Rainbow Dynamic Coloring in Some Corona Product Graphs

Gayathri Annasagaram, R. Murali, and Kulkarni Sunita Jagannatharao

***Abstract*—Consider a simple, non-trivial, connected graph G, determined by a coloring $c: V(G) \longrightarrow \{1, 2, 3, \ldots, k\}$ $k \in N$ of $V(G)$. In G, a rainbow dynamic coloring is a dynamic coloring where a minimum number of colors is needed such that every two vertices are connected by at least one path whose inner vertices are colored differently. Rainbow dynamic coloring of G, represented as $rdyc(G)$, is the minimum k for which the k-vertex coloring exists. In this work, we compute the $rdyc$ of certain graphs of the corona product. The critical property of the corona product graphs is also discussed.**

***Index Terms*—rainbow vertex connection number, dynamic coloring, corona product, rainbow dynamic coloring, p-critical.**

I. Introduction

THE graphs in this work are all finite, simple, connected, nontrivial, and undirected. In graph theory, two coloring issues occur. One is a vertex coloring and the other is an edge coloring. These problems have led to the introduction and detailed study of various coloring parameters, enriching our understanding of these types of problems. The Results related to these parameters are available in the literature, providing a comprehensive view of the research in this field.

Graph theory has numerous applications, including communication networks, network security, and more. One way to create a data structure is as a tree, which uses vertices and edges. Graphs are utilized to illustrate the computation flow. Graph transformation systems utilize rules to manipulate graphs stored in memory. Data structures that utilize graphs enable safe transactions, long-term storage, and querying of graph-structured data.

Bruce Montgomery introduced a relatively new concept in vertex coloring, called dynamic coloring, in 2001 [1]. A dynamic graph coloring $d(G)$ is a proper coloring of the vertex set, such that each vertex of degree at least two its neighbors receive at least two different colors. Krivelevich and Yuster proposed the theory of rainbow vertex coloring in 2010. In a connected graph, [2] the minimum number of colors needed to color its vertices is called the rainbow vertex connection number, or $rvc(G)$. At least one path connects each pair of vertices, whose internal vertices have distinct colors.

Manuscript received April 17, 2024; revised July 21, 2025.

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A rainbow dynamic coloring of a graph is not just a theoretical concept but a practical one. It is a [15] dynamic coloring in which a minimum number of colors is needed such that every two vertices are connected by at least one path whose inner vertices are colored differently. Rainbow dynamic coloring of G, represented as $rdyc(G)$, is the minimum k for which the k-vertex coloring exists.

In [3], Gologranc et al. examined the bounds of rainbow coloring for graph products such as direct and strong product graphs. For other results, we refer to [7], [9], [10], [11], [12], [14].

We start by providing a formal definition for the corona product graph.

A. Definition

[10] Given two graphs, G and H, which are connected, the corona product of G and H is as follows:

i) For a single copy of G, take $|V(G)|$ copies of graph H.

ii) Connect the y^{th} vertex of G to every vertex of the y^{th} copy of H.

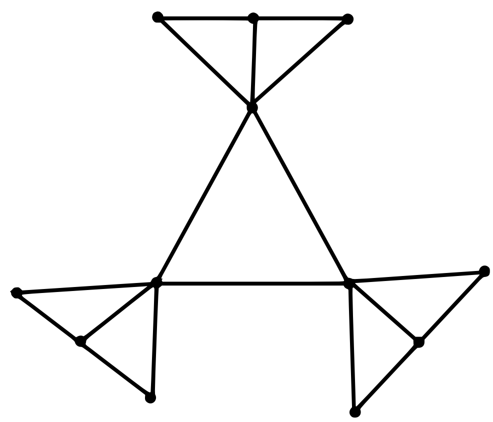

Fig. 1: $K_3 \circ P_3$

II. Results

This section contains the parameter $rdyc(G)$ for a few corona product graphs, such as the path with complete graph, path with star graph, star with complete graph, path with wheel graph, and K_1 with cycle graph.

We start with the corona product of the path with complete graph.

Proposition 1. $rdyc(P_2 \circ K_2) = 3$

Theorem 1. *For $n \geq 3$, $rdyc(P_n \circ K_n) = 2n$.*

Proof: Let $V(P_n) = \{u_y : 1 \leq y \leq n\}$ and let the vertex set of y copies of K_n be $V\{(K_n)_y\}$

$=\{v_{yz} : 1 \le y \le n, 1 \le z \le n\}$.

According to the corona product definition, every vertex of P_n is adjacent to every vertex of a copy of K_n, namely for $1 \le y \le n$ the vertex $u_y \in V(P_n)$ is adjacent to the set's vertices $\{v_{yz} : 1 \le z \le n\}$ in the y^{th} copy of K_n.

Consider $E(P_n \circ K_n)=\{E_1 \cup E_2 \cup E_3\}$ where $E_1=E(P_n)=\{e_y = (u_y, u_{y+1}); 1 \le y \le n-1\}$. E_2 be the edge set of $(K_n)_y$ for $1 \le y \le n$ and $E_3=\{(e_q)_y = (u_y, v_{yz});\ 1 \le y \le n, 1 \le q \le n$, and $1 \le z \le n\}$.

Color the vertices of $P_n \circ K_n$ in a rainbow dynamic pattern. For $1 \le y \le n$ allocate the colors $\{1, 2, \ldots, n\}$ to the vertices of $(K_n)_y$ and allocate the color $y+n$ to the vertices of P_n of $P_n \circ K_n$, from above allocation of colors, it shows that

$$\mathbf{rdyc}(P_n \circ K_n) \le \mathbf{2n} \tag{1}$$

To prove $rdyc(P_n \circ K_n) \ge 2n$.

Assume that $rdyc(P_n \circ K_n) = 2n-1$. Then, $2n-1$ colors must be allocated to the vertices of $P_n \circ K_n$ for proper rainbow dynamic coloring. As $P_n \circ K_n$ has n copies of K_n, we allocate n colors to each copy of K_n and the left-over $n-1$ colors to n vertices of P_n. A simple check exhibits that at least two vertices of P_n have the same color.

This contradicts, that at least one path of $P_n \circ K_n$ is not rainbow dynamic connected. Therefore

$$\mathbf{rdyc}(P_n \circ K_n) \ge \mathbf{2n} \tag{2}$$

Based on (1) and (2), it is obvious that $rdyc(P_n \circ K_n) = 2n$.

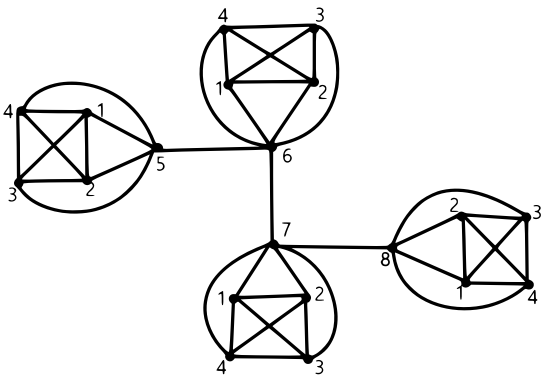

Fig. 2: Rainbow dynamic coloring in the graph $(P_4 \circ K_4)$.

■

The corona product of the star with the complete graph gives the following result.

Proposition 2. $rdyc(K_{1,2} \circ K_2) = 3$.

Theorem 2. *For* $n \ge 3$, $rdyc(K_{1,n} \circ K_n) = 2n+1$.

Proof: Let $V(K_{1,n})=\{u_y : 1 \le y \le n+1\}$ and let the vertex set of y copies of $(K_n)_y$ be $V\{(K_n)_y\} = \{v_{yz} : 1 \le y \le n+1, 0 \le z \le n-1\}$.

According to the corona product definition, every vertex of $K_{1,n}$ is adjacent to every vertex of a copy of K_n, namely, for $1 \le y \le n+1$ the vertex $u_y \in V(K_{1,n})$ is adjacent to the set's vertices $v_{yz} : 0 \le z \le n-1$ in the y^{th} copy of K_n.

Consider $E(K_{1,n} \circ K_n)=\{E_1 \cup E_2 \cup E_3\}$ where $E_1 = E(K_{1,n})=\{e_y : e_y = (u_1, u_{y+1});\ 1 \le y \le n\}$, E_2 be the edge set of $(K_n)_y$ for $1 \le y \le n+1$ and $E_3 = \{(e_q)_y = (u_y, v_{yz});\ 1 \le y \le n+1, 1 \le q \le n$, and $0 \le z \le n-1\}$.

Color the vertices of $K_{1,n} \circ K_n$ in a rainbow dynamic pattern. For $1 \le y \le n+1$ allocate the colors $\{1, 2, \ldots, n\}$ to the vertices of $(K_n)_y$ and for $1 \le y \le n+1$ allocate the color $y+n$ to the vertices of $K_{1,n}$ of $K_{1,n} \circ K_n$, from above allocation of colors, it shows that

$$\mathbf{rdyc}(K_{1,n} \circ K_n) \le \mathbf{2n+1} \tag{3}$$

To prove $rdyc(K_{1,n} \circ K_n) \ge 2n+1$.

Assume that $rdyc(K_{1,n} \circ K_n) = 2n$. Then, $2n$ colors must be allocated to the vertices of $K_{1,n} \circ K_n$ for proper rainbow dynamic coloring. As $K_{1,n} \circ K_n$ has n copies of K_n, we allocate n colors to each copy of K_n and allocate the left-over n colors to $n+1$ vertices of $K_{1,n}$. A simple check exhibits that at least two vertices of $K_{1,n}$ have the same color.

This contradicts, that at least one path of $K_{1,n} \circ K_n$ is not rainbow dynamic connected. Therefore

$$\mathbf{rdyc}(K_{1,n} \circ K_n) \ge \mathbf{2n+1} \tag{4}$$

Based on (3) and (4), it is obvious that $rdyc(K_{1,n} \circ K_n) = 2n+1$.

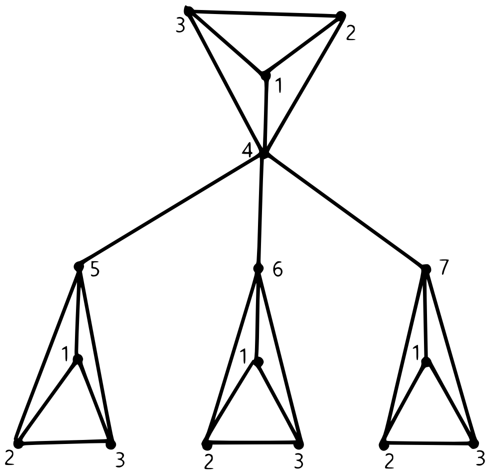

Fig. 3: Rainbow dynamic coloring in the graph $(K_{1,3} \circ K_3)$.

■

The corona product of the path with star graph gives the following result

Theorem 3. *For* $n \ge 2$, $rdyc(P_n \circ K_{1,n}) = n+2$.

Proof: Let $V(P_n) = \{u_y : 1 \le y \le n\}$ and let the vertex set of y copies of $K_{1,n}$ be $V\{(K_{1,n})_y\} = \{v_{yz} : 1 \le y \le n, 0 \le z \le n\}$.

According to the corona product definition, every vertex of P_n is adjacent to every vertex of a copy of $K_{1,n}$, namely, for $1 \le y \le n$ the vertex $u_y \in V(P_n)$ is adjacent to the set's vertices $\{v_{yz} : 0 \le z \le n\}$ in the y^{th} copy of $K_{1,n}$.

Consider $E(P_n \circ K_{1,n}) = \{E_1 \cup E_2 \cup E_3\}$ where $E_1 = E(P_n) = \{e_y : e_y = (u_y, u_{y+1}); 1 \leq y \leq n-1\}$, $E_2 = E(K_{1,n})_y = e_{yz} = (v_{y0}, v_{yz+1}); 1 \leq y \leq n, 0 \leq z \leq n-1$ and $E_3 = \{(e_q)_y = (u_y, v_{yz}); 1 \leq y \leq n, 1 \leq q \leq n+1, 0 \leq z \leq n\}$.

Color the vertices of $P_n \circ K_{1,n}$ in a rainbow dynamic pattern. For $1 \leq y \leq n$ allocate the color $v_{yz} = 1$ for $z = 0$ and $v_{yz} = 2$ for $1 \leq z \leq n$ to the vertices of $(K_{1,n})_y$ and for $1 \leq y \leq n$ allocate the color $y + n - 2$ to the vertices of P_n of $P_n \circ K_{1,n}$, from above allocation of colors, it shows that

$$\mathbf{rdyc}(P_n \circ K_{1,n}) \leq \mathbf{n+2} \tag{5}$$

To prove $rdyc(P_n \circ K_{1,n}) \geq n+2$.

Assume that $rdyc(P_n \circ K_{1,n}) = n+1$. Then, $n+1$ colors must be allocated to the vertices of $K_{1,n} \circ K_n$ for proper rainbow dynamic coloring. As $P_n \circ K_{1,n}$ has n copies of $K_{1,n}$, we allocate 2 colors to each copy of $(K_{1,n})_y$ and allocate the left-over $n-1$ colors to n vertices of P_n. A simple check exhibits that at least two vertices of P_n have the same color.

This contradicts, that at least one path of $P_n \circ K_{1,n}$ is not rainbow dynamic connected. Therefore

$$\mathbf{rdyc}(P_n \circ K_{1,n}) \geq \mathbf{n+2} \tag{6}$$

Based on (5) and (6), it is obvious that $rdyc(P_n \circ K_{1,n}) = n+2$.

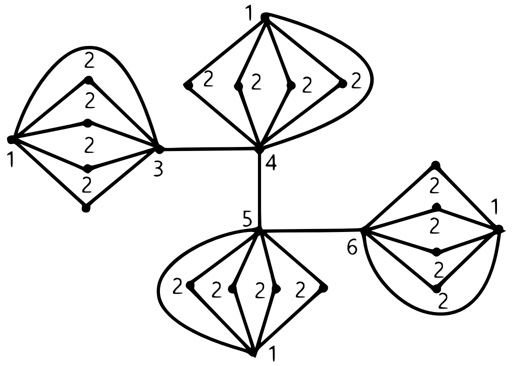

Fig. 4: Rainbow dynamic coloring in the graph $(P_4 \circ K_{1,4})$.

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The corona product of the path with wheel graph gives the following result.

Theorem 4.

$$rdyc(P_n \circ W_{1,n}) = \begin{cases} n+3 \; for \; all \; even \; n \geq 2 \\ n+4 \; for \; all \; odd \; n \geq 3 \end{cases}$$

Proof: Let $V(P_n) = \{u_y : 1 \leq y \leq n\}$ and let the vertex set of y copies of $W_{1,n}$. $\{W_{1,n}\}_y$ consists of n-cycle, $(C_n)_y = \{v_{y1}, v_{y2}, v_{y3}, v_{y4}, \ldots, v_{y(n+1)} = v_{y1}\}$ and one more vertex v_{y0}, connect to every vertex $(C_n)_y$ of $\{W_{1,n}\}_y$.

According to the corona product definition, every vertex of P_n is adjacent to every vertex of a copy of $W_{1,n}$, namely, for $1 \leq y \leq n$ the vertex $u_y \in V(P_n)$ is adjacent to the set's vertices $\{v_{yz} : 1 \leq z \leq n\}$ in the y^{th} copy of $W_{1,n}$.

Consider $E(P_n \circ W_{1,n}) = \{E_1 \cup E_2 \cup E_3\}$ where $E_1 = E(P_n) = \{e_y = (u_y, u_{y+1}); 1 \leq y \leq n-1\}$, E_2 be the edge set of $\{W_{1,n}\}_y$ where $E_2 = E_A \cup E_B$, $E_A = \{e_{yz} : e_{yz} = (v_{yz}, v_{yz+1}); 1 \leq y \leq n, 1 \leq z \leq n\}$, and $E_B = \{(e'_y)_{yz} : (e'_y)_{yz} = (v_{y0}, v_{yz}); 1 \leq y \leq n, 1 \leq z \leq n\}, E_3 = \{(e''_y)_{yz} : (e''_y)_{yz} = (u_y, v_{yz}); 1 \leq y \leq n, 0 \leq z \leq n\}$.

Case 1: n is even,

Color the vertices of $P_n \circ W_{1,n}$ in a rainbow dynamic pattern. Adjacent vertices of $(C_n)_y$ of $\{W_{1,n}\}_y$ for each y are colored again and again with the colors $\{2, 3, 2, 3, \ldots\ldots.\}$ and the same sequence is carried out till the last vertex, allocate the color '$1'$ to v_{y0}, of $\{W_{1,n}\}_y$ and for $1 \leq y \leq n$ allocate the color $y+n-1$ to $V(P_n)$ of $(P_n \circ W_{1,n})$. From the above allocation of colors, it shows that

$$\mathbf{rdyc}(P_n \circ W_{1,n}) \leq \mathbf{n+3} \tag{7}$$

To prove $rdyc(P_n \circ W_{1,n}) \geq n+3$ Assume that $rdyc(P_n \circ W_{1,n}) = n+2$. Then, $n+2$ colors must be allocated to the vertices of $P_n \circ W_{1,n}$ for proper rainbow dynamic coloring. As $P_n \circ W_{1,n}$ has n copies of $W_{1,n}$, we allocate 3 colors to each copy of $W_{1,n}$ and allocate the left-over $n-1$ colors to n vertices of P_n. A simple check exhibits that at least two vertices of P_n have the same color.

This contradicts, that at least one path of $P_n \circ W_{1,n}$ is not rainbow dynamic connected. Therefore

$$\mathbf{rdyc}(P_n \circ W_{1,n}) \geq \mathbf{n+3} \tag{8}$$

Based on (7) and (8), it is obvious that $rdyc(P_n \circ W_{1,n}) = n+3$.

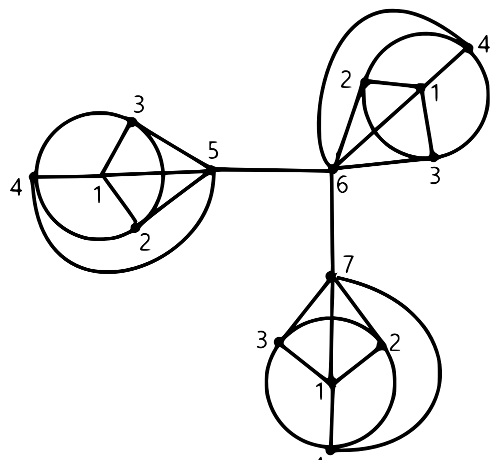

Fig. 5: Rainbow dynamic coloring in the graph $(P_3 \circ W_{1,3})$.

Case 2: n is odd,

Color the vertices of $P_n \circ W_{1,n}$ in a rainbow dynamic pattern as follows. Adjacent vertices of $(C_n)_y$ of $\{W_{1,n}\}_y$ for each y are colored again and again with the colors $\{2, 3, 2, 3, \ldots\ldots 4\}$ the same sequence is carried out with the end vertex as 4, allocate the color '$1'$ to v_{y0}, of $\{W_{1,n})\}_y$ and for $1 \leq y \leq n$ allocate the color $y+n-1$ to $V(P_n)$ of $P_n \circ W_{1,n}$. From the above allocation of colors, it shows that

$$\mathbf{rdyc}(P_n \circ W_{1,n}) \leq \mathbf{n+4} \tag{9}$$

To prove $rdyc(P_n \circ W_{1,n}) \geq n+4$

Assume that $rdyc(P_n \circ W_{1,n}) = n+4$. Then, $n+3$ colors must be allocated to the vertices of $P_n \circ W_{1,n}$ for proper rainbow dynamic coloring. As $P_n \circ W_{1,n}$ has n copies of $W_{1,n}$, we allocate 4 colors to each copy of $W_{1,n}$ and allocate the left-over $n-1$ colors to n vertices of P_n. A simple check exhibits that at least two vertices of P_n have the same color.

This contradicts, that at least one path of $P_n \circ W_{1,n}$ is not rainbow dynamic connected. Therefore

$$\mathbf{rdyc}(P_n \circ W_{1,n}) \geq \mathbf{n+4} \tag{10}$$

Based on (9) and (10), it is obvious that $rdyc(P_n \circ W_{1,n}) = n+4$.

■

The corona product of K_1 with cycle graph gives the following result.

Theorem 5. *For $n \geq 2$,*

$$rdyc(K_1 \circ C_n) = \begin{cases} 4 \ for \ n \ is \ odd \\ 3 \ for \ n \ is \ even \end{cases}$$

Proof: Let $V(K_1) = \{u_1\}$ and let the vertex set of (C_n) be $V(C_n) = \{v_z : 0 \leq z \leq n-1\}$.

According to the corona product definition, vertex of K_1 is adjacent to every vertex of C_n, namely, for the vertex $u_1 \in V(C_n)$ is adjacent to the set's vertices $\{v_z : 0 \leq z \leq n-1\}$.

Consider $E(K_1 \circ C_n) = \{E_1 \cup E_2\}$ where E_1 be the edge set of C_n and $E_2 = \{(e_q) = (u_1, v_z); 1 \leq q \leq n, 0 \leq z \leq n-1\}$.

Case 1: n is odd,

Color the vertices of $K_1 \circ C_n$ in a rainbow dynamic pattern. Allocate the color 1 to the vertex of K_1 and allocate the colors $\{2, 3, 2, 3....4\}$ and the same sequence is carried out till the last vertex where the end vertex is 4 to the vertices of (C_n) of $(K_1 \circ C_n)$, from above allocation of colors, it shows that

$$\mathbf{rdyc}(K_1 \circ C_n) \leq \mathbf{4} \tag{11}$$

To prove $rdyc(K_1 \circ C_n) \geq 4$.

Assume that $rdyc(K_1 \circ C_n) = 3$. Then 3 colors must be allocated to the vertices of $(K_1 \circ C_n)$ for proper rainbow dynamic coloring. We allocate 1 color to K_1 and leftover 2 colors to C_n. A simple check exhibits that at least two adjacent vertices have the same color.

This contradicts, that at least one path of $(K_1 \circ C_n)$ is not rainbow dynamic connected. Therefore

$$\mathbf{rdyc}(K_1 \circ C_n) \geq \mathbf{4} \tag{12}$$

Based on (11) and (12), it is obvious that $rdyc(K_1 \circ C_n) = 4$.

Case 2: n is even,

Color the vertices of $K_1 \circ C_n$ in a rainbow dynamic pattern. Allocate the color 1 to the vertex of K_1 and allocate the colors $\{2, 3, 2, 3....\}$ and the same sequence is carried out till the last vertex to the vertices of (C_n) of $(K_1 \circ C_n)$, from above allocation of colors, it shows that

$$\mathbf{rdyc}(K_1 \circ C_n) \leq \mathbf{3} \tag{13}$$

To prove $rdyc(K_1 \circ C_n) \geq 3$. Assume that $rdyc(K_1 \circ C_n) = 2$.

Then 2 colors must be allocated to the vertices of $(K_1 \circ C_n)$ for proper rainbow dynamic coloring. We allocate 1 color to K_1 and left-over 1 color to C_n. A simple check exhibits that at least two adjacent vertices have the same color. This contradicts, that at least one path of $(K_1 \circ C_n)$ is not rainbow dynamic connected. Therefore

$$\mathbf{rdyc}(K_1 \circ C_n) \geq \mathbf{3} \tag{14}$$

Based on (13) and (14), it is obvious that $rdyc(K_1 \circ C_n) = 3$. ■

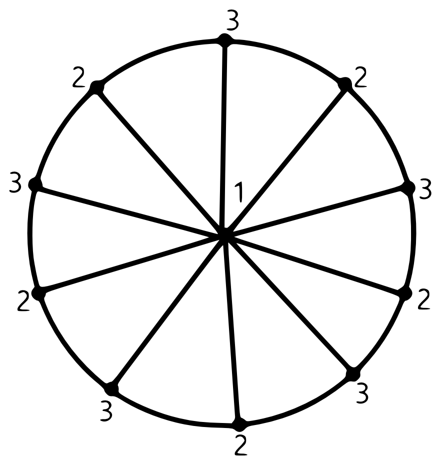

Fig. 6: Rainbow dynamic coloring in the graph $(K_1 \circ C_{10})$.

p **critical corona product graphs.**

In this section, we start with graph G, which is referred to in theorem 1, and analyze the p-criticalness property of the graphs of the corona product discussed in the preceding section. G becomes disconnected if any vertex in $V(P_n)$ is removed. For this reason, G is not p critical to $V(P_n)$. The outcome for the vertex set $V\{(K_n)_y\}$ is as follows.

Lemma 1. *For $n \geq 3$, $rdyc(P_n \circ K_n)$ is rainbow dynamic p critical for $V\{(K_n)_y\}$.*

Proof: Consider $G = P_n \circ K_n$. G is a rainbow dynamic p critical for $V\{(K_n)_y\}$ for $n \geq 3$. $v = (x, y)$ represents any vertex in $V\{(K_n)_y\}$. n colors may color the vertices in $V\{(K_n)_y\}$, If coloring is done according to theorem 1, $d(x, y) = n-3$ is the result of removing the vertex v from $V\{(K_n)_y\}$. In the set $V\{(K_n)_y\}$, let P represent the path from x to y. Then, it is possible to color the n vertices using $n-1$ colors. This is true for each vertex in $V\{(K_n)_y\}$. As a result, one color less than the necessary $2n$ colors already given in G is sufficient to provide a rainbow dynamic. This is true for each vertex v that is part of $V\{(K_n)_y\}$in G.

Consequently, $rdyc(G) = 2n-1$. According to $V\{(K_n)_y\}$, G is p critical. ■

Lemma 2. *For $n \geq 3$, $rdyc(P_n \circ K_{1,n})$ is rainbow dynamic p critical for $V\{(K_{1,n})_y\}$.*

Proof: Consider $G = P_n \circ K_{1,n}$. Next, G is a rainbow dynamic p-critical for $V\{(K_{1,n})_y\}$ for $n \geq 2$. $v = (x, y)$ represents any vertex in $V\{(K_{1,n})_y\}$. n colors may color the vertices in $V\{(K_{1,n})_y\}$, if coloring is done according to theorem 3, $d(x, y) = n-3$ is the result of removing the vertex v from $V\{(K_{1,n})_y\}$. In the set $V\{(K_{1,n})_y\}$, let P

represent the path from x to y. Then, it is possible to color the n vertices using 1 color. This is true for each vertex in $V\{(K_{1,n})_y\}$. As a result, one color less than the necessary $n+2$ colors already given in G is sufficient to provide a rainbow dynamic. This is true for each vertex v that is part of $V\{(K_{1,n})_y\}$in G.

Consequently, $rdyc(G) = n+1$.

According to $V\{(K_{1,n})_y\}$, G is p critical. ■

Lemma 3. *For $n \geq 2$, even and odd, $rdyc(P_n \circ W_{1,n})$ is rainbow dynamic p-critical for $V\{(W_{1,n})_y\}$.*

Proof: Consider $G = P_n \circ W_{1,n}$. G is a rainbow dynamic p critical for $V\{(W_{1,n})_y\}$ for $n \geq 3$. $v = (x, y)$ represents any vertex in $G = P_n \circ W_{1,n}$. n colors may color the vertices in $V\{(W_{1,n})_y\}$, if the coloring is done according to theorem 4, $d(x, y) = n-3$ is the result of removing the vertex v from $V\{(W_{1,n})_y\}$. In the set $V\{(W_{1,n})_y\}$, let P represent the path from x to y. Then, it is possible to color the n vertices using $n-1$ colors. This is true for each vertex in $V\{(W_{1,n})_y\}$. As a result, one color less than the necessary $n+3$ colors for even and $n+4$ colors for odd which are already given in G is sufficient to provide a rainbow dynamic. This is true for each vertex v that is part of $V\{(W_{1,n})_y\}$.

Consequently, $rdyc(G) = n+2$ for n is even and $rdyc(G) = n+3$ for n is odd.

According to $V\{(W_{1,n})_y\}$, G is p critical. ■

In the following lemma, G becomes disconnected if any vertex in $V(K_n)$ is removed. For this reason, G is not p critical to $V(K_{1,n})$. The outcome for the vertex set $V\{(K_n)_y\}$ is as follows.

Lemma 4. *For $n \geq 3$, $rdyc(K_{1,n} \circ K_n)$ is rainbow dynamic p critical for $V\{(K_n)_y\}$.*

Proof: Consider $G = K_{1,n} \circ K_n$. G is a rainbow dynamic p critical for $V\{(K_n)_y\}$ for $n \geq 2$. $v = (x, y)$ represents any vertex in $V\{(K_n)_y\}$. n colors may color the vertices in $V\{(K_n)_y\}$, if the coloring is done according to theorem 2, $d(x, y) = n-3$ is the result of removing the vertex v from $V\{(K_n)_y\}$. In the set $V\{(K_n)_y\}$, let P represent the path from x to y. Then, it is possible to color the n vertices using $n-1$ colors. This is true for each vertex in $V\{(K_n)_y\}$. As a result, one color less than the necessary $2n+1$ colors already given in G is sufficient to provide a rainbow dynamic. This is true for each vertex v that is part of $V\{(K_n)_y\}$in G.

Consequently, $rdyc(G) = 2n$.

According to $V\{(K_n)_y\}$, G is p critical. ■

In the following lemma, G becomes disconnected if K_1 is removed. For this reason, G is not p-critical about $V(K_1)$ and $V(C_n)$, if n is even. The outcome for the vertex set $V(C_n)$, if n is odd as follows.

Lemma 5. *For $n \geq 2$, $rdyc(K_1 \circ C_n)$ is rainbow dynamic p critical for $V(C_n)$, if n is odd.*

Proof: Consider $G = K_1 \circ C_n$. G is a rainbow dynamic p critical for $V(C_n)$ for $n \geq 2$. $v = (x, y)$ represents any vertex in $V(C_n)$. 3 colors color the vertices in $V(C_n)$, if coloring is done according to theorem 5, $d(x, y) = 2$ is the result of removing the vertex v from $V(C_n)$. In the set $V(C_n)$, let P represent the path from x to y. Then, it is possible to color the n vertices using 3 colors in $G = K_1 \circ C_n$. As a result, one color less than the necessary 4 colors already given in G is sufficient to provide a rainbow dynamic.

Consequently, $rdyc(G) = 3$.

According to $V(C_n)$, G is p critical. ■

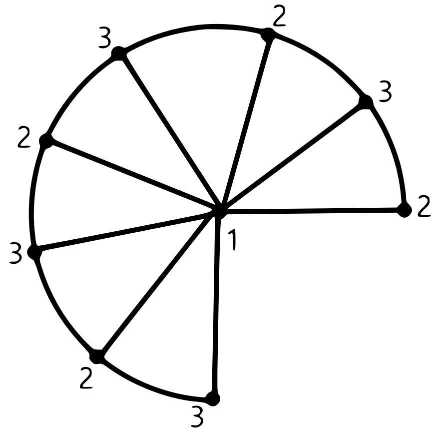

Fig. 7: Rainbow dynamic coloring in the p critical graph $(K_1 \circ C_9)$

Lemma 6. *The corona product of the complete graph K_1 with the cycle graph, i.e., $K_1 \circ C_n$ is the wheel graph $W_{1,n}$.*

Proof: Let G and H represent complete graph and cycle graph respectively. Let us take the corona product of K_1 and C_n, $K_1 \circ C_n$. Let v be the vertex of K_1 and $\{u_1, u_2, u_3, ...u_n\}$ be the vertices of cycle graph C_n, that is, $K_{1,n}$. Take a wheel graph $W_{1,n}$ with an internal vertex $'a'$ and cycle with vertices $\{w_1, w_2, w_3, ...w_n\}$.

Let us prove that $K_1 \circ C_n$ is isomorphic to $W_{1,n}$.

Assume g is a function such that $g(v) = a$, $g(u_1) = w_1$, $g(u_2) = w_2$, $g(u_3) = w_3$,.....$g(u_n) = w_n$. The vertices $v, u_1, u_2, u_3.....u_n$ and $a, w_1, w_2, w_3, ...w_n$ have been observed to be adjacent on map g. ■

III. Discussions

By the definition of rainbow dynamic coloring, $rdyc(G) \geq 3$ and $rvc(G) \geq rdyc(G) \geq d(G)$. Prop. 1 and Prop. 2 represent $rdyc(G) = 3$. The results for Theorem 1, 2 are obtained for $n \geq 3$, whereas in Theorem 3, 4, 5 the results are obtained for $n \geq 2$. The graph of $K_1 \circ C_n$ is observed to be equal to $W_{1,n}$. The properties of $K_1 \circ C_n$, $W_{1,n}$, and $K_{1,n}$ are the same. The $rdyc(W_{1,n})$ is the same as the $rdyc(K_1 \circ C_n)$ and $rdyc(K_{1,n})$ that is $rdyc(K_1 \circ C_n) = rdyc(W_{1,n}) = rdyc(K_{1,n})$ for $n \geq 2$. In all the above lemma's, the graph $G' = G \circ H$ we obtain a disconnected graph when G is removed from G'.

IV. Conclusion

In this study, we discover an idea of rainbow dynamic coloring for several kinds of corona product graphs, including combinations of path with complete graph, path with wheel graph, path with star graph, star with complete graph, and k_1 with cycle graph. We also describe the general problems that motivated this research. The field of graph theory is dynamic and impactful. Graphs can tackle complex challenges such as program analysis, cost reduction, and visualization. Network devices, such as switches and routers, utilize graphs to determine optimal traffic routing. The primary objective of this paper is to introduce recent advances in graph theory and its various applications within the engineering domain [4], [5], [6], [8], [10], [13].

Acknowledgement

The authors are deeply grateful to the management, R and D Center, Department of Mathematics, and Department of Mathematics staff, Dr. Ambedkar Institute of Technology, for their constant assistance and motivation.

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