The Congruence Extension Property of Quasi-MV* Algebras

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***Abstract*—Quasi-MV* algebras, introduced as a generalization of MV*-algebras and quasi-MV algebras, provide a general algebraic framework in the setting of many-valued logic and quantum computational logic. In this paper, we study the congruence extension property of quasi-MV* algebras. First, we present the subdirect product decomposition of any quasi-MV* algebra. Next, we prove that any MV*-algebra has the congruence extension property. Finally, we extend this property to quasi-MV* algebras.**

***Index Terms*—Congruences, Congruence extension property, Ideals, MV*-algebras, Quasi-MV* algebras.**

I. Introduction

NON-CLASSICAL mathematical logic, as the foundation of intelligent science, has received increasing attentions in recent years. It is well-known that the algebraic structures play a crucial role in the study of non-classical mathematical logic [3], [10], [13], [14], [15], [16], [18]. For example, in order to prove the completeness of Łukasiewicz's many-valued logic, Chang introduced MV-algebras in 1958 [3]. Since then, the algebraic structures of MV-algebras have been widely investigated. For another example, in order to characterize quantum computational logic, Ledda et al. introduced quasi-MV algebras in 2006 [10]. The study of the algebraic structures of quasi-MV algebras has played a positive role in quantum computational logic [1], [5], [8], [9], [17]. In 1965, to further characterize the structure of the real closed interval $[-1, 1]$ equipped with truncated addition $\varrho \uplus \varsigma = \max\{-1, \min\{1, \varrho + \varsigma\}\}$ and negation $-\varrho$, Chang introduced MV*-algebras in [4], paralleling similar work done for MV-algebras. Moreover, the logic associated with MV*-algebras was also investigated in [4], [12]. Recently, Jiang and Chen proposed quasi-MV* algebras in [7] as a unified framework for further research on quasi-MV algebras and MV*-algebras. The logic associated with quasi-MV* algebras has been preliminarily studied in [2]. To obtain additional characterizations of this logical system, we want to study more algebraic properties of quasi-MV* algebras.

The congruence extension property (CEP), an important property of varieties, characterizes whether a congruence on a subalgebra can be extended to the entire algebra. In 2005, Gispert and Mundici proved that the variety of MV-algebras satisfies CEP [6]. Subsequently, Paoli et al. generalized this result to quasi-MV algebras using the subdirect product decomposition of a quasi-MV algebra. Now, it is natural to ask whether quasi-MV* algebras, as a generalization of MV*-algebras and quasi-MV algebras, have CEP. We will give a positive answer in this paper.

The paper is organized as follows. In Section 2, we recall some definitions and results of MV*-algebras and quasi-MV* algebras. In Section 3, we present the subdirect product decomposition of a quasi-MV* algebra. Based on this decomposition, we establish the CEP for the variety of quasi-MV* algebras. Finally, a conclusion is given.

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II. Preliminaries

In this section, we recall some definitions and results of MV*-algebras and quasi-MV* algebras which will be used in what follows.

Definition 1: [4] Let $\mathbf{\Sigma} = \langle \Sigma; \uplus, -, 0, 1\rangle$ be an algebra of type $(2, 1, 0, 0)$. If the following conditions are satisfied for any $\varrho, \varsigma, \kappa \in \Sigma$,

(MV*1) $\varrho \uplus \varsigma = \varsigma \uplus \varrho$,
(MV*2) $(1 \uplus \varrho) \uplus (\varsigma \uplus (1 \uplus \kappa)) = ((1 \uplus \varrho) \uplus \varsigma) \uplus (1 \uplus \kappa)$,
(MV*3) $\varrho \uplus (-\varrho) = 0$,
(MV*4) $(\varrho \uplus 1) \uplus 1 = 1$,
(MV*5) $\varrho \uplus 0 = \varrho$,
(MV*6) $-(\varrho \uplus \varsigma) = (-\varrho) \uplus (-\varsigma)$,
(MV*7) $-(-\varrho) = \varrho$,
(MV*8) $\varrho \uplus \varsigma = (\varrho^+ \uplus \varsigma^+) \uplus (\varrho^- \uplus \varsigma^-)$,
(MV*9) $(-\varrho \uplus (\varrho \uplus \varsigma))^+ = -(\varrho^+) \uplus (\varrho^+ \uplus \varsigma^+)$,
(MV*10) $\varrho \vee \varsigma = \varsigma \vee \varrho$,
(MV*11) $\varrho \vee (\varsigma \vee \kappa) = (\varrho \vee \varsigma) \vee \kappa$,
(MV*12) $\varrho \uplus (\varsigma \vee \kappa) = (\varrho \uplus \varsigma) \vee (\varrho \uplus \kappa)$,

in which ones define $\varrho^+ = 1 \uplus (-1 \uplus \varrho)$, $\varrho^- = -1 \uplus (1 \uplus \varrho)$, and $\varrho \vee \varsigma = (\varrho^+ \uplus (-\varrho^+ \uplus \varsigma^+)^+) \uplus (\varrho^- \uplus (-\varrho^- \uplus \varsigma^-)^+)$, then $\mathbf{\Sigma} = \langle \Sigma; \uplus, -, 0, 1\rangle$ is called an *MV*-algebra*.

Example 1: Let $\Sigma = \{\varrho, \varsigma, 0, \vartheta, 1\}$ be a 5-element set and define operations on Σ as follows:

$\uplus$	ϱ	ς	0	ϑ	1
ϱ	ϱ	ϱ	ϱ	ς	0
ς	ϱ	ϱ	ς	0	ϑ
0	ϱ	ς	0	ϑ	1
ϑ	ς	0	ϑ	1	1
1	0	ϑ	1	1	1

	ϱ	ς	0	ϑ	1
$-$	1	ϑ	0	ς	ϱ

Then $\mathbf{\Sigma} = \langle \Sigma; \uplus, -, 0, 1\rangle$ is an MV*-algebra.

The variety of MV*-algebras is denoted by $\mathbb{MV}^*$. In the following, we abbreviate an MV*-algebra $\mathbf{\Sigma} = \langle \Sigma; \uplus, -, 0, 1\rangle$ as $\mathbf{\Sigma}$. Below we list some properties of ideals of any MV*-algebra.

Let $\mathbf{\Sigma}$ be an MV*-algebra. The operation $\ominus$ is defined by $\varrho \ominus \varsigma = \varrho \uplus (-\varsigma)$ for any $\varrho, \varsigma \in \Sigma$ in [4].

Definition 2: [11] Let $\mathbf{\Sigma}$ be an MV*-algebra. A non-empty subset Φ of Σ is called an *ideal* of $\mathbf{\Sigma}$, if the following conditions are satisfied:

(Φ1) If $\varrho, \varsigma \in \Phi$, then $\varrho \ominus \varsigma \in \Phi$,

(Φ2) If $\varrho \in \Phi$, then $\varrho^+ \in \Phi$,

(Φ3) If $\varrho, \kappa \in \Phi$ and $\varsigma \in \Sigma$ with $\varrho \leqslant \varsigma \leqslant \kappa$, then $\varsigma \in \Phi$.

Proposition 1: [11] Let $\mathbf{\Sigma}$ be an MV*-algebra and Φ be an ideal of $\mathbf{\Sigma}$. Then for any $\varrho, \varsigma, \kappa, \varepsilon, \vartheta \in \Sigma$, we have

(1) $0 \in \Phi$,

(2) If $\varrho \in \Phi$, then $-\varrho \in \Phi$,

(3) If $\varrho \in \Phi$, then $\varrho^- \in \Phi$,

(4) If $\varrho, \varsigma \in \Phi$, then $\varrho \uplus \varsigma \in \Phi$,

(5) If $\varrho \ominus \varsigma \in \Phi$ and $\varsigma \in \Phi$, then $\varrho \in \Phi$,

(6) If $\varrho \ominus \varsigma \in \Phi$ and $\kappa \in \Phi$, then $(\varrho \uplus \kappa) \ominus (\varsigma \uplus \kappa) \in \Phi$,

(7) If $\varrho \ominus \varsigma \in \Phi$ and $\varsigma \ominus \kappa \in \Phi$, then $\varrho \ominus \kappa \in \Phi$,

(8) If $\varrho \ominus \varsigma \in \Phi$ and $\varepsilon \ominus \vartheta \in \Phi$, then $(\varrho \uplus \varepsilon) \ominus (\varsigma \uplus \vartheta) \in \Phi$.

Theorem 1: [11] Let $\mathbf{\Sigma}$ be an MV*-algebra. Then the lattice of congruences on $\mathbf{\Sigma}$ and the lattice of ideals of $\mathbf{\Sigma}$ are isomorphic.

Now, we present the definition and related properties of a quasi-MV* algebra.

Definition 3: [7] Let $\mathbf{\Lambda} = \langle \Lambda; \uplus, -, ^+, ^- 0, 1 \rangle$ be an algebra of type $(2, 1, 1, 1, 0, 0)$. If the following conditions are satisfied for any $\varrho, \varsigma, \kappa \in \Lambda$,

(QMV*1) $\varrho \uplus \varsigma = \varsigma \uplus \varrho$,

(QMV*2) $(1 \uplus \varrho) \uplus (\varsigma \uplus (1 \uplus \kappa)) = ((1 \uplus \varrho) \uplus \varsigma) \uplus (1 \uplus \kappa)$,

(QMV*3) $(\varrho \uplus 1) \uplus 1 = 1$,

(QMV*4) $(\varrho \uplus \varsigma) \uplus 0 = \varrho \uplus \varsigma$,

(QMV*5) $0 = -0$,

(QMV*6) $\varrho \uplus (-\varrho) = 0$,

(QMV*7) $-(\varrho \uplus \varsigma) = -\varrho \uplus (-\varsigma)$,

(QMV*8) $-(-\varrho) = \varrho$,

(QMV*9) $\varrho^+ \uplus 0 = (\varrho \uplus 0)^+ = 1 \uplus (-1 \uplus \varrho)$ and $\varrho^- = -1 \uplus (1 \uplus \varrho)$,

(QMV*10) $\varrho \uplus \varsigma = (\varrho^+ \uplus \varsigma^+) \uplus (\varrho^- \uplus \varsigma^-)$,

(QMV*11) $(-\varrho \uplus (\varrho \uplus \varsigma))^+ = (-\varrho^+) \uplus (\varrho^+ \uplus \varsigma^+)$,

(QMV*12) $\varrho \vee \varsigma = \varsigma \vee \varrho$,

(QMV*13) $\varrho \vee (\varsigma \vee \kappa) = (\varrho \vee \varsigma) \vee \kappa$,

(QMV*14) $\varrho \uplus (\varsigma \vee \kappa) = (\varrho \uplus \varsigma) \vee (\varrho \uplus \kappa)$,

in which ones define $\varrho \vee \varsigma = (\varrho^+ \uplus (-\varrho^+ \uplus \varsigma^+)^+) \uplus (\varrho^- \uplus (-\varrho^- \uplus \varsigma^-)^+)$, then $\mathbf{\Lambda} = \langle \Lambda; \uplus, -, ^+, ^-, 0, 1 \rangle$ is called a *quasi-MV* algebra*.

Example 2: Let $\Lambda = \{\varrho, \varsigma, \kappa, 0, \varepsilon, \vartheta, 1\}$ be a 7-element set and define operations on Λ as follows:

$\uplus$	ϱ	ς	κ	0	ε	ϑ	1
ϱ	ϱ	ϱ	ϱ	ϱ	ς	ς	0
ς	ϱ	ϱ	ϱ	ς	0	0	ϑ
κ	ϱ	ϱ	ϱ	ς	0	0	ϑ
0	ϱ	ς	ς	0	ϑ	ϑ	1
ε	ς	0	0	ϑ	1	1	1
ϑ	ς	0	0	ϑ	1	1	1
1	0	ϑ	ϑ	1	1	1	1

	ϱ	ς	κ	0	ε	ϑ	1
$-$	1	ϑ	ε	0	κ	ς	ϱ
$+$	0	0	0	0	ε	ϑ	1
$-$	ϱ	ς	κ	0	0	0	0

Then $\mathbf{\Lambda} = \langle \Lambda; \uplus, -, ^+, ^-, 0, 1 \rangle$ is a quasi-MV* algebra.

Example 3: Let $\Lambda' = \{\varrho, \varsigma, \kappa, \varpi, 0, \varsigma, \varepsilon, \vartheta, 1\}$ be a 9-element set and define operations on Λ' as follows:

$\uplus$	ϱ	ς	κ	ϖ	0	ς	ε	ϑ	1
ϱ	ϱ	ϱ	ϱ	ϱ	ϱ	ς	ς	ς	0
ς	ϱ	ϱ	ϱ	ϱ	ς	0	0	0	ϑ
κ	ϱ	ϱ	ϱ	ϱ	ς	0	0	0	ϑ
ϖ	ϱ	ϱ	ϱ	ϱ	ς	0	0	0	ϑ
0	ϱ	ς	ς	ς	0	ϑ	ϑ	ϑ	1
ς	ς	0	0	0	ϑ	1	1	1	1
ε	ς	0	0	0	ϑ	1	1	1	1
ϑ	ς	0	0	0	ϑ	1	1	1	1
1	0	ϑ	ϑ	ϑ	1	1	1	1	1

	ϱ	ς	κ	ϖ	0	ς	ε	ϑ	1
$-$	1	ϑ	ε	ς	0	ϖ	κ	ς	ϱ
$+$	0	0	0	0	0	ς	ε	ϑ	1
$-$	ϱ	ς	κ	ϖ	0	0	0	0	0

Then $\mathbf{\Lambda}' = \langle \Lambda'; \uplus, -, ^+, ^-, 0, 1 \rangle$ is a quasi-MV* algebra.

The variety of quasi-MV* algebras is denoted by $\mathbb{QMV}^*$. In the following, we abbreviate a quasi-MV* algebra $\mathbf{\Lambda} = \langle \Lambda; \uplus, -, ^+, ^- 0, 1 \rangle$ as $\mathbf{\Lambda}$.

Obviously, any MV*-algebra is a quasi-MV* algebra. Conversely, for any quasi-MV* algebra $\mathbf{\Lambda}$, if $\varrho \uplus 0 = \varrho$ for any $\varrho \in \Lambda$, then it is an MV*-algebra. Moreover, let $\mathbf{\Lambda}$ be a quasi-MV* algebra and $\varrho \in \Lambda$. If $\varrho \uplus 0 = \varrho$, then ϱ is called *regular*. We denote that $\mathcal{R}(\Lambda)$ is the set of all regular elements in Λ. Then $\mathbf{R}_{\mathbf{\Lambda}} = \langle \mathcal{R}(\Lambda); \uplus, -, ^+, ^-, 0, 1 \rangle$ is an MV*-algebra, where the operations $\uplus, -, ^+$, and $^-$ are those of $\mathbf{\Lambda}$ restricted to $\mathcal{R}(\Lambda)$.

In any quasi-MV* algebra $\mathbf{\Lambda}$, we consider that the operations $^+$ and $^-$ (which have the same priority) have priority to operations $\uplus$ and $-$, the operation $-$ has priority to the operation $\uplus$.

Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. For any $\varrho, \varsigma \in \Lambda$, we define an operation $\varrho \wedge \varsigma = -((-\varrho) \vee (-\varsigma))$. We can also define a binary relation $\varrho \leqslant \varsigma$ iff $\varrho \vee \varsigma = \varsigma \uplus 0$. Then the following results hold.

Proposition 2: [7] Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then for any $\varrho, \varsigma, \kappa, \varepsilon, \vartheta \in \Lambda$, we have

(1) $0 \uplus 0 = 0$, $1 \uplus 0 = 1$, $-1 \uplus 0 = -1$, $1 \uplus 1 = 1$, $-1 \uplus (-1) = -1$,

(2) $-(\varrho \uplus 0) = -\varrho \uplus 0$,

(3) $\varrho \uplus \varsigma = (\varrho \uplus 0) \uplus \varsigma = \varrho \uplus (\varsigma \uplus 0) = (\varrho \uplus 0) \uplus (\varsigma \uplus 0)$,

(4) $\varrho \vee \varrho = \varrho \uplus 0 = \varrho \wedge \varrho$,

(5) $\varrho \wedge (\varsigma \wedge \kappa) = (\varrho \wedge \varsigma) \wedge \kappa$,

(6) $\varrho \uplus (\varsigma \wedge \kappa) = (\varrho \uplus \varsigma) \wedge (\varrho \uplus \kappa)$,

(7) If $\varrho \leqslant \varsigma$, then $-\varsigma \leqslant -\varrho$,

(8) If $\varrho \leqslant \varsigma$, then $\varrho^+ \leqslant \varsigma^+$ and $\varrho^- \leqslant \varsigma^-$,

(9) If $\varrho \leqslant \varsigma$ and $\varsigma \leqslant \varrho$, then $\varrho \uplus 0 = \varsigma \uplus 0$,

(10) If $\varrho \leqslant \varsigma$ and $\varepsilon \leqslant \vartheta$, then $\varrho \uplus \varepsilon \leqslant \varsigma \uplus \vartheta$,

(11) If $\varrho \leqslant \varsigma$ and $\varepsilon \leqslant \vartheta$, then $\varrho \wedge \varepsilon \leqslant \varsigma \wedge \vartheta$,

(12) If $\varrho \leqslant \varsigma$ and $\varepsilon \leqslant \vartheta$, then $\varrho \vee \varepsilon \leqslant \varsigma \vee \vartheta$.

III. Congruence extension property

In this section, we investigate the congruence extension properties of $\mathbb{QMV}^*$ mainly. To achieve it, we first discuss the subdirect product decomposition of a quasi-MV* algebra.

Definition 4: [2] Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then $\mathbf{\Lambda}$ is called *flat*, if it satisfies the equation $0 = 1$.

Remark 1: Let $\mathbf{\Lambda}$ be a flat quasi-MV* algebra. Then for any $\varrho, \varsigma \in \Lambda$, we have $\varrho \uplus \varsigma = ((\varrho \uplus \varsigma) \uplus 0) \uplus 0 = ((\varrho \uplus \varsigma) \uplus 1) \uplus 1 = 1 = 0$ by (QMV*4) and (QMV*3).

Example 4: Let $\Lambda_1 = \{\kappa, 0, \varepsilon\}$ be a 3-element set and define operations on Λ_1 as follows:

$\uplus$	κ	0	ε
κ	0	0	0
0	0	0	0
ε	0	0	0

	κ	0	ε
$-$	ε	0	κ
$^+$	0	0	ε
$^-$	κ	0	0

, and $1 = 0$.

Then $\mathbf{\Lambda_1} = \langle \Lambda_1; \uplus, -, ^+, ^-, 0, 1\rangle$ is a flat quasi-MV* algebra.

Example 5: Let $\Lambda_2 = \{\kappa, \varpi, 0, \varsigma, \varepsilon\}$ be a 5-element set and define operations on Λ_2 as follows:

$\uplus$	κ	ϖ	0	ς	ε
κ	0	0	0	0	0
ϖ	0	0	0	0	0
0	0	0	0	0	0
ς	0	0	0	0	0
ε	0	0	0	0	0

	κ	ϖ	0	ς	ε
$-$	ε	ς	0	ϖ	κ
$^+$	0	0	0	ς	ε
$^-$	κ	ϖ	0	0	0

,

and $1 = 0$.

Then $\mathbf{\Lambda_2} = \langle \Lambda_2; \uplus, -, ^+, ^-, 0, 1\rangle$ is a flat quasi-MV* algebra.

The variety of flat quasi-MV* algebras is denoted by $\mathbb{FQMV}^*$.

Definition 5: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. For any $\varrho, \varsigma \in \Lambda$, we define a binary relation

$$\langle \varrho, \varsigma\rangle \in \Re \text{ iff } \varrho \leqslant \varsigma \text{ and } \varsigma \leqslant \varrho.$$

Remark 2: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra and $\varrho, \varsigma \in \Lambda$. Then $\langle \varrho, \varsigma\rangle \in \Re$ iff $\varrho \uplus 0 = \varsigma \uplus 0$ by Proposition 2(9).

Lemma 1: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then $\Re$ is a congruence on $\mathbf{\Lambda}$.

Proof: For any $\varrho, \varsigma, \varepsilon \in \Lambda$, since $\varrho \uplus 0 = \varrho \uplus 0$, we have $\langle \varrho, \varrho\rangle \in \Re$. If $\langle \varrho, \varsigma\rangle \in \Re$, then $\varrho \uplus 0 = \varsigma \uplus 0$, it turns out that $\varsigma \uplus 0 = \varrho \uplus 0$ and then $\langle \varsigma, \varrho\rangle \in \Re$. If $\langle \varrho, \varsigma\rangle \in \Re$ and $\langle \varsigma, \varepsilon\rangle \in \Re$, then $\varrho \uplus 0 = \varsigma \uplus 0$ and $\varsigma \uplus 0 = \varepsilon \uplus 0$, it follows that $\varrho \uplus 0 = \varepsilon \uplus 0$ and then $\langle \varrho, \varepsilon\rangle \in \Re$, so $\Re$ is an equivalence relation on $\mathbf{\Lambda}$. Now, we prove that $\Re$ satisfies the compatibility property. For any $\varrho, \varsigma, \varepsilon, \vartheta \in \Lambda$, if $\langle \varrho, \varsigma\rangle \in \Re$ and $\langle \varepsilon, \vartheta\rangle \in \Re$, then $\varrho \uplus 0 = \varsigma \uplus 0$ and $\varepsilon \uplus 0 = \vartheta \uplus 0$. By Proposition 2(3), we have $(\varrho \uplus \varepsilon) \uplus 0 = (\varrho \uplus 0) \uplus (\varepsilon \uplus 0) = (\varsigma \uplus 0) \uplus (\vartheta \uplus 0) = (\varsigma \uplus \vartheta) \uplus 0$, so $\langle \varrho \uplus \varepsilon, \varsigma \uplus \vartheta\rangle \in \Re$. If $\langle \varrho, \varsigma\rangle \in \Re$, then $\varrho \uplus 0 = \varsigma \uplus 0$, it turns out that $-\varrho \uplus 0 = -(\varrho \uplus 0) = -(\varsigma \uplus 0) = -\varsigma \uplus 0$ by Proposition 2(2), so $\langle -\varrho, -\varsigma\rangle \in \Re$. Moreover, $\varrho^+ \uplus 0 = 1 \uplus (-1 \uplus \varrho) = 1 \uplus (-1 \uplus (\varrho \uplus 0)) = 1 \uplus (-1 \uplus (\varsigma \uplus 0)) = 1 \uplus (-1 \uplus \varsigma) = \varsigma^+ \uplus 0$ and $\varrho^- \uplus 0 = -1 \uplus (1 \uplus \varrho) = -1 \uplus (1 \uplus (\varrho \uplus 0)) = -1 \uplus (1 \uplus (\varsigma \uplus 0)) = -1 \uplus (1 \uplus \varsigma) = \varsigma^- \uplus 0$ by Proposition 2(3), so $\langle \varrho^+, \varsigma^+\rangle \in \Re$ and $\langle \varrho^-, \varsigma^-\rangle \in \Re$. Hence $\Re$ is a congruence on $\mathbf{\Lambda}$. ■

Let $\mathbf{\Lambda}$ be a quasi-MV* algebra and $\aleph$ be a congruence on $\mathbf{\Lambda}$. For any $\varrho \in \Lambda$, the equivalence class of ϱ with respect to $\aleph$ is denoted by $\varrho/\aleph = \{\varsigma \in \Lambda \mid \langle \varrho, \varsigma\rangle \in \aleph\}$. The set of all equivalence classes of elements in Λ is denoted by $\Lambda/\aleph$. For any $\varrho/\aleph, \varsigma/\aleph \in \Lambda/\aleph$, the operations on $\Lambda/\aleph$ are defined as follows: $(\varrho/\aleph) \uplus^{\mathbf{\Lambda}/\aleph} (\varsigma/\aleph) = (\varrho \uplus \varsigma)/\aleph$, $-^{\mathbf{\Lambda}/\aleph}(\varrho/\aleph) = (-\varrho)/\aleph$, $(\varrho/\aleph)^{+^{\mathbf{\Lambda}/\aleph}} = (\varrho^+)/\aleph$, and $(\varrho/\aleph)^{-^{\mathbf{\Lambda}/\aleph}} = (\varrho^-)/\aleph$. Then $\mathbf{\Lambda}/\aleph = \langle \Lambda/\aleph; \uplus^{\mathbf{\Lambda}/\aleph}, -^{\mathbf{\Lambda}/\aleph}, {}^{+^{\mathbf{\Lambda}/\aleph}}, {}^{-^{\mathbf{\Lambda}/\aleph}}, 0^{\mathbf{\Lambda}/\aleph}, 1^{\mathbf{\Lambda}/\aleph}\rangle$ is a quasi-MV* algebra and we call that $\mathbf{\Lambda}/\aleph$ is the quotient algebra of $\mathbf{\Lambda}$ with respect to $\aleph$. Furthermore, we discuss the quotient algebra of a quasi-MV* algebra with respect to $\Re$.

Lemma 2: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then $\mathbf{\Lambda}/\Re$ is an MV*-algebra.

Proof: We only need to prove that any element in $\Lambda/\Re$ is regular. For any $\varrho/\Re \in \mathbf{\Lambda}/\Re$, since $(\varrho \uplus 0) \uplus 0 = \varrho \uplus 0$ by (QMV*4), we have $\langle \varrho \uplus 0, \varrho\rangle \in \Re$, it turns out that $(\varrho \uplus 0)/\Re = \varrho/\Re$, so $(\varrho/\Re) \uplus^{\mathbf{\Lambda}/\Re} (0/\Re) = (\varrho \uplus 0)/\Re = \varrho/\Re$. Hence $\mathbf{\Lambda}/\Re$ is an MV*-algebra. ■

Likewise, we introduce a congruence which is called the *flat congruence* on any quasi-MV* algebra.

Definition 6: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. For any $\varrho, \varsigma \in \Lambda$, we define a binary relation

$$\langle \varrho, \varsigma\rangle \in \Im \text{ iff } \varrho = \varsigma \text{ or } \varrho, \varsigma \in \mathcal{R}(\Lambda).$$

Lemma 3: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then $\Im$ is a congruence on $\mathbf{\Lambda}$.

Proof: It is easy to see that $\Im$ is an equivalence relation on $\mathbf{\Lambda}$. Now, we prove that $\Im$ satisfies the compatibility property. For any $\varrho, \varsigma, \varepsilon, \vartheta \in \Lambda$, if $\langle \varrho, \varsigma\rangle \in \Im$ and $\langle \varepsilon, \vartheta\rangle \in \Im$, then $\varrho = \varsigma$ or $\varrho, \varsigma \in \mathcal{R}(\Lambda)$, and $\varepsilon = \vartheta$ or $\varepsilon, \vartheta \in \mathcal{R}(\Lambda)$. Since $\varrho \uplus \varepsilon$ and $\varsigma \uplus \vartheta \in \mathcal{R}(\Lambda)$, we have $\langle \varrho \uplus \varepsilon, \varsigma \uplus \vartheta\rangle \in \Im$. If $\langle \varrho, \varsigma\rangle \in \Im$, then $\varrho = \varsigma$ or $\varrho, \varsigma \in \mathcal{R}(\Lambda)$. We distinguish several cases to discuss. If $\varrho = \varsigma$, then $-\varrho = -\varsigma$, so $\langle -\varrho, -\varsigma\rangle \in \Im$. If $\varrho, \varsigma \in \mathcal{R}(\Lambda)$, then $\varrho \uplus 0 = \varrho$ and $\varsigma \uplus 0 = \varsigma$. Since $-\varrho \uplus 0 = -(\varrho \uplus 0) = -\varrho$ and $-\varsigma \uplus 0 = -(\varsigma \uplus 0) = -\varsigma$ by Proposition 2(2), we have $-\varrho, -\varsigma \in \mathcal{R}(\Lambda)$, so $\langle -\varrho, -\varsigma\rangle \in \Im$. Moreover, if $\varrho = \varsigma$, then $\varrho^+ = \varsigma^+$ and $\varrho^- = \varsigma^-$, so $\langle \varrho^+, \varsigma^+\rangle \in \Im$ and $\langle \varrho^-, \varsigma^-\rangle \in \Im$. If $\varrho, \varsigma \in \mathcal{R}(\Lambda)$, then $\varrho^+ \uplus 0 = (\varrho \uplus 0)^+ = \varrho^+$ and $\varsigma^+ \uplus 0 = (\varsigma \uplus 0)^+ = \varsigma^+$ by (QMV*9), it turns out that $\varrho^+, \varsigma^+ \in \mathcal{R}(\Lambda)$, so $\langle \varrho^+, \varsigma^+\rangle \in \Im$. Similarly, we have $\langle \varrho^-, \varsigma^-\rangle \in \Im$. Hence $\Im$ is a congruence on $\mathbf{\Lambda}$. ■

Lemma 4: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then $\mathbf{\Lambda}/\Im$ is a flat quasi-MV* algebra.

Proof: Since $0, 1 \in \mathcal{R}(\Lambda)$, we have $\langle 0, 1\rangle \in \Im$, it turns out that $0/\Im = 1/\Im$, so $\mathbf{\Lambda}/\Im$ is a flat quasi-MV* algebra. ■

Lemma 5: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then

(1) $\Re \cap \Im = \Delta$, where Δ is the diagonal relation,

(2) $\Re \vee \Im = \nabla$, where $\Re \vee \Im$ is the smallest congruence which contains $\Re \cup \Im$ and ∇ is the all relation.

Proof: (1) For any $\langle \varrho, \varsigma\rangle \in \Delta$, then $\varrho = \varsigma$, so $\Delta \subseteq \Re \cap \Im$. Conversely, for any $\langle \varrho, \varsigma\rangle \in \Re \cap \Im$, then $\langle \varrho, \varsigma\rangle \in \Re$ and $\langle \varrho, \varsigma\rangle \in \Im$, we have $\varrho = \varsigma$ or $\varrho, \varsigma \in \mathcal{R}(\Lambda)$. If $\varrho = \varsigma$, then $\langle \varrho, \varsigma\rangle \in \Delta$. If $\varrho, \varsigma \in \mathcal{R}(\Lambda)$, then $\varrho \uplus 0 = \varrho$ and $\varsigma \uplus 0 = \varsigma$. Since $\langle \varrho, \varsigma\rangle \in \Re$, we have $\varrho \uplus 0 = \varsigma \uplus 0$, it follows that $\varrho = \varsigma$, so $\langle \varrho, \varsigma\rangle \in \Delta$ and then $\Re \cap \Im \subseteq \Delta$. Hence, we have $\Re \cap \Im = \Delta$.

(2) It is clear that $\Re \vee \Im \subseteq \nabla$. Now, we prove that $\nabla \subseteq \Re \vee \Im$. For any $\varrho, \varsigma \in \Lambda$, if $\langle \varrho, \varsigma\rangle \in \Re$ or $\langle \varrho, \varsigma\rangle \in \Im$, then we have $\langle \varrho, \varsigma\rangle \in \Re \vee \Im$. If $\langle \varrho, \varsigma\rangle \notin \Re$ and $\langle \varrho, \varsigma\rangle \notin \Im$, then $\varrho \uplus 0 \neq \varsigma \uplus 0$, $\varrho \neq \varsigma$, and $\varrho, \varsigma \notin \mathcal{R}(\Lambda)$, it turns out that there exist $\varrho \uplus 0, \varsigma \uplus 0 \in \mathcal{R}(\Lambda)$ such that $\varrho \uplus 0 = (\varrho \uplus 0) \uplus 0$ and $\varsigma \uplus 0 = (\varsigma \uplus 0) \uplus 0$, it means that $\langle \varrho, \varrho \uplus 0\rangle \in \Re$, $\langle \varsigma, \varsigma \uplus 0\rangle \in \Re$, $\langle \varrho \uplus 0, \varsigma \uplus 0\rangle \in \Im$, and then $\langle \varrho, \varrho \uplus 0\rangle \in \Re \vee \Im$, $\langle \varsigma, \varsigma \uplus 0\rangle \in \Re \vee \Im$, $\langle \varrho \uplus 0, \varsigma \uplus 0\rangle \in \Re \vee \Im$. Since $\Re \vee \Im$ is a congruence on $\mathbf{\Lambda}$, we have $\langle \varrho, \varsigma\rangle \in \Re \vee \Im$, it turns out that $\nabla \subseteq \Re \vee \Im$. Hence $\Re \vee \Im = \nabla$. ■

Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then $\mathbf{\Lambda}$ is called *simple*, if the set of all congruences on $\mathbf{\Lambda}$ is $\{\Delta, \nabla\}$.

Lemma 6: Let $\mathbf{\Lambda}$ be a simple quasi-MV* algebra. Then either $\mathbf{\Lambda}$ is an MV*-algebra or $\mathbf{\Lambda}$ is a flat quasi-MV* algebra.

Proof: Let $\mathbf{\Lambda}$ be a simple quasi-MV* algebra. Then either $\Re = \Delta$ or $\Re = \nabla$. If it is the former, then $\mathbf{\Lambda}$ is an

MV*-algebra. If it is the latter, then $\mathbf{\Lambda}$ is a flat quasi-MV* algebra. ■

Definition 7: Let $\mathbf{\Lambda_1}$ and $\mathbf{\Lambda_2}$ be quasi-MV* algebras. A mapping $f : \Lambda_1 \to \Lambda_2$ is called a *quasi-MV* algebra homomorphism* from $\mathbf{\Lambda_1}$ to $\mathbf{\Lambda_2}$, if $f(0^{\mathbf{\Lambda_1}}) = 0^{\mathbf{\Lambda_2}}$, $f(1^{\mathbf{\Lambda_1}}) = 1^{\mathbf{\Lambda_2}}$, $f(\varrho_1 \uplus^{\mathbf{\Lambda_1}} \varrho_2) = f(\varrho_1) \uplus^{\mathbf{\Lambda_2}} f(\varrho_2)$, $f(-^{\mathbf{\Lambda_1}}\varrho_1) = -^{\mathbf{\Lambda_2}} f(\varrho_1)$, $f(\varrho_1^{+^{\mathbf{\Lambda_1}}}) = f(\varrho_1)^{+^{\mathbf{\Lambda_2}}}$, $f(\varrho_1^{-^{\mathbf{\Lambda_1}}}) = f(\varrho_1)^{-^{\mathbf{\Lambda_2}}}$ for any $\varrho_1, \varrho_2 \in \Lambda_1$. Moreover, if the mapping f is injective, then f is called *monomorphism*. Such a mapping f is also called an *embedding*.

Proposition 3: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra. Then there exist an MV*-algebra $\mathbf{\Sigma}$ and a flat quasi-MV* algebra $\mathbf{\Gamma}$ such that $\mathbf{\Lambda}$ can be embedded into the direct product $\mathbf{\Sigma} \times \mathbf{\Gamma}$.

Proof: Denote $\mathbf{\Sigma}=\mathbf{\Lambda}/\Re$ and $\mathbf{\Gamma} = \mathbf{\Lambda}/\Im$. Then we have that $\mathbf{\Sigma}$ is an MV*-algebra by Lemma 2 and $\mathbf{\Gamma}$ is a flat MV*-algebra by Lemma 4. Define a mapping $f : \Lambda \to \Lambda/\Re \times \Lambda/\Im$ by $f(\varrho) = \langle \varrho/\Re, \varrho/\Im \rangle$ for any $\varrho \in \Lambda$. We show that f is a homomorphism. Obviously, we have $f(0) = \langle 0/\Re, 0/\Im \rangle$, $f(1) = \langle 1/\Re, 1/\Im \rangle$. For any $\varrho, \varsigma \in \Lambda$, we have $f(\varrho) \uplus^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im} f(\varsigma) = \langle \varrho/\Re, \varrho/\Im \rangle \uplus^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im} \langle \varsigma/\Re, \varsigma/\Im \rangle = \langle \varrho/\Re \uplus^{\mathbf{\Lambda}/\Re} \varsigma/\Re, \varrho/\Im \uplus^{\mathbf{\Lambda}/\Im} \varsigma/\Im \rangle = \langle (\varrho \uplus \varsigma)/\Re, (\varrho \uplus \varsigma)/\Im \rangle = f(\varrho \uplus \varsigma)$. For any $\varrho \in \Lambda$, we have $-^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im} f(\varrho) = -^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im} \langle \varrho/\Re, \varrho/\Im \rangle = \langle -^{\mathbf{\Lambda}/\Re}(\varrho/\Re), -^{\mathbf{\Lambda}/\Im}(\varrho/\Im) \rangle = \langle (-\varrho)/\Re, (-\varrho)/\Im \rangle = f(-\varrho)$. Moreover, we have $f(\varrho)^{+^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im}} = \langle \varrho/\Re, \varrho/\Im \rangle^{+^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im}} = \langle (\varrho/\Re)^{+^{\mathbf{\Lambda}/\Re}}, (\varrho/\Im)^{+^{\mathbf{\Lambda}/\Im}} \rangle = \langle \varrho^+/\Re, \varrho^+/\Im \rangle = f(\varrho^+)$ and $f(\varrho)^{-^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im}} = \langle \varrho/\Re, \varrho/\Im \rangle^{-^{\mathbf{\Lambda}/\Re \times \mathbf{\Lambda}/\Im}} = \langle (\varrho/\Re)^{-^{\mathbf{\Lambda}/\Re}}, (\varrho/\Im)^{-^{\mathbf{\Lambda}/\Im}} \rangle = \langle \varrho^-/\Re, \varrho^-/\Im \rangle = f(\varrho^-)$. Finally, if $f(\varrho) = f(\varsigma)$, then $\langle \varrho/\Re, \varrho/\Im \rangle = \langle \varsigma/\Re, \varsigma/\Im \rangle$, it turns out that $\varrho/\Re = \varsigma/\Re$ and $\varrho/\Im = \varsigma/\Im$, which means that $\langle \varrho, \varsigma \rangle \in \Re$ and $\langle \varrho, \varsigma \rangle \in \Im$. By Lemma 5(1), we have $\varrho = \varsigma$, so f is injective. Hence $\mathbf{\Lambda}$ can be embedded into the direct product $\mathbf{\Sigma} \times \mathbf{\Gamma}$. ■

Corollary 1: Let $\mathbf{\Lambda}$ be a simple quasi-MV* algebra. Then the embedding in Proposition 3 is an isomorphism.

Based on the subdirect product decomposition of a quasi-MV* algebra, we can transform the study on the CEP of quasi-MV* algebras into the study on the CEPs of MV*-algebras and flat quasi-MV* algebras, respectively.

Definition 8: [17] A variety $\mathbb{K}$ is called to have the *congruence extension property* (for short, CEP), iff for any $\mathbf{\Lambda} \in \mathbb{K}$, any subalgebra $\mathbf{\Upsilon}$ of $\mathbf{\Lambda}$ and for any congruence $\aleph$ on $\mathbf{\Upsilon}$, there exists a congruence φ on $\mathbf{\Lambda}$ such that $\aleph = \varphi \cap \Upsilon^2$.

According to Theorem 1, we have that $\mathbb{MV}^*$ has the CEP iff $\mathbb{MV}^*$ has the ideal extension property, i.e., iff for any $\mathbf{\Sigma} \in \mathbb{MV}^*$, any subalgebra $\mathbf{\Upsilon}$ of $\mathbf{\Sigma}$ and for any ideal Φ of $\mathbf{\Upsilon}$, there exists an ideal Ψ of $\mathbf{\Sigma}$ such that $\Phi = \Psi \cap \Upsilon$.

Lemma 7: Let $\mathbf{\Sigma}$ be an MV*-algebra, $\mathbf{\Upsilon}$ be a subalgebra of $\mathbf{\Sigma}$, and Φ be an ideal of $\mathbf{\Upsilon}$. Then $(\Phi] = \{\varrho \in \Sigma : a_1 \uplus \cdots \uplus a_n \leqslant \varrho \leqslant b_1 \uplus \cdots \uplus b_m$, for some $a_1, ..., a_n, b_1, ..., b_m \in \Phi\}$ is an ideal of $\mathbf{\Sigma}$.

Proof: Since $\Phi \subseteq (\Phi]$, we have that the set $(\Phi]$ is non empty. If $\varrho, \varsigma \in (\Phi]$, then there exist $a_1, ..., a_n$, $b_1, ..., b_m, c_1, ..., c_\ell, d_1, ..., d_\jmath \in \Phi$ such that $a_1 \uplus \cdots \uplus a_n \leqslant \varrho \leqslant b_1 \uplus \cdots \uplus b_m$ and $c_1 \uplus \cdots \uplus c_\ell \leqslant \varsigma \leqslant d_1 \uplus \cdots \uplus d_\jmath$. By Proposition 1(4) and Proposition 2(7), we have $a_1 \uplus \cdots \uplus a_n$, $b_1 \uplus \cdots \uplus b_m$, $c_1 \uplus \cdots \uplus c_\ell$, $d_1 \uplus \cdots \uplus d_\jmath \in \Phi$, and $-(d_1 \uplus \cdots \uplus d_\jmath) \leqslant -\varsigma \leqslant -(c_1 \uplus \cdots \uplus c_\ell)$. Since Φ is an ideal of $\mathbf{\Upsilon}$, we have $(a_1 \uplus \cdots \uplus a_n) \ominus (d_1 \uplus \cdots \uplus d_\jmath) \in \Phi$ and $(b_1 \uplus \cdots \uplus b_m) \ominus (c_1 \uplus \cdots \uplus c_\ell) \in \Phi$ by (Φ1). Meanwhile, we have $(a_1 \uplus \cdots \uplus a_n) \uplus (-(d_1 \uplus \cdots \uplus d_\jmath)) \leqslant \varrho \uplus (-\varsigma) \leqslant (b_1 \uplus \cdots \uplus b_m) \uplus (-(c_1 \uplus \cdots \uplus c_\ell))$ by Lemma 2(7), it turns out that $(a_1 \uplus \cdots \uplus a_n) \ominus (d_1 \uplus \cdots \uplus d_\jmath) \leqslant \varrho \ominus \varsigma \leqslant (b_1 \uplus \cdots \uplus b_m) \ominus (c_1 \uplus \cdots \uplus c_\ell)$, so $\varrho \ominus \varsigma \in (\Phi]$ by (Φ3). If $\varrho \in (\Phi]$, then there exist $a_1, ..., a_n, b_1, ..., b_m \in \Phi$ such that $a_1 \uplus \cdots \uplus a_n \leqslant \varrho \leqslant b_1 \uplus \cdots \uplus b_m$. By Proposition 1(4), we have $a_1 \uplus \cdots \uplus a_n \in \Phi$ and $b_1 \uplus \cdots \uplus b_m \in \Phi$. Since Φ is an ideal of $\mathbf{\Upsilon}$, we have $(a_1 \uplus \cdots \uplus a_n)^+ \in \Phi$ and $(b_1 \uplus \cdots \uplus b_m)^+ \in \Phi$ by (Φ2). By Proposition 2(8), we have $(a_1 \uplus \cdots \uplus a_n)^+ \leqslant \varrho^+ \leqslant (b_1 \uplus \cdots \uplus b_m)^+$, so $\varrho^+ \in (\Phi]$ by (Φ3). If $\varrho, \kappa \in (\Phi]$ and $\varsigma \in \Sigma$ with $\varrho \leqslant \varsigma \leqslant \kappa$, then there exist $a_1, ..., a_n, b_1, ..., b_m, c_1, ..., c_\ell, d_1, ..., d_\jmath \in \Phi$ such that $a_1 \uplus \cdots \uplus a_n \leqslant \varrho \leqslant b_1 \uplus \cdots \uplus b_m$ and $c_1 \uplus \cdots \uplus c_\ell \leqslant \kappa \leqslant d_1 \uplus \cdots \uplus d_\jmath$, it turns out that $a_1 \uplus \cdots \uplus a_n \leqslant \varsigma \leqslant d_1 \uplus \cdots \uplus d_\jmath$, so $\varsigma \in (\Phi]$ by (Φ3). Hence $(\Phi]$ is an ideal of $\mathbf{\Sigma}$. ■

Theorem 2: The variety $\mathbb{MV}^*$ has the CEP.

Proof: For any $\mathbf{\Sigma} \in \mathbb{MV}^*$, $\mathbf{\Upsilon}$ is a subalgebra of $\mathbf{\Sigma}$, and Φ is an ideal of $\mathbf{\Upsilon}$. Then we have that $(\Phi]$ is an ideal of $\mathbf{\Sigma}$ by Lemma 7. Below we prove that $\Phi = (\Phi] \cap \Upsilon$. For any $\varrho \in \Phi$, since $\varrho \leqslant \varrho \leqslant \varrho$, we have $\varrho \in (\Phi]$ by (Φ3) and then $\varrho \in (\Phi] \cap \Upsilon$, so $\Phi \subseteq (\Phi] \cap \Upsilon$. For any $\varrho \in (\Phi] \cap \Upsilon$, then there exist $a_1, ..., a_n, b_1, ..., b_m \in \Phi$ such that $a_1 \uplus \cdots \uplus a_n \leqslant \varrho \leqslant b_1 \uplus \cdots \uplus b_m$. Since Φ is an ideal of $\mathbf{\Upsilon}$, we have $a_1 \uplus \cdots \uplus a_n \in \Phi$ and $b_1 \uplus \cdots \uplus b_m \in \Phi$ by Proposition 1(4), it turns out that $\varrho \in \Phi$ by (Φ3), so $(\Phi] \cap \Upsilon \subseteq \Phi$ and then $(\Phi] \cap \Upsilon = \Phi$. Hence the variety $\mathbb{MV}^*$ has the CEP. ■

Since $\mathbb{MV}^*$ has the CEP, we next need to discuss the variety $\mathbb{FQMV}^*$.

Lemma 8: The variety $\mathbb{FQMV}^*$ has the CEP.

Proof: For any $\mathbf{\Lambda} \in \mathbb{FQMV}^*$, $\mathbf{\Upsilon}$ is a subalgebra of $\mathbf{\Lambda}$, and $\aleph$ is a congruence on $\mathbf{\Upsilon}$. We define a binary relation $\aleph' = \{\langle \varrho, \varsigma \rangle \in \Lambda^2 \mid \langle \varrho, \varsigma \rangle \in \aleph$ or $\varrho = \varsigma\}$. Then $\aleph'$ is the congruence on $\mathbf{\Lambda}$ such that $\aleph = \aleph' \cap \Upsilon^2$. Indeed, it is easy to see that $\aleph'$ is an equivalence on $\mathbf{\Lambda}$. Suppose that $\langle \varrho, \varsigma \rangle \in \aleph'$ and $\langle \varepsilon, \vartheta \rangle \in \aleph'$. Since $\mathbf{\Lambda}$ is flat, we have $\langle \varrho \uplus \varepsilon, \varsigma \uplus \vartheta \rangle = \langle 0, 0 \rangle \in \aleph'$. For any $\langle \varrho, \varsigma \rangle \in \aleph'$, then $\langle \varrho, \varsigma \rangle \in \aleph$ or $\varrho = \varsigma$. If $\langle \varrho, \varsigma \rangle \in \aleph$, since $\aleph$ is a congruence on $\mathbf{\Upsilon}$, we have $\langle -\varrho, -\varsigma \rangle \in \aleph$, so $\langle -\varrho, -\varsigma \rangle \in \aleph'$. If $\varrho = \varsigma$, then $-\varrho = -\varsigma$, we also have $\langle -\varrho, -\varsigma \rangle \in \aleph'$. Moreover, if $\varrho = \varsigma$, then $\varrho^+ = \varsigma^+$ and $\varrho^- = \varsigma^-$, so $\langle \varrho^+, \varsigma^+ \rangle \in \aleph'$ and $\langle \varrho^-, \varsigma^- \rangle \in \aleph'$. If $\langle \varrho, \varsigma \rangle \in \aleph$, since $\aleph$ is a congruence on $\mathbf{\Upsilon}$, we have $\langle \varrho^+, \varsigma^+ \rangle \in \aleph$ and $\langle \varrho^-, \varsigma^- \rangle \in \aleph$, so $\langle \varrho^+, \varsigma^+ \rangle \in \aleph'$ and $\langle \varrho^-, \varsigma^- \rangle \in \aleph'$. Hence $\aleph'$ is a congruence on $\mathbf{\Lambda}$. For any $\langle \varrho, \varsigma \rangle \in \aleph$, then $\langle \varrho, \varsigma \rangle \in \Upsilon^2$ and $\langle \varrho, \varsigma \rangle \in \aleph'$, it turns out that $\langle \varrho, \varsigma \rangle \in \aleph' \cap \Upsilon^2$, so $\aleph \subseteq \aleph' \cap \Upsilon^2$. Conversely, for any $\langle \varrho, \varsigma \rangle \in \aleph' \cap \Upsilon^2$, then $\langle \varrho, \varsigma \rangle \in \Upsilon^2$ and $\langle \varrho, \varsigma \rangle \in \aleph'$, it turns out that $\varrho = \varsigma$ or $\langle \varrho, \varsigma \rangle \in \aleph$. If $\langle \varrho, \varsigma \rangle \in \aleph$, then the result is true. If $\varrho = \varsigma$, since $\aleph$ is a congruence on $\mathbf{\Upsilon}$, we also have $\langle \varrho, \varsigma \rangle \in \aleph$, so $\aleph' \cap \Upsilon^2 \subseteq \aleph$ and then $\aleph = \aleph' \cap \Upsilon^2$. Hence the variety $\mathbb{FQMV}^*$ has the CEP. ■

Lemma 9: Let $\mathbf{\Lambda}$ be a quasi-MV* algebra and $\aleph$ be a congruence on $\mathbf{\Lambda}$. Then there exist a congruence $\aleph_1$ on $\mathbf{\Lambda}/\Re$ and a congruence $\aleph_2$ on $\mathbf{\Lambda}/\Im$ such that $\langle \varrho, \varsigma \rangle \in \aleph$ iff $\langle \langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle \rangle \in \aleph_1 \times \aleph_2$.

Proof: Define a binary relation $\aleph' = \{\langle \langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle \rangle : \langle \varrho, \varsigma \rangle \in \aleph\}$. Then $\aleph'$ is a congruence on $\mathbf{\Lambda}$ such that $\langle \varrho, \varsigma \rangle \in \aleph$ iff $\langle \langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle \rangle \in \aleph'$. It is clear that $\aleph'$ is an equivalent relation. For any $\langle \langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle \rangle \in \aleph'$ and $\langle \langle \varepsilon/\Re, \varepsilon/\Im \rangle, \langle \vartheta/\Re, \vartheta/\Im \rangle \rangle \in \aleph'$, then $\langle \varrho, \varsigma \rangle \in \aleph$

and $\langle \varepsilon, \vartheta \rangle \in \aleph$. Since $\aleph$ is a congruence on $\boldsymbol{\Lambda}$, we have $\langle \varrho \uplus \varepsilon, \varsigma \uplus \vartheta \rangle \in \aleph$, it turns out that $\langle\langle (\varrho \uplus \varepsilon)/\Re, (\varrho \uplus \varepsilon)/\Im \rangle, \langle (\varsigma \uplus \vartheta)/\Re, (\varsigma \uplus \vartheta)/\Im \rangle\rangle \in \aleph'$, so $\langle\langle (\varrho/\Re) \uplus^{\boldsymbol{\Lambda}/\Re} (\varepsilon/\Re), (\varrho/\Im) \uplus^{\boldsymbol{\Lambda}/\Im} (\varepsilon/\Im) \rangle, \langle (\varsigma/\Re) \uplus^{\boldsymbol{\Lambda}/\Re} (\vartheta/\Re), (\varsigma/\Im) \uplus^{\boldsymbol{\Lambda}/\Im} (\vartheta/\Im) \rangle\rangle \in \aleph'$. For any $\langle\langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle\rangle \in \aleph'$, then $\langle \varrho, \varsigma \rangle \in \aleph$. Since $\aleph$ is a congruence on $\boldsymbol{\Lambda}$, we have $\langle -\varrho, -\varsigma \rangle \in \aleph$, it turns out that $\langle\langle (-\varrho)/\Re, (-\varrho)/\Im \rangle, \langle (-\varsigma)/\Re, (-\varsigma)/\Im \rangle\rangle \in \aleph'$, which means that $\langle\langle -^{\boldsymbol{\Lambda}/\Re}(\varrho/\Re), -^{\boldsymbol{\Lambda}/\Im}(\varrho/\Im) \rangle, \langle -^{\boldsymbol{\Lambda}/\Re}(\varsigma/\Re), -^{\boldsymbol{\Lambda}/\Im}(\varsigma/\Im) \rangle\rangle \in \aleph'$. Moreover, we have $\langle \varrho^+, \varsigma^+ \rangle \in \aleph$ and $\langle \varrho^-, \varsigma^- \rangle \in \aleph$, it turns out that $\langle\langle (\varrho^+)/\Re, (\varrho^+)/\Im \rangle, \langle (\varsigma^+)/\Re, (\varsigma^+)/\Im \rangle \in \aleph$ and $\langle\langle (\varrho^-)/\Re, (\varrho^-)/\Im \rangle, \langle (\varsigma^-)/\Re, (\varsigma^-)/\Im \rangle \in \aleph$, which means that $\langle\langle (\varrho/\Re)^{+^{\boldsymbol{\Lambda}/\Re}}, (\varrho/\Im)^{+^{\boldsymbol{\Lambda}/\Im}} \rangle, \langle (\varsigma/\Re)^{+^{\boldsymbol{\Lambda}/\Re}}, (\varsigma/\Im)^{+^{\boldsymbol{\Lambda}/\Im}} \rangle \in \aleph$ and $\langle\langle (\varrho/\Re)^{-^{\boldsymbol{\Lambda}/\Re}}, (\varrho/\Im)^{-^{\boldsymbol{\Lambda}/\Im}} \rangle, \langle (\varsigma/\Re)^{-^{\boldsymbol{\Lambda}/\Re}}, (\varsigma/\Im)^{-^{\boldsymbol{\Lambda}/\Im}} \rangle \in \aleph$. So $\aleph'$ is a congruence on $\boldsymbol{\Lambda}$ and $\langle \varrho, \varsigma \rangle \in \aleph$ iff $\langle\langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle\rangle \in \aleph'$ for any $\varrho, \varsigma \in \Lambda$.

Define a binary relation $\aleph_1$ on $\boldsymbol{\Lambda}/\Re$ by $\langle \varrho/\Re, \varsigma/\Re \rangle \in \aleph_1$ iff $\langle\langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle\rangle \in \aleph'$. Define a binary relation $\aleph_2$ on $\boldsymbol{\Lambda}/\Im$ by $\langle \varrho/\Im, \varsigma/\Im \rangle \in \aleph_2$ iff $\langle\langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle\rangle \in \aleph'$. It is clear that $\aleph_1$ is a congruence on $\boldsymbol{\Lambda}/\Re$ and $\aleph_2$ is a congruence on $\boldsymbol{\Lambda}/\Im$. Moreover, $\langle \varrho, \varsigma \rangle \in \aleph$ iff $\langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle\rangle \in \aleph'$, iff $\langle \varrho/\Re, \varsigma/\Re \rangle \in \aleph_1$ and $\langle \varrho/\Im, \varsigma/\Im \rangle \in \aleph_2$, iff $\langle\langle \varrho/\Re, \varrho/\Im \rangle, \langle \varsigma/\Re, \varsigma/\Im \rangle\rangle \in \aleph_1 \times \aleph_2$. ■

Lemma 10: Let $\boldsymbol{\Lambda}$ be a quasi-MV* algebra and $\boldsymbol{\Upsilon}$ be a subalgebra of $\boldsymbol{\Lambda}$. Then the congruence $\Re^{\boldsymbol{\Upsilon}}$ on $\boldsymbol{\Upsilon}$ extends to the congruence $\Re^{\boldsymbol{\Lambda}}$ on $\boldsymbol{\Lambda}$ and the congruence $\Im^{\boldsymbol{\Upsilon}}$ on $\boldsymbol{\Upsilon}$ extends to the congruence $\Im^{\boldsymbol{\Lambda}}$ on $\boldsymbol{\Lambda}$.

Proof: We only prove that $\Re^{\boldsymbol{\Upsilon}} = \Re^{\boldsymbol{\Lambda}} \cap \Upsilon^2$ and $\Im^{\boldsymbol{\Upsilon}} = \Im^{\boldsymbol{\Lambda}} \cap \Upsilon^2$. For any $\langle \varrho, \varsigma \rangle \in \Re^{\boldsymbol{\Upsilon}}$, then $\langle \varrho, \varsigma \rangle \in \Upsilon^2$ and $\varrho \uplus 0 = \varsigma \uplus 0$. Because $\boldsymbol{\Upsilon}$ is a subalgebra of $\boldsymbol{\Lambda}$, we have $\langle \varrho, \varsigma \rangle \in \Lambda^2$, it turns out that $\langle \varrho, \varsigma \rangle \in \Re^{\boldsymbol{\Lambda}}$ and then $\langle \varrho, \varsigma \rangle \in \Re^{\boldsymbol{\Lambda}} \cap \Upsilon^2$, so $\Re^{\boldsymbol{\Upsilon}} \subseteq \Re^{\boldsymbol{\Lambda}} \cap \Upsilon^2$. For any $\langle \varrho, \varsigma \rangle \in \Re^{\boldsymbol{\Lambda}} \cap \Upsilon^2$, then $\langle \varrho, \varsigma \rangle \in \Upsilon^2$ and $\varrho \uplus 0 = \varsigma \uplus 0$, it turns out that $\langle \varrho, \varsigma \rangle \in \Re^{\boldsymbol{\Upsilon}}$, so $\Re^{\boldsymbol{\Lambda}} \cap \Upsilon^2 \subseteq \Re^{\boldsymbol{\Upsilon}}$ and then $\Re^{\boldsymbol{\Upsilon}} = \Re^{\boldsymbol{\Lambda}} \cap \Upsilon^2$. Similarly, for any $\langle \varrho, \varsigma \rangle \in \Im^{\boldsymbol{\Upsilon}}$, then $\langle \varrho, \varsigma \rangle \in \Upsilon^2$, and $\varrho = \varsigma$ or $\varrho, \varsigma \in \mathcal{R}(\Upsilon)$. Because $\boldsymbol{\Upsilon}$ is a subalgebra of $\boldsymbol{\Lambda}$, we have $\langle \varrho, \varsigma \rangle \in \Lambda^2$, it turns out that $\langle \varrho, \varsigma \rangle \in \Im^{\boldsymbol{\Lambda}}$ and then $\langle \varrho, \varsigma \rangle \in \Im^{\boldsymbol{\Lambda}} \cap \Upsilon^2$, so $\Im^{\boldsymbol{\Upsilon}} \subseteq \Im^{\boldsymbol{\Lambda}} \cap \Upsilon^2$. For any $\langle \varrho, \varsigma \rangle \in \Im^{\boldsymbol{\Lambda}} \cap \Upsilon^2$, then $\langle \varrho, \varsigma \rangle \in \Upsilon^2$, and $\varrho = \varsigma$ or $\varrho, \varsigma \in \mathcal{R}(\Lambda)$, it turns out that $\langle \varrho, \varsigma \rangle \in \Im^{\boldsymbol{\Upsilon}}$, so $\Im^{\boldsymbol{\Lambda}} \cap \Upsilon^2 \subseteq \Im^{\boldsymbol{\Upsilon}}$ and then $\Im^{\boldsymbol{\Upsilon}} = \Im^{\boldsymbol{\Lambda}} \cap \Upsilon^2$. Hence the congruence $\Re^{\boldsymbol{\Upsilon}}$ on $\boldsymbol{\Upsilon}$ extends to the congruence $\Re^{\boldsymbol{\Lambda}}$ on $\boldsymbol{\Lambda}$ and the congruence $\Im^{\boldsymbol{\Upsilon}}$ on $\boldsymbol{\Upsilon}$ extends to the congruence $\Im^{\boldsymbol{\Lambda}}$ on $\boldsymbol{\Lambda}$. ■

Lemma 11: Let $\boldsymbol{\Lambda}$ be a quasi-MV* algebra and $\boldsymbol{\Upsilon}$ be a subalgebra of $\boldsymbol{\Lambda}$. Then $\boldsymbol{\Upsilon}/\Re^{\boldsymbol{\Upsilon}}$ is a subalgebra of $\boldsymbol{\Lambda}/\Re^{\boldsymbol{\Lambda}}$ and $\boldsymbol{\Upsilon}/\Im^{\boldsymbol{\Upsilon}}$ is a subalgebra of $\boldsymbol{\Lambda}/\Im^{\boldsymbol{\Lambda}}$.

Proof: Let $\boldsymbol{\Lambda} = \langle A; \uplus^{\boldsymbol{\Lambda}}, -^{\boldsymbol{\Lambda}}, {}^{+^{\boldsymbol{\Lambda}}}, {}^{-^{\boldsymbol{\Lambda}}}, 0, 1 \rangle$ be a quasi-MV* algebra and $\boldsymbol{\Upsilon} = \langle \Upsilon; \uplus^{\boldsymbol{\Upsilon}}, -^{\boldsymbol{\Upsilon}}, {}^{+^{\boldsymbol{\Upsilon}}}, {}^{-^{\boldsymbol{\Upsilon}}}, 0, 1 \rangle$ be a subalgebra of $\boldsymbol{\Lambda}$. Then $\boldsymbol{\Lambda}/\Re^{\boldsymbol{\Lambda}} = \langle A/\Re^{\boldsymbol{\Lambda}}; \uplus_1, -_1, {}^{+_1}, {}^{-_1}, 0/\Re^{\boldsymbol{\Lambda}}, 1/\Re^{\boldsymbol{\Lambda}} \rangle$ and $\boldsymbol{\Upsilon}/\Re^{\boldsymbol{\Upsilon}} = \langle \Upsilon/\Re^{\boldsymbol{\Upsilon}}; \uplus_2, -_2, {}^{+_2}, {}^{-_2}, 0/\Re^{\boldsymbol{\Upsilon}}, 1/\Re^{\boldsymbol{\Upsilon}} \rangle$ are MV* algebras. For any $\varrho/\Re^{\boldsymbol{\Upsilon}} \in \Upsilon/\Re^{\boldsymbol{\Upsilon}}$, then $\varrho/\Re^{\boldsymbol{\Upsilon}} = \varrho/(\Re^{\boldsymbol{\Lambda}} \cap \Upsilon^2) = (\varrho/\Re^{\boldsymbol{\Lambda}}) \cap \Upsilon \subseteq \varrho/\Re^{\boldsymbol{\Lambda}}$ by Lemma 10. Since $\boldsymbol{\Upsilon}$ is a subalgebra of $\boldsymbol{\Lambda}$, we have $\Upsilon/\Re^{\boldsymbol{\Lambda}} \subseteq \Lambda/\Re^{\boldsymbol{\Lambda}}$ and then $\Upsilon/\Re^{\boldsymbol{\Upsilon}} \subseteq \Lambda/\Re^{\boldsymbol{\Lambda}}$. For any $\varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \in \boldsymbol{\Upsilon}/\Re^{\boldsymbol{\Upsilon}}$, we have $(\varrho/\Re^{\boldsymbol{\Upsilon}}) \uplus_1 (\varsigma/\Re^{\boldsymbol{\Upsilon}}) = (\varrho \uplus^{\boldsymbol{\Lambda}} \varsigma)/\Re^{\boldsymbol{\Upsilon}} = (\varrho \uplus^{\boldsymbol{\Upsilon}} \varsigma)/\Re^{\boldsymbol{\Upsilon}} = (\varrho/\Re^{\boldsymbol{\Upsilon}}) \uplus_2 (\varsigma/\Re^{\boldsymbol{\Upsilon}})$. For any $\varrho/\Re^{\boldsymbol{\Upsilon}} \in \boldsymbol{\Upsilon}/\Re^{\boldsymbol{\Upsilon}}$, we have $-_1(\varrho/\Re^{\boldsymbol{\Upsilon}}) = (-^{\boldsymbol{\Lambda}}\varrho)/\Re^{\boldsymbol{\Upsilon}} = (-^{\boldsymbol{\Upsilon}}\varrho)/\Re^{\boldsymbol{\Upsilon}} = -_2(\varrho/\Re^{\boldsymbol{\Upsilon}})$. Moreover, we have $(\varrho/\Re^{\boldsymbol{\Upsilon}})^{+_1} = (\varrho^{+^{\boldsymbol{\Lambda}}})/\Re^{\boldsymbol{\Upsilon}} = (\varrho^{+^{\boldsymbol{\Upsilon}}})/\Re^{\boldsymbol{\Upsilon}} = (\varrho/\Re^{\boldsymbol{\Upsilon}})^{+_2}$ and $(\varrho/\Re^{\boldsymbol{\Upsilon}})^{-_1} = (\varrho^{-^{\boldsymbol{\Lambda}}})/\Re^{\boldsymbol{\Upsilon}} = (\varrho^{-^{\boldsymbol{\Upsilon}}})/\Re^{\boldsymbol{\Upsilon}} = (\varrho/\Re^{\boldsymbol{\Upsilon}})^{-_2}$. Hence $\boldsymbol{\Upsilon}/\Re^{\boldsymbol{\Upsilon}}$ is a subalgebra of $\boldsymbol{\Lambda}/\Re^{\boldsymbol{\Lambda}}$. Similarly, $\boldsymbol{\Upsilon}/\Im^{\boldsymbol{\Upsilon}}$ is a subalgebra of $\boldsymbol{\Lambda}/\Im^{\boldsymbol{\Lambda}}$. ■

Theorem 3: The variety $\mathbb{QMV}^*$ has the CEP.

Proof: For any $\boldsymbol{\Lambda} \in \mathbb{QMV}^*$, $\boldsymbol{\Upsilon}$ is a subalgebra of $\boldsymbol{\Lambda}$ and $\aleph$ is a congruence on $\boldsymbol{\Upsilon}$. Then there exist a congruence $\aleph_1$ on $\boldsymbol{\Upsilon}/\Re^{\boldsymbol{\Upsilon}}$ and a congruence $\aleph_2$ on $\boldsymbol{\Upsilon}/\Im^{\boldsymbol{\Upsilon}}$ such that $\langle \varrho, \varsigma \rangle \in \aleph$ iff $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in \aleph_1 \times \aleph_2$ for any $\varrho, \varsigma \in \Upsilon$ by Lemma 9. Moreover, since the CEP holds for $\mathbb{MV}^*$ and $\mathbb{FQMV}^*$ by Theorem 2 and Lemma 8, respectively, we have a congruence $\aleph_1'$ on $\boldsymbol{\Lambda}/\Re^{\boldsymbol{\Lambda}}$ and a congruence $\aleph_2'$ on $\boldsymbol{\Lambda}/\Im^{\boldsymbol{\Lambda}}$ such that $\aleph_1 = \aleph_1' \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}})$ and $\aleph_2 = \aleph_2' \cap (\Upsilon/\Im^{\boldsymbol{\Upsilon}})$. Now, we only show that $\aleph_1 \times \aleph_2 = (\aleph_1' \times \aleph_2') \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}})$. For any $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in \aleph_1 \times \aleph_2$, then $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \aleph_1$ and $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \aleph_2$. Since $\aleph_1 = \aleph_1' \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}})$ and $\aleph_2 = \aleph_2' \cap (\Upsilon/\Im^{\boldsymbol{\Upsilon}})$, we have $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \aleph_1' \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}})$ and $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \aleph_2' \cap (\Upsilon/\Im^{\boldsymbol{\Upsilon}})$, it turns out that $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \aleph_1'$, $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \Upsilon/\Re^{\boldsymbol{\Upsilon}}$, and $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \aleph_2'$, $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \Upsilon/\Im^{\boldsymbol{\Upsilon}}$, which imply that $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in \aleph_1' \times \aleph_2'$ and $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in \Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}}$. So $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in (\aleph_1' \times \aleph_2') \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}})$, and then we get $\aleph_1 \times \aleph_2 \subseteq (\aleph_1' \times \aleph_2') \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}})$. For any $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}} \times \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}} \times \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in (\aleph_1' \times \aleph_2') \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}})$, then $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}} \times \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}} \times \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in \aleph_1' \times \aleph_2'$ and $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}} \times \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}} \times \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in \Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}}$, it turns out that $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \aleph_1'$, $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \aleph_2'$, and $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \Upsilon/\Re^{\boldsymbol{\Upsilon}}$, $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \Upsilon/\Im^{\boldsymbol{\Upsilon}}$, which imply that $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \aleph_1' \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}})$ and $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \aleph_2' \cap (\Upsilon/\Im^{\boldsymbol{\Upsilon}})$. Since $\aleph_1 = \aleph_1' \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}})$ and $\aleph_2 = \aleph_2' \cap (\Upsilon/\Im^{\boldsymbol{\Upsilon}})$, we have $\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Re^{\boldsymbol{\Upsilon}} \rangle \in \aleph_1$ and $\langle \varrho/\Im^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle \in \aleph_2$, it turns out that $\langle\langle \varrho/\Re^{\boldsymbol{\Upsilon}}, \varrho/\Im^{\boldsymbol{\Upsilon}} \rangle, \langle \varsigma/\Re^{\boldsymbol{\Upsilon}}, \varsigma/\Im^{\boldsymbol{\Upsilon}} \rangle\rangle \in \aleph_1 \times \aleph_2$, so $(\aleph_1' \times \aleph_2') \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}}) \subseteq \aleph_1 \times \aleph_2$, and then $(\aleph_1' \times \aleph_2') \cap (\Upsilon/\Re^{\boldsymbol{\Upsilon}} \times \Upsilon/\Im^{\boldsymbol{\Upsilon}}) = \aleph_1 \times \aleph_2$. Hence the variety $\mathbb{QMV}^*$ has the CEP. ■

At the end of this section, we demonstrate the congruence extension property of quasi-MV* algebras with an illustrative example.

Remark 3: Let $\boldsymbol{\Lambda}'$ be the algebra defined in Example 3 and $\boldsymbol{\Lambda}$ be the algebra defined in Example 2. Then $\boldsymbol{\Lambda}$ is the subalgebra of $\boldsymbol{\Lambda}'$, where the operations $\uplus$, $-$, $^+$, $^-$ of $\boldsymbol{\Lambda}$ are those of $\boldsymbol{\Lambda}'$ restricted to Λ and $\boldsymbol{\Lambda} \cong \boldsymbol{\Sigma} \times \boldsymbol{\Lambda_1}$, where $\boldsymbol{\Sigma}$ is the algebra defined in Example 1 and $\boldsymbol{\Lambda_1}$ is the algebra defined in Example 4. Define a congruence $\aleph$ on $\boldsymbol{\Lambda}$ by $\langle \varrho, \varsigma \rangle \in \aleph$ iff $\varrho \vee 0 = \varsigma \vee 0$ for any $\varrho, \varsigma \in \Lambda$. Then we have a congruence $\aleph_1$ on $\boldsymbol{\Sigma}$ and a congruence $\aleph_2$ on $\boldsymbol{\Lambda_1}$ following Lemma 9. Define a binary relation $\aleph_2'$ on $\boldsymbol{\Lambda_2}$ by $\aleph_2' = \aleph \cup \Delta$, where $\boldsymbol{\Lambda_2}$ is defined in Example 5. Then $\aleph_2'$ is a congruence on $\boldsymbol{\Lambda_2}$. Denote $\aleph' = \aleph_1 \times \aleph_2'$. Then $\aleph'$ is a congruence on $\boldsymbol{\Lambda}'$ and $\aleph = \aleph' \cap \Lambda^2$.

IV. Conclusion

In this paper, we have proved that the variety of quasi-MV* algebras has the congruence extension property. To complete this work, we have first shown that the subdirect product decomposition of a quasi-MV* algebra, and then proved that MV*-algebras and flat quasi-MV* algebras have the CEP. These results mean that in these algebras, the congruence on a subalgebra can be extended to the entire algebra, which is helpful to the study of algebraic structures. Consider that quasi-MV* algebras are the new non-classical logical algebras arising from many-valued logic and quantum computational logic, their theoretical research could be applied to fields such as artificial intelligence and quantum computation. Thus, future work will discuss more properties of quasi-MV* algebras in order to characterize the logical system associated with quasi-MV* algebras.

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