1,2,3,4,5\}$. For example, $(24351)^2 = 2435124351$. Let

$$V = \{v_i : 0 \leq i \leq tp - 1\},$$
$$E_1 = \{v_i v_{(i+1) mod\, n} : 0 \leq i \leq tp - 1\},$$
$$E_2 = \{v_i v_{(i+t\mu+\lambda) mod\, n} : 0 \leq i \leq tp - 1\}.$$

Next, we will provide constructive proofs for Theorems 1 through 10 using these notations.

**Theorem 1**. $\chi''(C_{8p}(1,8\mu + \lambda)) = 5$ for $\lambda = 2,6,7$.

**Proof:** We construct $\sigma(C_{8p}(1, 8\mu + \lambda))$ for $\lambda = 2,6,7$ as follows.

$$\sigma(V) = \begin{cases} (43542532)^p, & \lambda = 2,6, \\ (23543245)^p, & \lambda = 7, \end{cases}$$

$$\sigma(E_1) = \begin{cases} (12131415)^p, & \lambda = 2,6, \\ (12121313)^p, & \lambda = 7, \end{cases}$$

$$\sigma(E_2) = \begin{cases} (35425324)^p, & \lambda = 2, \\ (24354253)^p, & \lambda = 6, \\ (45435452)^p, & \lambda = 7. \end{cases}$$

Fig.1(1)-(2) show $\sigma(C_8(1,2))$ and $\sigma(C_{16}(1,6))$.

To demonstrate that the above construction satisfies the requirements of a total 5-coloring, we examine the colors assigned to all adjacent or incident elements of $C_{8p}(1,8\mu + \lambda)$ for $\lambda = 2,6,7$.

First, consider the vertex $v_i$. Its adjacent vertices are $v_{i-(8\mu+\lambda)}, v_{i-1}, v_{i+1}$ and $v_{i+(8\mu+\lambda)}$. The construction indicates $\sigma(v_i) \neq \sigma(v_{i-(8\mu+\lambda)}), \sigma(v_{i-1}), \sigma(v_{i+1}), \sigma(v_{i+(8\mu+\lambda)})$, which means that two adjacent vertices receive different colors.

Second, consider the edges $v_i v_{i+1}$ and $v_i v_{i+(8\mu+\lambda)}$. The adjacent edges of $v_i v_{i+1}$ are $v_{i-(8\mu+\lambda)} v_i, v_{i-1} v_i, v_i v_{i+(8\mu+\lambda)}, v_{i-(8\mu+\lambda)+1} v_{i+1}, v_{i+1} v_{i+2}$ and $v_{i+1} v_{i+(8\mu+\lambda)+1}$, and the adjacent edges of $v_i v_{i+(8\mu+\lambda)}$ are $v_{i-(8\mu+\lambda)} v_i, v_{i-1} v_i, v_i v_{i+1}, v_{i+(8\mu+\lambda)-1} v_{i+(8\mu+\lambda)}, v_{i+(8\mu+\lambda)} v_{i+(8\mu+\lambda)+1}$ and $v_{i+(8\mu+\lambda)} v_{i+2(8\mu+\lambda)}$. The construction indicates $\sigma(v_i v_{i+1}) \neq \sigma(v_{i-(8\mu+\lambda)} v_i), \sigma(v_{i-1} v_i), \sigma(v_i v_{i+(8\mu+\lambda)}), \sigma(v_{i-(8\mu+\lambda)+1} v_{i+1}), \sigma(v_{i+1} v_{i+2}), \sigma(v_{i+1} v_{i+(8\mu+\lambda)+1})$, and $\sigma(v_i v_{i+(8\mu+\lambda)}) \neq \sigma(v_{i-(8\mu+\lambda)} v_i), \sigma(v_{i-1} v_i), \sigma(v_i v_{i+1}), \sigma(v_{i+(8\mu+\lambda)-1} v_{i+(8\mu+\lambda)}), \sigma(v_{i+(8\mu+\lambda)} v_{i+(8\mu+\lambda)+1}), \sigma(v_{i+(8\mu+\lambda)} v_{i+2(8\mu+\lambda)})$, which means that two adjacent edges receive different colors.

Third, the edges incident to the vertex $v_i$ are $v_{i-(8\mu+\lambda)} v_i, v_{i-1} v_i, v_i v_{i+1}$ and $v_i v_{i+(8\mu+\lambda)}$. The construction indicates $\sigma(v_i) \neq \sigma(v_{i-(8\mu+\lambda)} v_i), \sigma(v_{i-1} v_i), \sigma(v_i v_{i+1}), \sigma(v_i v_{i+(8\mu+\lambda)})$, which means that a vertex receives a different color from its incident edges.

Therefore, the above construction fufils all the requirements of a total 5-coloring of $C_{8p}(1,8\mu + \lambda)$ for $\lambda = 2,6,7$. We then have $\chi''(C_{8p}(1,8\mu + \lambda)) \leq 5$ for $\lambda = 2,6,7$. On the other hand, there is $\chi''(C_{8p}(1,8\mu + \lambda)) \geq 5$. Hence, $\chi''(C_{8p}(1,8\mu + \lambda)) = 5$ for $\lambda = 2,6,7$.

**Theorem 2.** $\chi''(C_{11p}(1,11\mu + \lambda)) = 5$ for $\lambda = 2,3,4,5,6,8,9$.

**Proof:** We construct $\sigma(C_{11p}(1,11\mu + \lambda))$ for $\lambda = 2,3,4,5,6,8,9$ as follows.

$$\sigma(V) = \begin{cases} (34534534512)^p, & \lambda = 2,9, \\ (25354543431)^p, & \lambda = 3, \\ (23431251231)^p, & \lambda = 4, \\ (25435343541)^p, & \lambda = 5, \\ (24534353451)^p, & \lambda = 6, \\ (24345453531)^p, & \lambda = 8, \end{cases}$$

$$\sigma(E_1) = \begin{cases} (12121212345)^p, & \lambda = 2,9, \\ (12121212123)^p, & \lambda = 3,5,6,8, \\ (12123123123)^p, & \lambda = 4, \end{cases}$$

$$\sigma(E_2) = \begin{cases} (45345345123)^p, & \lambda = 2, \\ (43435354545)^p, & \lambda = 3, \\ (45345445455)^p, & \lambda = 4, \\ (43543554354)^p, & \lambda = 5, \\ (45345534534)^p, & \lambda = 6, \\ (45453534345)^p, & \lambda = 8, \\ (23453453451)^p, & \lambda = 9. \end{cases}$$

Fig.1(3)-(4) show $\sigma(C_{11}(1,2))$ and $\sigma(C_{11}(1,3))$.

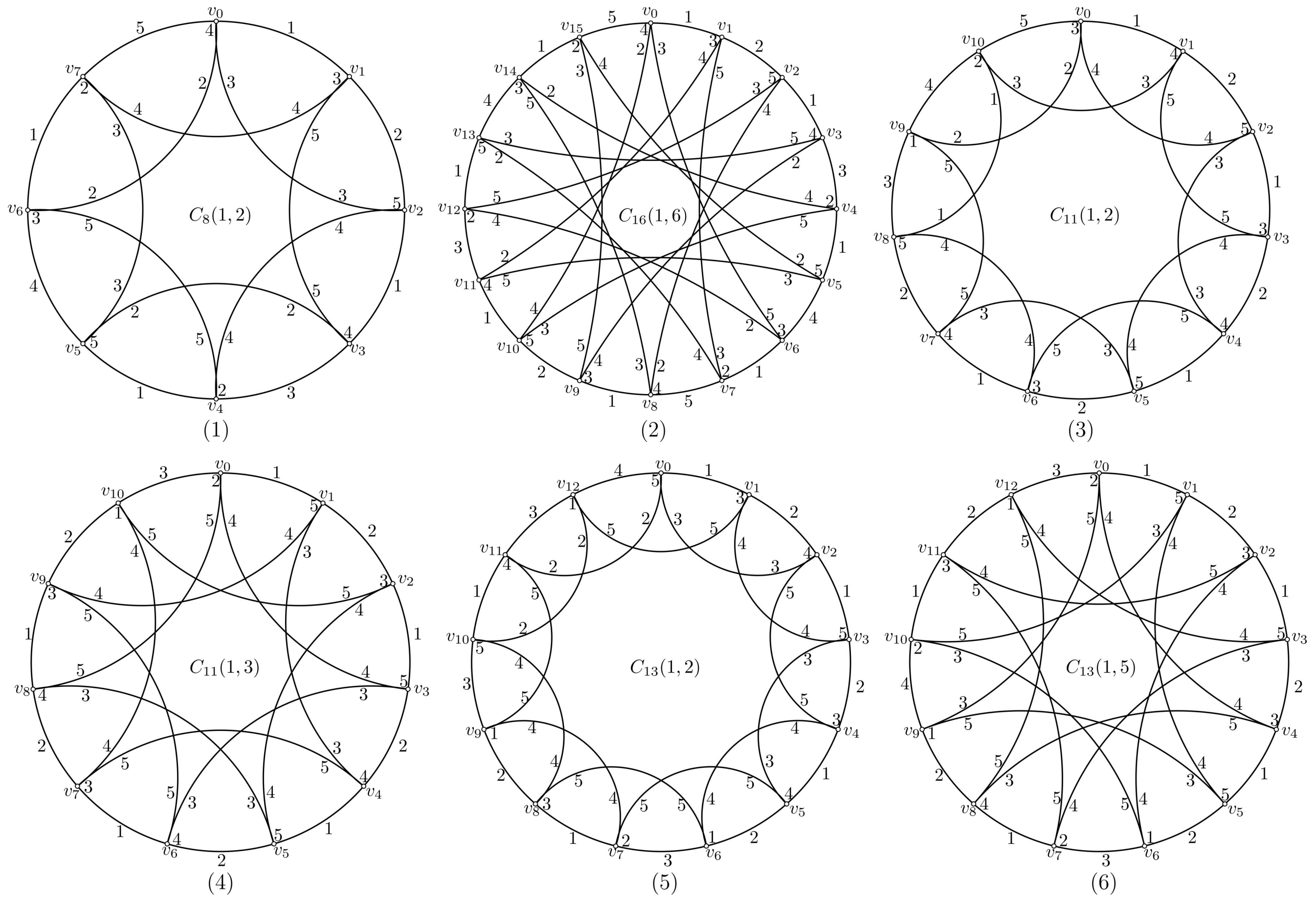


Fig. 1. $\sigma(C_8(1,2))$, $\sigma(C_{16}(1,6))$, $\sigma(C_{11}(1,2))$, $\sigma(C_{11}(1,3))$, $\sigma(C_{13}(1,2))$ and $\sigma(C_{13}(1,5))$

Similarly to the proof of Theorem 1, we can verify that the construction fulfills the requirements of a total 5-coloring by examining the colors assigned to all adjacent or incident elements of $C_{11p}(1,11\mu+\lambda)$ for $\lambda=2,3,4,5,6,8,9$. First, the vertices adjacent to $v_i$ are $v_{i-(11\mu+\lambda)}, v_{i-1}, v_{i+1}$ and $v_{i+(11\mu+\lambda)}$. It holds that $\sigma(v_i)\neq\sigma(v_{i-(11\mu+\lambda)}), \sigma(v_{i-1}), \sigma(v_{i+1}), \sigma(v_{i+(11\mu+\lambda)})$, ensuring distinct colors for adjacent vertices. Second, the edges adjacent to $v_iv_{i+1}$ are $v_{i-(11\mu+\lambda)}v_i, v_{i-1}v_i, v_iv_{i+(11\mu+\lambda)}, v_{i-(11\mu+\lambda)+1}v_{i+1}, v_{i+1}v_{i+2}$ and $v_{i+1}v_{i+(11\mu+\lambda)+1}$, and the edges adjacent to $v_iv_{i+(11\mu+\lambda)}$ are $v_{i-(11\mu+\lambda)}v_i, v_{i-1}v_i, v_iv_{i+1}, v_{i+(11\mu+\lambda)-1}v_{i+(11\mu+\lambda)}, v_{i+(11\mu+\lambda)}v_{i+(11\mu+\lambda)+1}$ and $v_{i+(11\mu+\lambda)}v_{i+2(11\mu+\lambda)}$. It holds that $\sigma(v_iv_{i+1})\neq\sigma(v_{i-(11\mu+\lambda)}v_i), \sigma(v_{i-1}v_i), \sigma(v_iv_{i+(11\mu+\lambda)}), \sigma(v_{i-(11\mu+\lambda)+1}v_{i+1}), \sigma(v_{i+1}v_{i+2}), \sigma(v_{i+1}v_{i+(11\mu+\lambda)+1})$, and $\sigma(v_iv_{i+(11\mu+\lambda)})\neq\sigma(v_{i-(11\mu+\lambda)}v_i), \sigma(v_{i-1}v_i), \sigma(v_iv_{i+1}), \sigma(v_{i+(11\mu+\lambda)-1}v_{i+(11\mu+\lambda)}), \sigma(v_{i+(11\mu+\lambda)}v_{i+(11\mu+\lambda)+1}), \sigma(v_{i+(11\mu+\lambda)}v_{i+2(11\mu+\lambda)})$, guaranteeing different colors for all adjacent edges. Third, the edges incident to the vertex $v_i$ are $v_{i-(11\mu+\lambda)}v_i, v_{i-1}v_i, v_iv_{i+1}$ and $v_iv_{i+(11\mu+\lambda)}$. It holds that $\sigma(v_i)\neq\sigma(v_{i-(11\mu+\lambda)}v_i), \sigma(v_{i-1}v_i), \sigma(v_iv_{i+1}), \sigma(v_iv_{i+(11\mu+\lambda)})$, thus each vertex receives a color different from its incident edges.

So, for $\lambda=2,3,4,5,6,8,9$, $C_{11p}(1,11\mu+\lambda)$ admits a total 5-coloring based on the above construction. We then obtain $\chi''(C_{11p}(1,11\mu+\lambda))\leq 5$ for $\lambda=2,3,4,5,6,8,9$. On the other hand, there is $\chi''(C_{11p}(1,11\mu+\lambda))\geq 5$. Hence, $\chi''(C_{11p}(1,11\mu+\lambda))=5$ for $\lambda=2,3,4,5,6,8,9$.

**Theorem 3.** $\chi''(C_{13p}(1,13\mu+\lambda))=5$ for $\lambda=2,5,6$.

**Proof:** We construct $\sigma(C_{13p}(1,13\mu+\lambda))$ for $\lambda=2,5,6$ as follows.

$$\sigma(V)=\begin{cases}(5345341231541)^p, & \lambda=2,\\ (2535351241231)^p, & \lambda=5,\\ (2353424542153)^p, & \lambda=6,\end{cases}$$

$$\sigma(E_1)=\begin{cases}(1212123123134)^p, & \lambda=2,\\ (1212123124123)^p, & \lambda=5,\\ (1212131213214)^p, & \lambda=6,\end{cases}$$

$$\sigma(E_2)=\begin{cases}(3453455445225)^p, & \lambda=2,\\ (4443535553544)^p, & \lambda=5,\\ (5435542354435)^p, & \lambda=6.\end{cases}$$

Fig.1(5)-(6) show $\sigma(C_{13}(1,2))$ and $\sigma(C_{13}(1,5))$.

Following a similar approach to the proofs of Theorems 1 and 2, we can also verify that the above construction is a total 5-coloring of $C_{13p}(1,13\mu+\lambda)$ for $\lambda=2,5,6$. Consequently, we can establish that $\chi''(C_{13p}(1,13\mu+\lambda))=5$ for $\lambda=2,5,6$.

The method employed in the proofs of Theorems 1 and 2 can be used to prove the subsequent theorems. Therefore, we will only provide the construction $\sigma(C_{tp}(1,t\mu+\lambda))$ for Theorems 4 through 10.

**Theorem 4.** $\chi''(C_{14p}(1,14\mu+4))=5$.

**Proof:** We construct $\sigma(C_{14p}(1,14\mu+4)$ as follows.

$$\sigma(V)=(25434352143431)^p,$$

$$\sigma(E_1) = (12121213212124)^p,$$
$$\sigma(E_2) = (34345545435355)^p.$$

Fig.2(1) shows $\sigma(C_{14}(1,4))$.

**Theorem 5.** $\chi''(C_{15p}(1,15\mu+5)) = 5$.

**Proof:** We construct $\sigma(C_{15p}(1,15\mu+5))$ as follows.

$$\sigma(V) = (253545454543431)^p,$$
$$\sigma(E_1) = (121212121212123)^p,$$
$$\sigma(E_2) = (434333535454545)^p.$$

Fig.2(2) shows $\sigma(C_{15}(1,5))$.

**Theorem 6.** $\chi''(C_{16p}(1,16\mu+4)) = 5$.

**Proof:** We construct $\sigma(C_{16p}(1,16\mu+4))$ as follows.

$$\sigma(V) = (2534345253134145)^p,$$
$$\sigma(E_1) = (1212121314515314)^p,$$
$$\sigma(E_2) = (5345453422223453)^p.$$

Fig.2(3) shows $\sigma(C_{16}(1,4))$.

**Theorem 7.** $\chi''(C_{17p}(1,17\mu+\lambda)) = 5$ for $\lambda = 2,4,5,8$.

**Proof:** We construct $\sigma(C_{17p}(1,17\mu+\lambda))$ for $\lambda = 2,4,5,8$ as follows.

$$\sigma(V) = \begin{cases} (34534534534534512)^p, & \lambda = 2, \\ (24535345453434531)^p, & \lambda = 4, \\ (25353545454343431)^p, & \lambda = 5, \\ (25435435343543541)^p, & \lambda = 8, \end{cases}$$

$$\sigma(E_1) = \begin{cases} (12121212121212345)^p, & \lambda = 2, \\ (12121212121212123)^p, & \lambda = 4,5,8, \end{cases}$$

$$\sigma(E_2) = \begin{cases} (45345345345345123)^p, & \lambda = 2, \\ (45343453534545345)^p, & \lambda = 4, \\ (43434353535454545)^p, & \lambda = 5, \\ (43543543554354354)^p, & \lambda = 8. \end{cases}$$

Fig.2(4)-(5) show $\sigma(C_{17}(1,2))$ and $\sigma(C_{17}(1,4))$.

**Theorem 8.** $\chi''(C_{19p}(1,19\mu+\lambda)) = 5$ for $\lambda = 2,4,5,6,9$.

**Proof:** We construct $\sigma(C_{19p}(1,19\mu+\lambda))$ for $\lambda = 2,4,5,6,9$ as follows.

$$(V) = \begin{cases} (5345345345341231541)^p, & \lambda = 2, \\ (2545345343121535321)^p, & \lambda = 4, \\ (2454345354534253231)^p, & \lambda = 5, \\ (2534124125312341251)^p, & \lambda = 6, \\ (4354353435243542152)^p, & \lambda = 9, \end{cases}$$

$$\sigma(E_1) = \begin{cases} (1212121212123123134)^p, & \lambda = 2, \\ (1212121212342142145)^p, & \lambda = 4, \\ (1212121212123121423)^p, & \lambda = 5, \\ (1212312341241253123)^p, & \lambda = 6, \\ (1212121213121213213)^p, & \lambda = 9, \end{cases}$$

$$\sigma(E_2) = \begin{cases} (3453453453455445225)^p, & \lambda = 2, \\ (4334554534515323453)^p, & \lambda = 4, \\ (5343534543451545354)^p, & \lambda = 5, \\ (5453553512435434544)^p, & \lambda = 6, \\ (2435435544554354435)^p, & \lambda = 9. \end{cases}$$

Fig.2(6), 3(1) show $\sigma(C_{19}(1,2))$ and $\sigma(C_{19}(1,4))$.

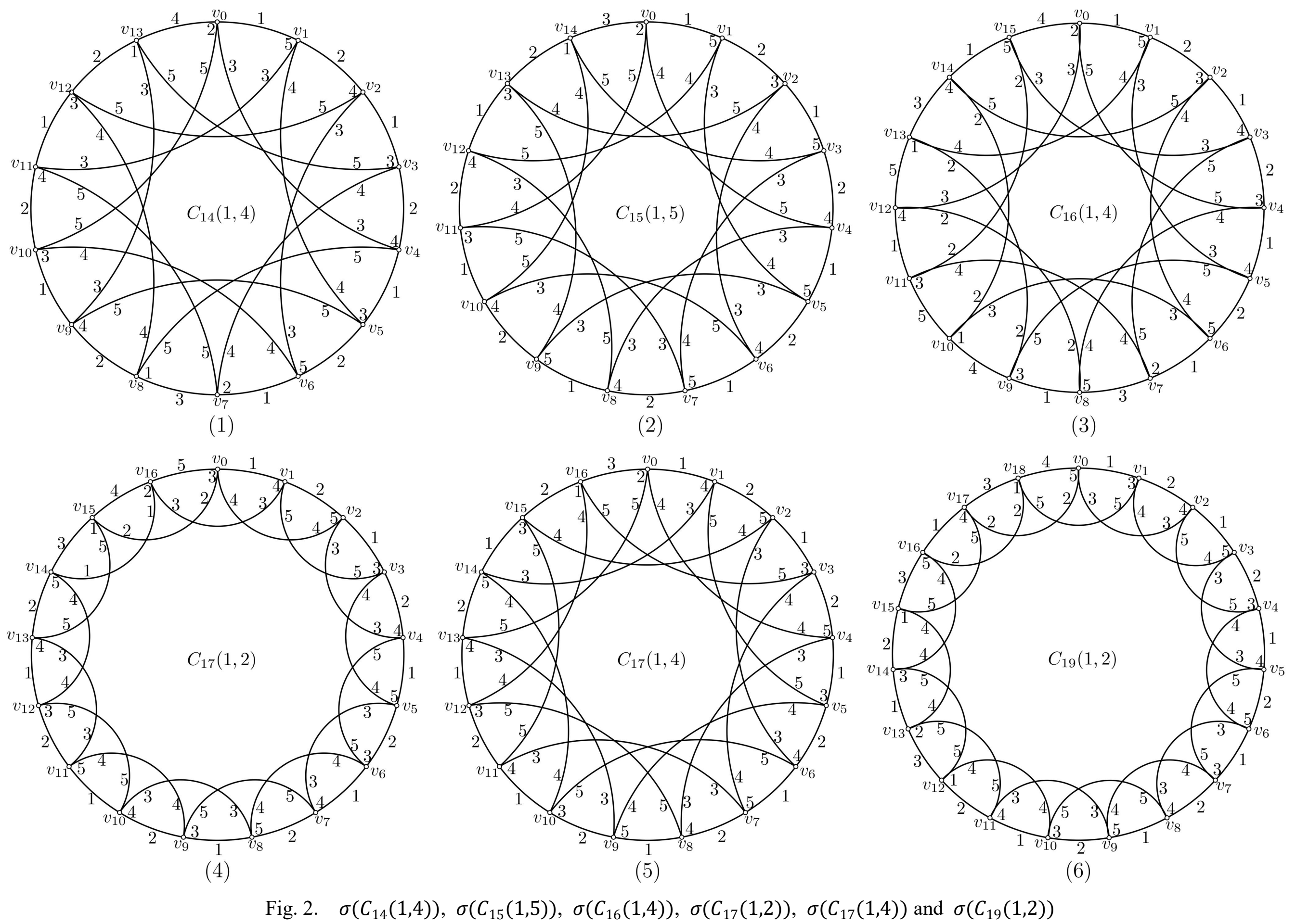


Fig. 2. $\sigma(C_{14}(1,4))$, $\sigma(C_{15}(1,5))$, $\sigma(C_{16}(1,4))$, $\sigma(C_{17}(1,2))$, $\sigma(C_{17}(1,4))$ and $\sigma(C_{19}(1,2))$

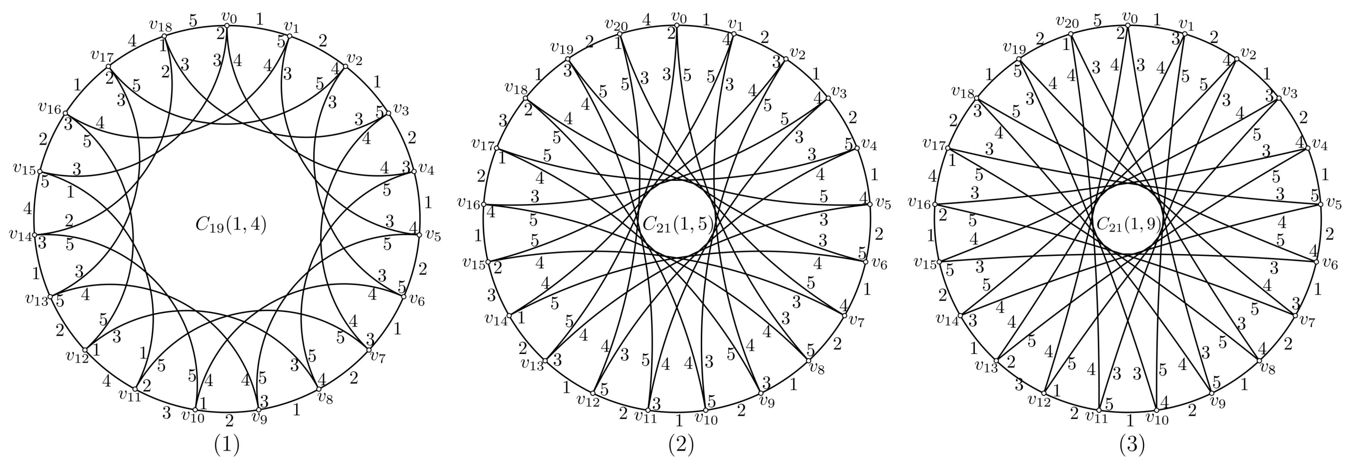


Fig. 3. $\sigma(C_{19}(1,4))$, $\sigma(C_{21}(1,5))$ and $\sigma(C_{21}(1,9))$

**Theorem 9.** $\chi''(C_{21p}(1,21\mu+\lambda)) = 5$ for $\lambda = 5,9$.

**Proof:** We construct $\sigma(C_{21p}(1,21\mu+\lambda))$ for $\lambda = 5,9$ as follows.

$$\sigma(\mathrm{V}) = \begin{cases}(243454545353531241231)^p, & \lambda = 5,\\ (234345434545123521351)^p, & \lambda = 9,\end{cases}$$

$$\sigma(E_1) = \begin{cases}(121212121212123123124)^p, & \lambda = 5,\\ (121212121212312142125)^p, & \lambda = 9,\end{cases}$$

$$\sigma(E_2) = \begin{cases}(535343434545454535453)^p, & \lambda = 5,\\ (353554355434445433543)^p, & \lambda = 9.\end{cases}$$

Fig.3(2)-(3) show $\sigma(C_{21}(1,5))$ and $\sigma(C_{21}(1,9))$.

**Theorem 10.** $\chi''(C_{23p}(1,23\mu+5)) = 5$.

**Proof:** We construct $\sigma(C_{23p}(1,23\mu+5))$ as follows.

$$\sigma(V) = (24545453535213214243231)^p,$$

$$\sigma(E_1) = (12121212121321323121424)^p,$$

$$\sigma(E_2) = (53434345454545451535353)^p.$$

Fig.4 shows $\sigma(C_{23}(1,5))$.

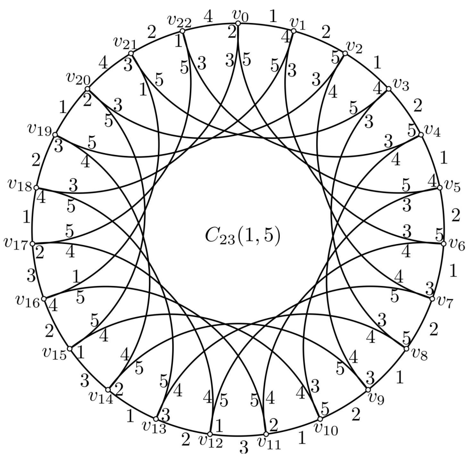


Fig. 4. $\sigma(C_{23}(1,5))$

Note that for $p \pmod 2 = 1$, the results $\chi''(C_{15p}(1,15\mu+5)) = 5$, $\chi''(C_{21p}(1,21\mu+5)) = 5$, and $\chi''(C_{21p}(1,21\mu+9)) = 5$ with the additional condition $\frac{7p}{gcd(7p,7\mu+3)} = 3s$ can be obtained too in [17].

## IV. Conclusion

In conclusion, we have determined the total chromatic numbers for several families of 4-regular circulant graphs $C_{tp}(1,t\mu+\lambda)$. These results support the conjecture proposed by Khennoufa and Togni [14], which states that except for a finite number of Type II graphs, 4-regular circulant graphs are all Type I graphs. We summarize our results as Table II. There are still several 4-regular circulant graphs, especially those of the form $C_{tp}(1,t\mu+\lambda)$, that remain to be studied. We hope to obtain results on the total chromatic numbers of $C_{tp}(1,t\mu+\lambda)$ for more $t$ and $\lambda$ later.

TABLE II
FAMILIS OF TYPE I 4-REGULAR CIRCULANT GRAPHS OBTAINED IN THIS PAPER

| | |
|---|---|
| $C_{tp}(1,t\mu+\lambda)$ | $t = 8$, $\lambda = 2,6,7$ |
| | $t = 11$, $\lambda = 2,3,4,5,6,8,9$ |
| | $t = 13$, $\lambda = 2,5,6$ |
| | $t = 14$, $\lambda = 4$ |
| | $t = 15$, $\lambda = 5$ |
| | $t = 16$, $\lambda = 4$ |
| | $t = 17$, $\lambda = 2,4,5,8$ |
| | $t = 19$, $\lambda = 2,4,5,6,9$ |
| | $t = 21$, $\lambda = 5,9$ |
| | $t = 23$, $\lambda = 5$ |

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