A Review on Some Numerical Methods on Solving the Fractional Initial Value Problems of Fractional Differential Equations

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***Abstract*—Over the past few years, there has been an increasing interest in fractional differential equations (FDEs), owing to the enhanced effectiveness of fractional calculus compared to traditional calculus. Fractional calculus, with its non-integer order derivatives, provides more accurate models for complex phenomena in science and engineering, capturing memory effects and hereditary properties in processes like viscoelasticity, anomalous diffusion, and signal processing. This has made FDEs a valuable tool for advancing the modeling of intricate systems. This paper aims to examine various numerical methods for solving fractional initial value problems associated with FDEs. A comparative analysis of the Fractional Explicit Adams Method of Order 3, the Fractional Adams Method of Explicit Order 2, Implicit Order 2, the Fourth-order 2-point Fractional Block Backward Differentiation Formula, the Fractional Explicit Method and the PECE method of Adams-Bashforth-Moulton type are presented with respect to their performance against the exact solution. The comparison focuses on key metrics, including convergence and accuracy of the methods. Three numerical problems were successfully solved to evaluate the methods. The results showed that all of the five methods are reliable when solving FDEs.**

***Index Terms*—fractional differential equations, fractional initial value problems, numerical method, convergence, accuracy.**

I. Introduction

FRACTIONAL calculus is a mathematical domain focused on the study of derivatives and integrals with non-integer orders, extending to both real and complex numbers. This field extends the principles of classical calculus by adapting its foundational concepts to accommodate fractional orders. As a result, many of the core properties from classical calculus are retained in fractional calculus, albeit with modifications to handle the nuances introduced by fractional orders. The origins of fractional calculus can be traced back to notable mathematicians, including Leibniz, L'Hôpital, Abel, Liouville, Riemann, and others, who laid the groundwork for its development. In essence, fractional calculus provides tools and methodologies to solve differential equations involving fractional derivatives of unknown functions, commonly known as fractional differential equations (FDEs).

FDEs play a significant role across a broad spectrum of disciplines, such as speech signal modelling [1], viscoelastic materials [2], diffusion process modelling [3] and control theory [4]. According to [5], FDEs have shown exceptional potential in modeling complex phenomena in various scientific and engineering domains, primarily due to their unique property of nonlocality. Unlike conventional differential equations, which are influenced by the local behavior of a system, FDEs consider the system's global evolution, offering a more comprehensive view of its dynamics. This feature enables FDEs to provide more accurate approximations of real-world behaviors, surpassing the precision of traditional derivatives. Consequently, the solutions to FDEs have garnered significant interest among researchers and practitioners, reflecting their critical applications in numerous fields and underscoring the value of fractional calculus in advancing our understanding of complex systems. Besides, FDEs enable the representation of memory effects, as mentioned in [6]. Several mathematical models that utilize FDEs are the Pharmacokinetics [7], the SIR [8], the Economic Growth [9], FitzHugh–Nagumo [10], the COVID-19 [11] and the Viscoelastic models [12].

Many methods have been proposed to solve FDEs including block and non-block methods. This article concentrates on four prominent non-block methods: the Adams-Bashforth method Fractional Explicit Adams Method of Order 3 (FEAM3), the Adams-Moulton method Fractional Adams Method of Explicit Order 2, Implicit Order 2 (FAM22), Fractional Explicit Method (FE) and Fractional Explicit Method and the PECE method of Adams-Bashforth-Moulton type (FDE12) alongside one significant block method: the Block Backward Differentiation Formula (Fourth-order 2-point Fractional Block Backward Differentiation Formula (2FBBDF(4)). The convergence and accuracy of these methods will be examined and compared to understand their strengths and suitability for solving FDEs.

FEAM3 [20] is a non-block multistep method which is explicit inspired by the principles of the Adams-Bashforth method. It serves as an effective approximation technique for solving FDEs. The main benefit of the FEAM3 method is that it can handle both linear and nonlinear FDEs, making it versatile to solve different types of applications in mathematical modeling and scientific computations. Despite its advantages, the method has certain limitations. Specifically, it encounters challenges when applied to solve systems of FDEs. FAM22 [21] is a non-block implicit multistep method that derived by using the concept of Adam-Moulton method to solve FDEs. Similar to FEAM3, FAM22 method is capable of effectively solving both linear and nonlinear FDEs. This makes it versatile for addressing a wide range of problems in fractional calculus. Moreover, FAM22 has a distinct ad-

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vantage over FEAM3 whereby it is specifically designed to handle systems of FDEs. FE [22] is an explicit method that is formulated based on the concept of a second-order Adam-Bashforth method by implementing Lagrange interpolation for fractional case. FE is an appropriate method to solve different types of FDEs, similar to FEAM3 and FAM22. FDE12 initially developed by [13], offers a reliable and effective numerical method for addressing fractional-order initial value problems. The method was derived according to the concept of predictor-corrector Adams-Bashforth-Moulton in order to deal with the singular kernel and memory effects typical in fractional systems. Lastly, 2FBBDF(4) [23] is a 2-point implicit multistep block method derived to solve FDEs. Similar to other methods, it is also capable of solving various types of FDEs, including linear, nonlinear and system. The advantage of 2FBBDF(4) lies in its ability to compute multiple solution points simultaneously within a single step. This approach significantly reduces computational time and requires fewer steps compared to non-block methods that calculate one solution point at a time.

Although several numerical techniques are available for addressing FDEs, there is a deficiency in the literature on hybrid methods that integrate the advantages of both block and non-block methods. Although the advantages of individual block and non-block methods are well-established, there has been insufficient research on the integration of these methods to improve computational efficiency and accuracy. Thus, the objective of this review paper is to investigate the potential of hybrid methods by comparing the strengths and weaknesses of each method. Thus, the review of FEAM3, FAM22, FE, 2FBBDF(4) and FDE12 for solving linear and non-linear FDEs is presented. The novelty of this review lies in its systematic evaluation of both block and non-block numerical methods for solving fractional differential equations. This study compares different approaches inside a shared framework by employing similar tested problems and error metrics. The majority of prior research have focused on the performance of individual methods while they were in isolation. This approach offers a deeper comprehension of the selection of methods based on several criteria, including accuracy, convergence, and efficiency. Thus, this study provides an extensive review of various numerical approaches for addressing both linear and nonlinear FDEs, with a particular focus on five distinct approaches: FEAM3, FAM22, 2FBBDF(4), FE and FDE12. The article is structured as follows: section 2 discusses the preliminaries, section 3 discusses the existence and uniqueness of the solution, section 4 analyzes the methods, section 5 presents three illustrative examples and section 6 discusses the results and concludes the findings.

II. PRELIMINARIES

Fractional calculus encompasses various types of fractional differential operators, including the Hilfer, Riemann-Liouville, Caputo, Caputo-Fabrizio and Atangana-Baleanu among others. Despite this diversity, the Riemann-Liouville and Caputo operators are the most frequently utilized in FDEs. Many researchers prefer to use the Caputo definition of fractional derivatives in their studies. According to [16], one of the main advantages of employing the Caputo definition is its ability to provide a clear, interpretable physical meaning for fractional derivatives, which can often be directly linked to measurable quantities. Due to its interpretability, the Caputo derivative is highly effective in real-world processes, as it allows researchers to connect mathematical formulations with observable phenomena. The fractional Caputo's derivative operator of order α, D^{α} is defined as [14]:

$$D^{\alpha}y(t)=\frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}\frac{y^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau, \quad m-1<\alpha<m, \quad m\in\mathbb{Z}^{+}. \tag{1}$$

According to [14], $cD^{\alpha}t_0y(t)={}_{\mathrm{RL}}D_{t_0}^{\alpha}y(t)-y(t_0)$ where ${}_{\mathrm{RL}}D_{t}^{\alpha}y(t)$ represents the Rieman-Liouville differential operator, defined as follows:

$${}_{\mathrm{RL}}D_{t_0}^{\alpha}y(t)=\frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^{m}\int_{t_0}^{t}\frac{y(\tau)}{(t-\tau)^{\alpha-m+1}}\,d\tau, \quad \alpha>0, \quad m=[\alpha]. \tag{2}$$

III. EXISTENCE AND UNIQUENESS OF THE SOLUTION

According to [15], the fractional initial value problems (FIVP) for a system of FDEs are of the form:

$$cD_{t_0}^{\alpha}y(t)=f(t,y(t)),y(t_0)=y_0. \tag{3}$$

Theorem 1: [16] (Existence of solution) Let $D:=[0,X^*]\times[y_0^{(0)}-\alpha,y_0^{(0)}-\alpha]$ where $X^*>0$ and some $\alpha>0$, and assume that the function $f:D\rightarrow\mathbb{R}$ be continuous. Furthermore, define $X:=\min\left(X^*,(\alpha\Gamma(q+1)/\|f\|_{\infty})^{1/q}\right)$. Under these conditions, there exists a function $y:[0,X^*]\rightarrow\mathbb{R}$ that solves the FIVP in (3).

Theorem 2: [16] (Uniqueness of solution) Consider $D:=[0,X^*]\times[y_0^{(0)}-\alpha,y_0^{(0)}-\alpha]$ where $X^*>0$ and some $\alpha>0$. Moreover, suppose that the function $f:D\rightarrow\mathbb{R}$ is bounded on D and satisfies a Lipschitz condition in relation to its second variable:

$$|f(x,y)-f(x,z)|\leq K|y-z|. \tag{4}$$

with constant $K>0$ that does not depend on x, y, and z. Then, letting X as specified in Theorem 1, there exists a unique function $y:[0,X]\rightarrow\mathbb{R}$ that solves the FIVP in (3).

Lemma 1: [16] Provided that f is continuous, the FIVP in (3) can be reformulated into an equivalent Volterra integral equation, which is expressed as

$$y(x)=\sum_{k=0}^{m-1}\frac{x^k}{k!}y^{(k)}(0)+\frac{1}{\Gamma(p)}\int_{0}^{x}(x-z)^{p-1}f(z,y(z))\,dz. \tag{5}$$

with $m-1<p\leq m$. In other words, any solution to the original initial value problem (3) is also a solution of the Volterra equation.

Theorem 3: [17] Suppose that $f(t,y)$ is Lipschitz continuous at each point (t,y) in the region R, defined by

$$a\leq t\leq b, \quad -\infty<y<\infty. \tag{6}$$

where a and b are finite. Assume a constant L exists such that for all values of t, y, y^*, both the coordinates (t, y) and (t, y^*) lie within the region R,

$$|f(t,y) - f(t,y^*)| \leq L|y - y^*|. \tag{7}$$

Theorem 4: [15], [17], [18] A linear multistep method is considered convergent if, for all initial value problems that satisfy the assumptions outlined in Theorem 3 as $t \in [a,b]$ and $0 < \alpha < 1$, the following condition holds:

$$|y - y*| \leq M \cdot t^{1+\alpha} h^p. \tag{8}$$

where M is a constant depending exclusively on p and α, where p is between 0 and 1. Moreover,

$$\lim_{h\to\infty} y_n = y^*(t_n). \tag{9}$$

Theorem 5: [19] Convergence of the method will be satisfied when the method achieves both consistency and zero stability.

Definition 1: [19] When the order is is greater than or equal to p, where $p \geq 1$, the fractional linear multistep method (FLMM) is said to be consistent.

Definition 2: [19] When no root of the polynomial's initial features have modulus greater than one, the FLMM is said to be zero stable, and each root with a modulus of one is a simple root.

IV. ANALYSIS OF THE METHODS

This section presents the analysis of the five methods under consideration.

A. Fractional Explicit Adams Method of Order 3 (FEAM3)

FEAM3 utilizes Lagrange interpolation technique to build on the third-order Adam–Bashforth numerical scheme by adapting it for fractional calculus. The fractional derivative used is Caputo sense. The solution to the FIVP of FDEs is obtained from the following scheme [20]:

$$\begin{aligned} y(t_{n+1}) = y(t_n) + \frac{h}{\Gamma(\alpha)}\Bigg[&\Bigg(\frac{3(n+1)^\alpha - (n)^\alpha}{\alpha} + \\ &\frac{4(n+1)^{\alpha+1} - 2(n)^{\alpha+1}}{\alpha+1} + \frac{(n)^{\alpha+2} - (n+1)^{\alpha+2}}{\alpha+2}\Bigg) y_n + \\ &\Bigg(\frac{-3(n+1)^\alpha}{\alpha} + \frac{4(n+1)^{\alpha+1} - 2(n)^{\alpha+1}}{\alpha+1} + \\ &\frac{(n)^{\alpha+2} - (n+1)^{\alpha+2}}{\alpha+2}\Bigg) y_{n-1} + \\ &\Bigg(\frac{(n+1)^\alpha}{\alpha} + \frac{(n)^{\alpha+1} - 3(n+1)^{\alpha+1}}{2\alpha+2} + \\ &\frac{(n+1)^{\alpha+2} - (n)^{\alpha+2}}{2\alpha+4}\Bigg) y_{n-2}\Bigg]. \end{aligned} \tag{10}$$

Based on Theorem 2, FEAM3 is said to be convergent if and only if $|y - y^*| \leq K t^{\alpha-1} h^p$, by which K is a constant and $\lim_{h\to\infty} y_n = y(t_n)$. From (10), let

$$\begin{aligned} P =& \frac{3(n+1)^\alpha - (n)^\alpha}{\alpha} + \frac{3(n)^{\alpha+1} - 5(n+1)^{\alpha+1}}{2\alpha+2} + \\ &\frac{(n+1)^{\alpha+2} - (n)^{\alpha+2}}{2\alpha+4}, \\ Q =& \frac{-3(n+1)^{\alpha+4}}{\alpha} + \frac{4(n+1)^{\alpha+1} - 2(n)^{\alpha+1}}{\alpha+1} + \\ &\frac{(n)^{\alpha+2} - (n+1)^{\alpha+2}}{\alpha+2}, \\ R =& \frac{(n+1)^\alpha}{\alpha} + \frac{(n)^{\alpha+1} - 3(n+1)^{\alpha+1}}{2\alpha+2} + \\ &\frac{(n+1)^{\alpha+2} - (n)^{\alpha+2}}{2\alpha+4}. \end{aligned} \tag{11}$$

Substituting equation (11) into Equation (10) yield the following exact form:

$$\begin{aligned} y^*(t_{n+1}) - y^*(t_n) =& \frac{h^\alpha}{\Gamma(\alpha)}(P)F_n^* + \frac{h^\alpha}{\Gamma(\alpha)}(Q)F_{n-1}^* + \\ &\frac{h^\alpha}{\Gamma(\alpha)}(R)F_{n-2}^* + \frac{3}{8}h^4 y^{*(4)}\epsilon, \end{aligned} \tag{12}$$

and the following approximate form:

$$\begin{aligned} y(t_{n+1}) - y(t_n) =& \frac{h^\alpha}{\Gamma(\alpha)}(P)F_n + \frac{h^\alpha}{\Gamma(\alpha)}(Q)F_{n-1} + \\ &\frac{h^\alpha}{\Gamma(\alpha)}(R)F_{n-2.} \end{aligned} \tag{13}$$

Subtracting equation (12) from equation (13) leads to:

$$\begin{aligned} y(t_{n+1}) - y^*(t_{n+1}) = y(t_n) - y^*(t_n) + \\ \frac{h^\alpha}{\Gamma(\alpha)}(P)[f(t_n, y_n) - f(t_n^*, y_n^*)] + \\ \frac{h^\alpha}{\Gamma(\alpha)}(Q)[f(t_{n-1}, y_{n-1}) - f(t_{n-1}^*, y_{n-1}^*)] + \\ \frac{h^\alpha}{\Gamma(\alpha)}(R)[f(t_{n-2}, y_{n-2}) - f(t_{n-2}^*, y_{n-2}^*)] + \\ \frac{3}{8}h^4 y^{*(4)}\epsilon. \end{aligned} \tag{14}$$

Let

$$\begin{aligned} |d_{n+1} &= |y_{n+1} - y_{n+1}^*|, \\ |d_n| &= |y_n - y_n^*|, \\ |d_{n-1}| &= |y_{n-1} - y_{n-1}^*|, \\ |d_{n-2}| &= |y_{n-2} - y_{n-2}^*|. \end{aligned} \tag{15}$$

By using the assumption outlined in equation (15) and Theorem 3, along with the application of the Lipschitz condition, we obtain:

$$\begin{aligned} |d_{n+1}| \leq \left(1 + \frac{h^\alpha P}{\Gamma(\alpha)}\right)|d_n| + \frac{h^\alpha Q}{\Gamma(\alpha)}|d_{n-1}| + \\ \frac{h^\alpha R}{\Gamma(\alpha)}|d_{n-2}| + \frac{3}{8}h^4 y^{*(4)}\epsilon. \end{aligned} \tag{16}$$

Rewriting equation (16) based on Theorem 4 yield:

$$|d_{n+1}| \leq (1+Kh^{\alpha})|d_n| + Kh^{\alpha}|d_{n-1}| + Kh^{\alpha}|d_{n-2}| + \frac{3}{8}h^4 y^{*(4)}\epsilon. \tag{17}$$

Hence, the initial value goes to zero when h is small enough or gets closer to zero, that is, it is proven that $|d_{n+1} \leq |d_n|$. This leads to the conclusion that $|y_{n+1}| = |y^*_{n+1}|$ and $|y_n| = |y^*_n|$. Therefore, since Theorem 2 is satisfied, it follows that FEAM3 is convergent.

B. Fractional Adams Method of Explicit Order 2, Implicit Order 2 (FAM22)

FAM22 is developed by combining the principles of Lagrange interpolation with the concepts of the Adams–Moulton method, specifically adapted for the fractional case. The numerical approach for solving the FIVP of FDEs is expressed as follows [21]:

$$y(t_{n+1}) = y(t_n) + \frac{h}{\Gamma(\alpha)}\left[\left(\frac{(n+1)^{\alpha}}{\alpha} + \frac{(n)^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}\right)y_{n+1} + \left(\frac{-(n)^{\alpha}}{\alpha} + \frac{(n+1)^{\alpha+1} - (n)^{\alpha+1}}{\alpha+1}\right)y_n\right]. \tag{18}$$

According to Theorem 4, FAM22 is said to be convergent if and only if $|y - y^*| \leq Mt^{\alpha-1}h^p$, where M is a constant and $\lim_{h\to\infty} y_n = y^*(t_n)$. Based on equation (18), let

$$P = \frac{(n+1)^{\alpha}}{\alpha} + \frac{(n)^{\alpha+1} - (n+1)^{\alpha+1}}{\alpha+1}, \quad Q = \frac{-(n)^{\alpha}}{\alpha} + \frac{(n+1)^{\alpha+1} - (n)^{\alpha+1}}{\alpha+1}. \tag{19}$$

Substituting equation (19) into equation (18) yield the exact and approximate form of the system as shown in equation (20) and equation (21), respectively.

$$y^*(t_{n+1}) - y^*(t_n) = \frac{h^{\alpha}}{\Gamma(\alpha)}(P)F^*_{n+1} + \frac{h^{\alpha}}{\Gamma(\alpha)}(Q)F^*_n - \frac{1}{12}h^3 y^{*(3)}\epsilon. \tag{20}$$

$$y(t_{n+1}) - y(t_n) = \frac{h^{\alpha}}{\Gamma(\alpha)}(P)F_{n+1} + \frac{h^{\alpha}}{\Gamma(\alpha)}(Q)F_n. \tag{21}$$

Subtracting equation (20) from equation (21) leads to:

$$y(t_{n+1}) - y^*(t_{n+1}) = y(t_n) - y^*(t_n) + \frac{h^{\alpha}}{\Gamma(\alpha)}(P)[f(t_{n+1}, y_{n+1}) - f(t^*_{n+1}, y^*_{n+1})] + \frac{h^{\alpha}}{\Gamma(\alpha)}(Q)[f(t_n, y_n) - f(t^*_n, y^*_n)] - \frac{1}{12}h^3 y^{*(3)}\epsilon. \tag{22}$$

Let

$$|d_{n+1}| = |y_{n+1} - y^*_{n+1}|, \quad |d_n| = |y_n - y^*_n|. \tag{23}$$

Applying the assumption from equation (23) and Theorem 3, and utilizing the Lipschitz condition, we obtain:

$$\left(1 - \frac{h^{\alpha}P}{\Gamma(\alpha)}\right)|d_{n+1}| \leq \left(1 + \frac{h^{\alpha}Q}{\Gamma(\alpha)}\right)|d_n| - \frac{1}{12}h^3 y^{*(3)}\epsilon. \tag{24}$$

By reformulating equation (24) in accordance with Theorem 4, we obtain:

$$(1 - Kh^{\alpha})|d_{n+1}| \leq (1 + Kh^{\alpha})|d_n| - \frac{1}{12}h^3 y^{*(3)}\epsilon. \tag{25}$$

Based on the above analysis, the initial value goes to zero when h is small enough or gets closer to zero, hence, it is proven that $|d_{n+1} \leq |d_n|$. This implies that $|y_{n+1}| = |y^*_{n+1}|$ and $|y_n| = |y^*_n|$. Therefore, Theorem 2 is fulfilled, confirming that FAM22 is convergent.

C. Fractional Explicit Method (FE)

FE is formulated by involving the approximation at the point t_n and t_{n-2} by which the step size is $2h$. The method is derived by integrating the concept of Adams-Bashforth method of fractional case and implementing Lagrange interpolation for fractional case. The numerical solution to the FIVP of FDEs is presented as follows [22]:

$$y(t_{n+1}) = y(t_n) + \frac{h}{\Gamma(\alpha)}\left[\left(\frac{3(n+1)^{\alpha} - 2(n)^{\alpha}}{2\alpha} + \frac{(n)^{\alpha+1} - (n+1)^{\alpha+1}}{2(\alpha+1)}\right)F_n + \left(\frac{-(n+1)^{\alpha}}{2\alpha} + \frac{(n+1)^{\alpha+1} - (n)^{\alpha+1}}{2(\alpha+1)}\right)F_{n-2}\right]. \tag{26}$$

According to Theorem 4, FE is said to be convergent if and only if $|y - y^*| \leq Mt^{\alpha-1}h^p$, where M is a constant and $\lim_{h\to\infty} y_n = y^*(t_n)$. Based on equation (26), let

$$X = \frac{3(n+1)^{\alpha} - 2(n)^{\alpha}}{2\alpha} + \frac{(n)^{\alpha+1} - (n+1)^{\alpha+1}}{2(\alpha+1)}, \quad Y = \frac{-(n+1)^{\alpha}}{2\alpha} + \frac{(n+1)^{\alpha+1} - (n)^{\alpha+1}}{2(\alpha+1)}. \tag{27}$$

Substituting equation (27) into equation (26) generates the exact and approximate forms of the system, given by equation (28) and equation (29), respectively.

$$y^*(t_{n+1}) - y^*(t_n) = \frac{h^{\alpha}}{\Gamma(\alpha)}(X)F^*_n + \frac{h^{\alpha}}{\Gamma(\alpha)}(Y)F^*_{n-2} - \frac{2}{3}h^3 y^{*(3)}\epsilon. \tag{28}$$

$$y(t_{n+1}) - y(t_n) = \frac{h^{\alpha}}{\Gamma(\alpha)}(X)F_n + \frac{h^{\alpha}}{\Gamma(\alpha)}(Y)F_{n-2}. \tag{29}$$

Subtracting equation (28) from equation (29) leads to:

$$
\begin{aligned}
y(t_{n+1})-y^*(t_{n+1}) = y(t_n) - y^*(t_n)+ \\
\frac{h^\alpha}{\Gamma(\alpha)}(X)[f(t_{n+1},y_{n+1}) - f(t^*_{n+1},y^*_{n+1})]+ \\
\frac{h^\alpha}{\Gamma(\alpha)}(Y)[f(t_{n-2},y_{n-2}) - f(t^*_{n-2},y^*_{n-2})]- \\
\frac{2}{3}h^3y^{*(3)}\epsilon.
\end{aligned} \tag{30}
$$

Let

$$
\begin{aligned}
|d_{n+1}| &= |y_{n+1} - y^*_{n+1}|, \\
|d_n| &= |y_n - y^*_n|, \\
|d_{n-2}| &= |y_{n-2} - y^*_{n-2}|.
\end{aligned} \tag{31}
$$

Applying the assumption from equation (31) along with Theorem 3, and utilizing the Lipschitz condition, we obtain:

$$
|d_{n+1}| \leq \left(1 + \frac{h^\alpha X}{\Gamma(\alpha)}\right)|d_n| + \frac{h^\alpha Y}{\Gamma(\alpha)}|d_{n-2}| + \frac{2}{3}h^3y^{*(3)}\epsilon. \tag{32}
$$

Reformulating equation (32) in accordance with Theorem 4, we obtain:

$$
|d_{n+1}| \leq (1 + Kh^\alpha)|d_n| + Kh^\alpha|d_{n-2}| - \frac{2}{3}h^3y^{*(3)}\epsilon. \tag{33}
$$

Based on the above analysis, the initial value tends to closer to 0 when h is approaching to 0 or very small. This proves that $|d_{n+1} \leq |d_n|$; hence, $|y_{n+1}| = |y^*_{n+1}|$ and $|y_n| = |y^*_n|$. Therefore, FE is confirmed to be convergent.

D. Fourth-order 2-point Fractional Block Backward Differentiation Formula (2FBBDF(4))

2FBBDF(4) is formulated by combining the fractional linear multistep method (FLMM) with the linear difference operator. The numerical solution to the FIVP of FDEs is presented as follows [23]:

$$
\begin{aligned}
y(t_{n+1}) =& -\frac{\alpha(\alpha^2-8\alpha+13)}{4(2\alpha^3-22\alpha^2+77\alpha-72)}y_{n-2}+ \\
&\frac{\alpha(2\alpha^2-14\alpha+21)}{2\alpha^3-22\alpha^2+77\alpha-72}y_{n-1}- \\
&\frac{3(5\alpha^2-35\alpha+48)}{2(2\alpha^3-22\alpha^2+77\alpha-72)}y_n+ \\
&\frac{\alpha(\alpha^2-10\alpha+27)}{4(2\alpha^3-22\alpha^2+77\alpha-72)}y_{n+2}- \\
&\frac{3\Gamma(5-\alpha))}{2\alpha^3-22\alpha^2+77\alpha-72}h^\alpha cD^\alpha_{t_0}y_{n+1}. \\
y(t_{n+2}) =& \frac{\alpha(\alpha^2-7\alpha-12)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-2}- \\
&\frac{8\alpha(\alpha^2-5\alpha-8)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n-1}+ \\
&\frac{12(5\alpha^2-35\alpha+12)}{\alpha^3-11\alpha^2+16\alpha+144}y_n+ \\
&\frac{8\alpha(\alpha^2-13\alpha+48)}{\alpha^3-11\alpha^2+16\alpha+144}y_{n+1}+ \\
&\frac{12(2^{\alpha-1})\Gamma(5-\alpha)}{\alpha^3-11\alpha^2+16\alpha+144}h^\alpha cD^\alpha_{t_0}y_{n+2}.
\end{aligned} \tag{34}
$$

According to [23], 2FBBDF(4) demonstrates an error constant localized at C_5 , which confirm the method to be order 4. Based on Definition 1, the method is proven to be consistent due to the presence of the order of the method. By referring to Definition 2, [23] has proved that the method is zero stable due to the existence of the roots, $|r_s|$, satisfying $|r_s| \leq 1$. From Theorem 5, the combination of zero stability and consistency of this method implies convergence.

E. PECE Method of Adams-Bashforth-Moulton Type (FDE12)

According to [13], the primary concept behind developing the method is to convert the fractional differential equation into integral equation with a weak singularity. A product trapezoidal quadrature formula is used to solve this integral equation. Thus, the numerical solution to the FIVP of FDEs is presented as follows:

$$
y(t_{n+1}) = y(t_n) + \frac{1}{\Gamma(\alpha)}\left(\sum_{j=0}^{n} a_j^{(n+1)}F_n + a_{n+1}F^{(p)}_{n+1}\right). \tag{35}
$$

This method is proven to be second-order convergence [13]. Second-order convergence may not seem very good, because the fractional order of the differential equations are usually known in two or three decimal places. Due to the limited accuracy of the input, using a higher-order method is often not a good idea. In fact, higher-order methods often are less stable than their lower-order methods, which makes them less reliable.

In numerical techniques, the term "convergence" refers to the degree to which a numerical solution approaches the exact answer even if the discretisation parameters such asthe time step are improved or made more precise. As the approximation grows ever more exact, it essentially measures the degree to which the computer model is able to accurately reflect the real-life behaviour of the system. Several approaches, such as non-block methods which is FEAM3, have been shown to converge more rapidly than block methods such as 2FBBDF(4). In general, non-block approaches are more efficient in terms of the number of computation steps required to reach a high degree of accuracy. These methods analyse data in an orderly way. Block methods, on the other hand, which process data in bigger blocks, may be slower to converge, taking more steps to obtain a same degree of precision. FAM22, FE and FDE12 have been analyzed for its stability and convergence. Research indicates that while they are stable, their convergence rates may be slower compared to explicit methods.

V. ILLUSTRATIVE EXAMPLE

To showcase the methods' effectiveness and practical applicability, several FIVP of FDEs are sourced directly for comparative analysis. The evaluation approach included a careful comparison of the numerical results derived from the methods with the corresponding exact answers, with the findings clearly presented in Tables I - VIII and Figures 1-12. Tables I until III present the absolute error of different methods at t with $h = 0.1$, $h = 0.01$ and $h = 0.001$ for

Problem 1 respectively. Next, the absolute error of different methods at various t with $\alpha = 0.70$ and $\alpha = 0.50$ for solving Problem 2 are presented in Tables IV and V. Tables VI until VIII present the absolute error of different methods at t with $h = 0.1$, $h = 0.01$ and $h = 0.001$ for Problem 3 respectively. Furthermore, the plots of the exact and approximate solution for Problem 1 with $h = 0.1$, $h = 0.01$ and $h = 0.001$ are figured out as in Figures 1 until 3 while Figure 4 shows the efficiency curves for Problem 1. Moreover, Figures 5 until 6 show the plots of the exact and approximate solutions for Problem 2 with $\alpha = 0.70$ and $\alpha = 0.50$ respectively. Also, Figures 7 and 8 show the efficiency curves with $\alpha = 0.70$ and $\alpha = 0.50$ respectively. Besides, the plots of the exact and approximate solution for Problem 3 with $h = 0.1$, $h = 0.01$ and $h = 0.001$ are figured out as in Figures 9 until 11 while Figure 12 shows the efficiency curves for Problem 3. This comparison allows for a comprehensive assessment of both accuracy and efficiency. The following are the notations that are used in this study:

h	:	Step size
AbsE	:	Absolute error
Method	:	Comparison method
FEAM3	:	Fractional Explicit Adams Method of Order 3 [20]
FAM22	:	Fractional Adams Method of Explicit Order 2, Implicit Order 2 [21]
2EBBDF(4)	:	Fourth-order 2-point Fractional Block Backward Differentiation Formula [23]
FE	:	Fractional Explicit Method [22]
FDE12	:	PECE Method of Adams-Bashforth-Moulton Type - Available in MATLAB

Problem 1:
The following is an application problem of Riccati FDEs [24, 25]:

$$D^{\alpha}y(t) = -y^2 + 1, \quad y(0) = 0, \quad t \in [0, 1].$$

where the exact solution when $\alpha = 1.00$ is

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Problem 2:
The linear fractional order equation is given as [26]:

$$D^{\alpha}y(t) = -y, \quad y(0) = 1, \quad t \in [0, 1].$$

where the exact solution is

$$y(t) = E_{\alpha}(-t^{\alpha}).$$

and $E_{\alpha}(z)$ is defined as Mittag-Leffler function:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Problem 3:
The nonlinear fractional order equation is considered as follows [27]:

$$D^{\alpha}y(t) = (1 - y)^4, \quad y(0) = 0, \quad t \in [0, 1].$$

where the exact solution when $\alpha = 1.00$ is

$$y(t) = \frac{1 + 3t - (1 + 6t + 9t^2)^{\frac{1}{3}}}{2 + 3t}.$$

TABLE I
ABSOLUTE ERROR OF DIFFERENT METHODS AT t WITH $h = 0.1$ IN SOLVING PROBLEM 1

t	Method	Absolute Error
0.2	FEAM3	1.6247E-06
	FAM22	9.5000E-05
	2FBBDF(4)	1.6247E-03
	FE	1.6247E-03
	FDE12	3.4559E-04
0.4	FEAM3	5.3423E-04
	FAM22	2.5921E-04
	2FBBDF(4)	6.3860E-03
	FE	6.3861E-03
	FDE12	7.5174E-04
0.6	FEAM3	4.5129E-04
	FAM22	3.9363E-04
	2FBBDF(4)	5.9010E-03
	FE	1.2137E-02
	FDE12	1.2023E-03
0.8	FEAM3	3.3488E-04
	FAM22	5.0366E-04
	2FBBDF(4)	3.9650E-03
	FE	1.6670E-02
	FDE12	1.6033E-03
1.0	FEAM3	2.3869E-04
	FAM22	5.9372E-04
	2FBBDF(4)	1.9556E-03
	FE	1.8847E-02
	FDE12	1.8586E-03

TABLE II
ABSOLUTE ERROR OF DIFFERENT METHODS AT t WITH $h = 0.01$ IN SOLVING PROBLEM 1

t	Method	Absolute Error
0.2	FEAM3	3.7449E-06
	FAM22	8.7304E-06
	2FBBDF(4)	6.7106E-06
	FE	1.8822E-04
	FDE12	3.4563E-06
0.4	FEAM3	1.3975E-05
	FAM22	2.2295E-05
	2FBBDF(4)	2.5380E-06
	FE	6.6446E-04
	FDE12	7.4298E-06
0.6	FEAM3	2.1961E-05
	FAM22	4.4568E-05
	2FBBDF(4)	2.1296E-05
	FE	1.2111E-03
	FDE12	1.1644E-05
0.8	FEAM3	2.7323E-05
	FAM22	5.4616E-05
	2FBBDF(4)	4.3499E-05
	FE	1.6295E-03
	FDE12	1.5153E-05
1.0	FEAM3	3.0198E-05
	FAM22	6.2843E-05
	2FBBDF(4)	6.1566E-05
	FE	1.8277E-03
	FDE12	1.7168E-05

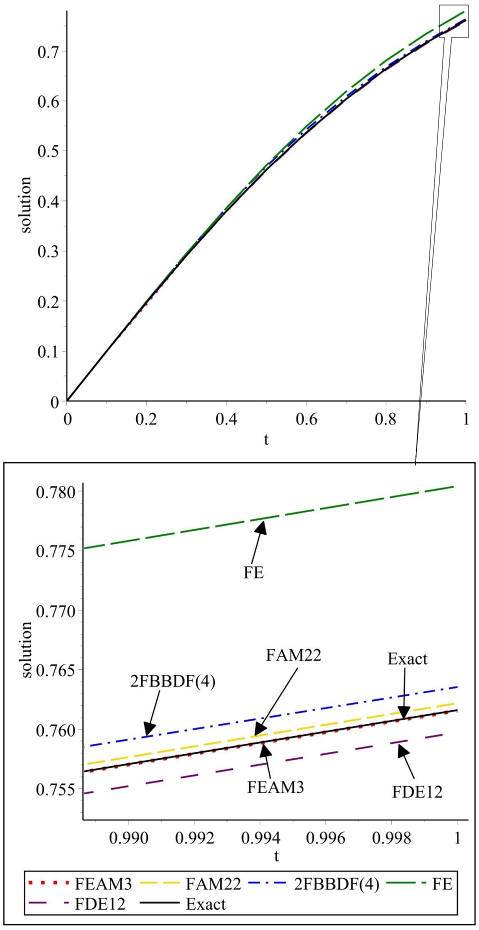

Fig. 1. Plots of the exact and approximate solutions for Problem 1 with $h = 0.1$

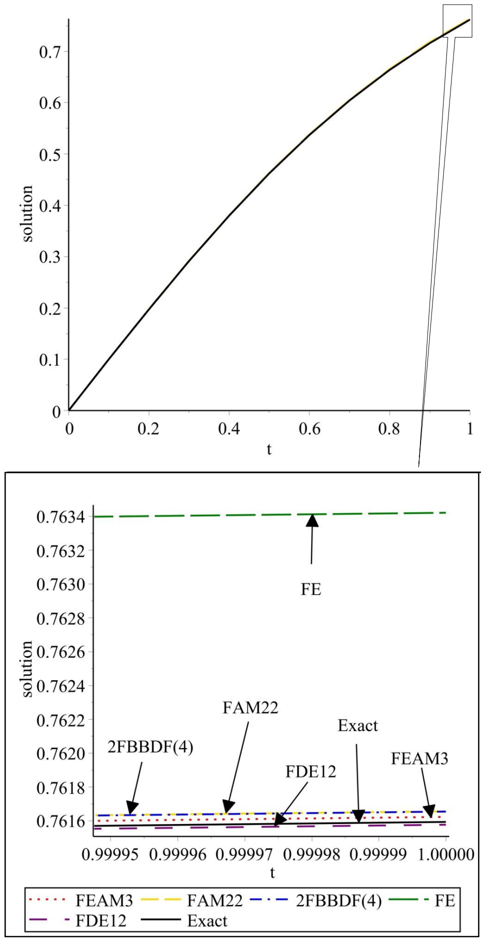

Fig. 2. Plots of the exact and approximate solutions for Problem 1 with $h = 0.01$

VI. DISCUSSION

A detailed comparative analysis of five methods can be performed by evaluating their performance against exact solutions of the FDEs in problems 1 until 3. For problems 1 and 3, accuracy of each method is increased when N is increased, as can be observed in tables 1-3 and 6-8. This indicates that the approximate solution converges towards the exact solution. Based on figures 1-3 and 9-11, the convergence of each numerical method is demonstrated using graphical comparisons between the approximate and exact solutions. These visual representations clearly illustrate the gradual alignment of the approximate solutions with the exact solutions, highlighting the accuracy of the methods. According to Table I until Table III, it can be observed that FEAM3 performs better when $h = 0.1$. However, when $h = 0.01$ and $h = 0.001$, FDE12 produce the smallest absolute error. Furthermore, according to Table VI until Table VIII, FEAM3 is the most accurate method in terms of absolute error for all the step size. Not only that, the computational performance of each method is demonstrated by plotting the efficiency curve of each method. From Figure 4, it is demonstrated that FEAM3 method outperforms FAM22, 2FBBDF(4), FE and FDE12 methods. For problem 2, the approximate solution converges when α increases. From Table VI and Table V, FEAM3 generated more comparable results when compared to FAM22, 2FBBDF(4), FE and FDE12 methods. Based on Figure 5 and Figure 6, all methods closely approximate the exact solution as the value of alpha varies. FEAM3 can be seen to have a higher accuracy method by plotting the efficiency curve of the each method. Based on Figure 7 and Figure 8, FEAM3 produces smaller error as compared to FAM22, 2FBBDF(4), FE and FDE12 methods. In short, the non-block methods which are FEAM3, FAM22 and FDE12 perform better in terms of accuracy than block method which is 2FBBDF(4). FE method is the least accurate method when compared to FEAM3, FAM22, 2FBBDF(4) and FDE12 due to its step size of interpolating points, which is $2h$, while step size of FEAM3, FAM22, 2FBBDF(4) and FDE12 is h. For FEAM3, FAM22 and FDE12, FEAM3 outperforms FAM22 and 2FBBDF(4) when solving fractional order ordinary differential equations because it converges much closer to the exact solution, exhibiting an exceptionally small error in comparison to FAM22 and FDE12. FEAM3 also has higher order than FAM22. In short, non-block method such as FEAM3 has been highlighted for its accuracy and efficiency in solving both linear and nonlinear FDEs while block method which is 2FBBDF(4) allows for simultaneous computation of multiple solution points, where less number of steps are involved in the computations.

VII. CONCLUSIONS

In this paper, several numerical methods for solving FDEs were selected to be reviewed. They are the FEAM3, FAM22, 2FBBDF(4), FE and FDE12. Three numerical problems were

TABLE III
ABSOLUTE ERROR OF DIFFERENT METHODS AT t WITH $h = 0.001$ IN SOLVING PROBLEM 1

t	Method	Absolute Error
0.2	FEAM3	1.0653E-07
	FAM22	9.8045E-07
	2FBBDF(4)	9.3634E-09
	FE	1.9067E-05
	FDE12	3.4333E-08
0.4	FEAM3	2.0725E-06
	FAM22	2.2901E-06
	2FBBDF(4)	1.1229E-07
	FE	6.6675E-05
	FDE12	7.1956E-08
0.6	FEAM3	2.8479E-06
	FAM22	4.5064E-06
	2FBBDF(4)	3.0807E-07
	FE	1.2107E-04
	FDE12	1.1900E-07
0.8	FEAM3	3.3549E-06
	FAM22	5.5022E-06
	2FBBDF(4)	5.3024E-07
	FE	1.6259E-04
	FDE12	1.5038E-07
1.0	FEAM3	3.6079E-06
	FAM22	6.3175E-06
	2FBBDF(4)	7.0278E-07
	FE	1.8224E-04
	FDE12	1.7432E-07

TABLE IV
ABSOLUTE ERROR OF DIFFERENT METHODS AT VARIOUS t WITH α= 0.70 FOR SOLVING PROBLEM 2

t	Method	Absolute Error
0.2	FEAM3	9.7441E-05
	FAM22	4.3898E-06
	2FBBDF(4)	2.8507E-02
	FE	1.7272E-01
	FDE12	1.0212E-01
0.4	FEAM3	9.0656E-05
	FAM22	5.0362E-06
	2FBBDF(4)	4.4815E-02
	FE	2.9831E-02
	FDE12	7.5208E-02
0.6	FEAM3	8.0133E-05
	FAM22	3.2049E-06
	2FBBDF(4)	4.6510E-02
	FE	5.1524E-03
	FDE12	5.1374E-02
0.8	FEAM3	6.6562E-05
	FAM22	2.1809E-06
	2FBBDF(4)	4.0086E-02
	FE	8.8992E-04
	FDE12	2.2517E-02
1.0	FEAM3	7.0611E-06
	FAM22	3.9177E-05
	2FBBDF(4)	2.9183E-02
	FE	1.5370E-04
	FDE12	2.5679E-04

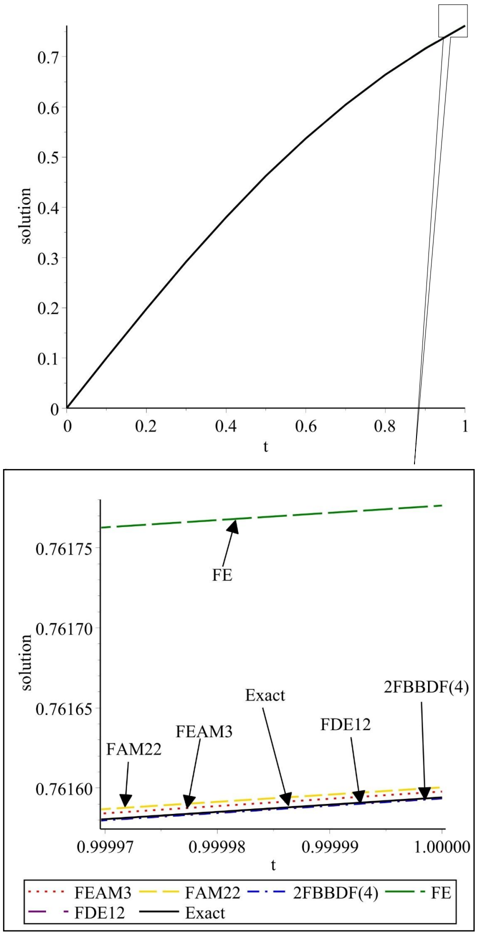

Fig. 3. Plots of the exact and approximate solutions for Problem 1 with $h = 0.001$

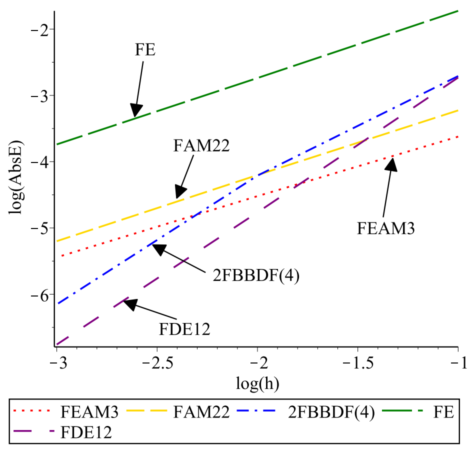

Fig. 4. Efficiency curves for Problem 1

presented in order to determine the performance of each method in terms of accuracy. It is observed that non-block method which are FEAM3, FAM22 and FDE12 perform better than block method which is 2FBBDF(4). For non-block method, FEAM3 outperforms than FAM22 in terms of accuracy. However, FE is the least accurate method among the five.

Although a variety of numerical methods have been developed to address FDEs, including finite difference, spectral methods, and variational approaches, there is a notable lack

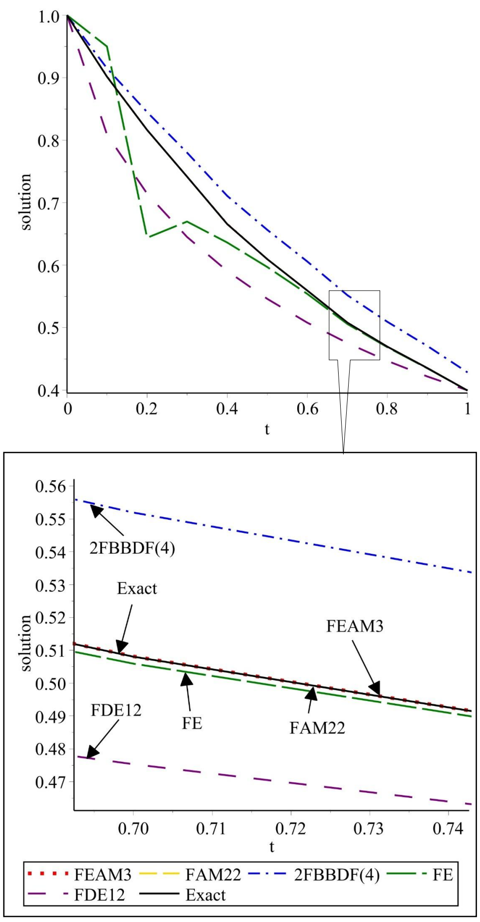

Fig. 5. Plots of the exact and approximate solutions for Problem 2 with $\alpha = 0.70$

of comprehensive studies that investigate a wider range of techniques. Thus, in future, the exploration of hybrid methods, which combine the assets of various techniques to improve both computational efficiency and accuracy, is a promising research path. For instance, the convergence and stability of solutions could be enhanced by integrating conventional numerical schemes, such as the finite difference method, with contemporary methods, such as fractional Adam Bashforth-Moulton method or fractional block backward differentiation method. Researchers would be able to more effectively address the unique challenges presented by various varieties of FDEs and applications by utilising a broader range of numerical methods.

Besides, when applied to large systems, particularly in high-dimensional settings, numerous existing methods of FDEs for solving fractional differential equations may not scale effectively. This becomes more challenging as FDEs are frequently employed for modelling complicated, multidimensional phenomena in disciplines such as epidemiology, image processing, and fluid dynamics. The practical applicability of numerous existing numerical methods is restricted by the rapid increase in computational cost associated with solving large systems of FDEs. Thus, future research is required to create algorithms that are more computationally efficient and scalable, and that are capable of resolving high-dimensional fractional problems. Advancements in distributed computing and parallel computing could be instrumental in overcoming this obstacle. Additionally, the improvement of efficiency

TABLE V
ABSOLUTE ERROR OF DIFFERENT METHODS AT VARIOUS t WITH α= 0.50 FOR SOLVING PROBLEM 2

t	Method	Absolute Error
0.2	FEAM3	9.5845E-04
	FAM22	1.7297E-04
	2FBBDF(4)	7.5473E-03
	FE	1.7272E-01
	FDE12	1.7356E-01
0.4	FEAM3	8.7032E-04
	FAM22	9.5146E-05
	2FBBDF(4)	3.1492E-02
	FE	2.9831E-02
	FDE12	1.2951E-01
0.6	FEAM3	7.4739E-05
	FAM22	3.8351E-05
	2FBBDF(4)	3.7311E-02
	FE	5.1524E-03
	FDE12	8.3832E-02
0.8	FEAM3	6.0223E-05
	FAM22	2.2925E-05
	2FBBDF(4)	3.2808E-02
	FE	8.8992E-04
	FDE12	4.5417E-02
1.0	FEAM3	4.4374E-05
	FAM22	6.3798E-05
	2FBBDF(4)	2.2799E-02
	FE	1.5370E-04
	FDE12	1.4437E-02

TABLE VI
ABSOLUTE ERROR OF DIFFERENT METHODS AT t WITH $h = 0.1$ IN SOLVING PROBLEM 3

t	Method	Absolute Error
0.2	FEAM3	8.5496E-05
	FAM22	6.2234E-02
	2FBBDF(4)	1.9738E-01
	FE	2.0598E-02
	FDE12	6.2913E-04
0.4	FEAM3	7.6884E-05
	FAM22	3.7348E-02
	2FBBDF(4)	1.2772E-01
	FE	2.1113E-02
	FDE12	1.6726E-04
0.6	FEAM3	7.0944E-05
	FAM22	2.6024E-02
	2FBBDF(4)	1.6046E-02
	FE	1.9345E-02
	FDE12	6.3448E-04
0.8	FEAM3	6.6498E-05
	FAM22	1.9641E-02
	2FBBDF(4)	1.2560E-02
	FE	1.7416E-02
	FDE12	8.7997E-04
1.0	FEAM3	6.2990E-04
	FAM22	1.5588E-02
	2FBBDF(4)	1.0427E-02
	FE	1.5693E-02
	FDE12	1.0172E-03

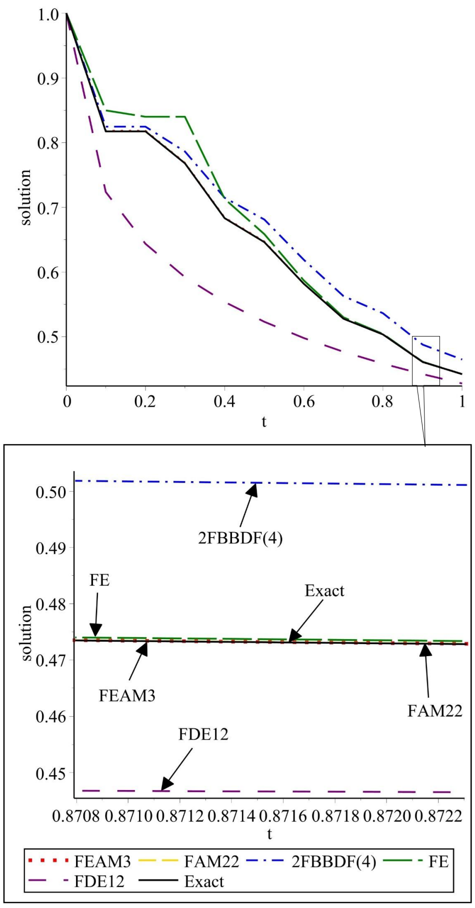

Fig. 6. Plots of the exact and approximate solutions for Problem 2 with $\alpha = 0.50$

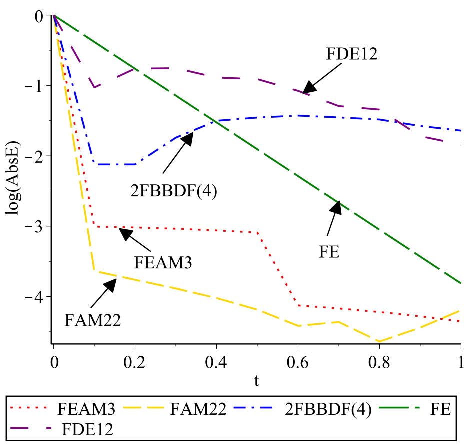

Fig. 7. Efficiency curves for Problem 2 with $\alpha = 0.70$

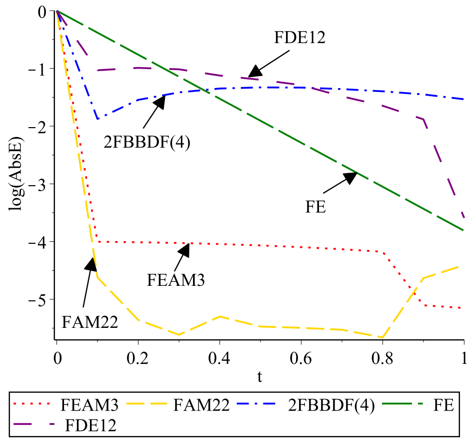

Fig. 8. Efficiency curves for Problem 2 with $\alpha = 0.50$

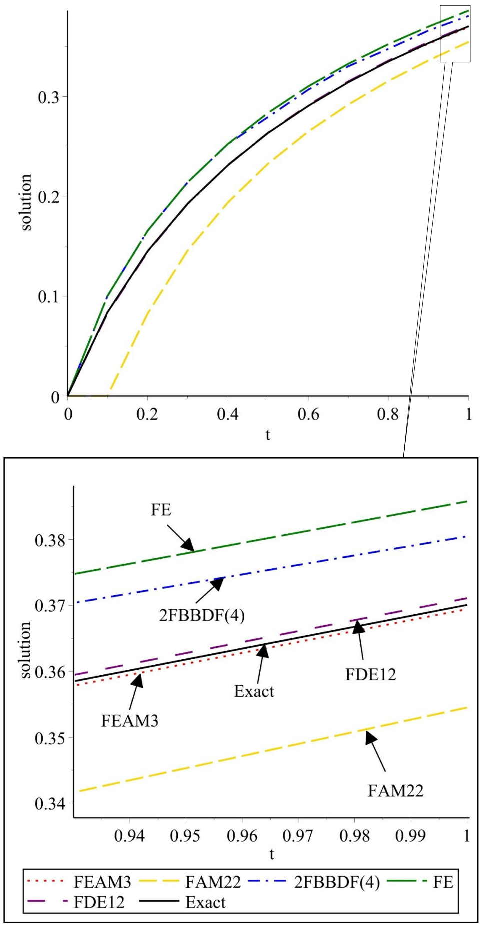

Fig. 9. Plots of the exact and approximate solutions for Problem 3 with $h = 0.1$

and accuracy for large systems may be achieved by the development of improved multigrid and multilevel methods, which would enable the resolution of more intricate real-world problems within a reasonable computational budget.

Furthermore, there is a substantial gap in research that tailors numerical methods to specific applications, despite the widespread application of FDEs in a variety of fields. The majority of the present methods are generic, and while they may be effective for certain issues, they may not completely address the unique challenges of application-specific scenarios. For example, the accuracy and efficiency of a solution can be substantially influenced by the numerical method selected in biological modelling, where FDEs are employed to simulate complex processes such as cell growth or diffusion. Similarly, in finance, where fractional calculus is employed to simulate anomalous diffusion in asset prices or volatility, specialised methods that consider the stochastic nature of financial systems are required. Future research could concentrate on the development of numerical methods that are specific to the application and that are tailored

TABLE VII
ABSOLUTE ERROR OF DIFFERENT METHODS AT t WITH $h = 0.01$ IN SOLVING PROBLEM 3

t	Method	Absolute Error
0.2	FEAM3	4.3778E-06
	FAM22	5.4235E-03
	2FBBDF(4)	4.5628E-04
	FE	1.7050E-03
	FDE12	3.2091E-05
0.4	FEAM3	4.1394E-06
	FAM22	3.5526E-03
	2FBBDF(4)	3.3346E-04
	FE	1.8602E-03
	FDE12	3.9051E-05
0.6	FEAM3	1.0483E-05
	FAM22	2.5881E-03
	2FBBDF(4)	2.6830E-04
	FE	1.7561E-03
	FDE12	4.5799E-05
0.8	FEAM3	1.5097E-05
	FAM22	2.0109E-03
	2FBBDF(4)	2.2627E-04
	FE	1.6084E-03
	FDE12	5.0438E-05
1.0	FEAM3	1.8343E-05
	FAM22	1.6317E-03
	2FBBDF(4)	1.9629E-04
	FE	1.4654E-03
	FDE12	4.1005E-05

TABLE VIII
ABSOLUTE ERROR OF DIFFERENT METHODS AT t WITH $h = 0.001$ IN SOLVING PROBLEM 3

t	Method	Absolute Error
0.2	FEAM3	4.3778E-06
	FAM22	5.4235E-03
	2FBBDF(4)	4.5628E-04
	FE	1.6774E-04
	FDE12	2.8013E-05
0.4	FEAM3	4.1394E-06
	FAM22	3.5526E-03
	2FBBDF(4)	3.3346E-04
	FE	1.8393E-04
	FDE12	4.0941E-05
0.6	FEAM3	1.0483E-05
	FAM22	2.5881E-03
	2FBBDF(4)	2.6830E-04
	FE	1.7410E-04
	FDE12	5.1648E-05
0.8	FEAM3	1.5097E-05
	FAM22	2.0109E-03
	2FBBDF(4)	2.2627E-04
	FE	1.5970E-04
	FDE12	5.8577E-05
1.0	FEAM3	1.8343E-05
	FAM22	1.6317E-03
	2FBBDF(4)	1.9629E-04
	FE	1.4565E-04
	FDE12	5.0431E-05

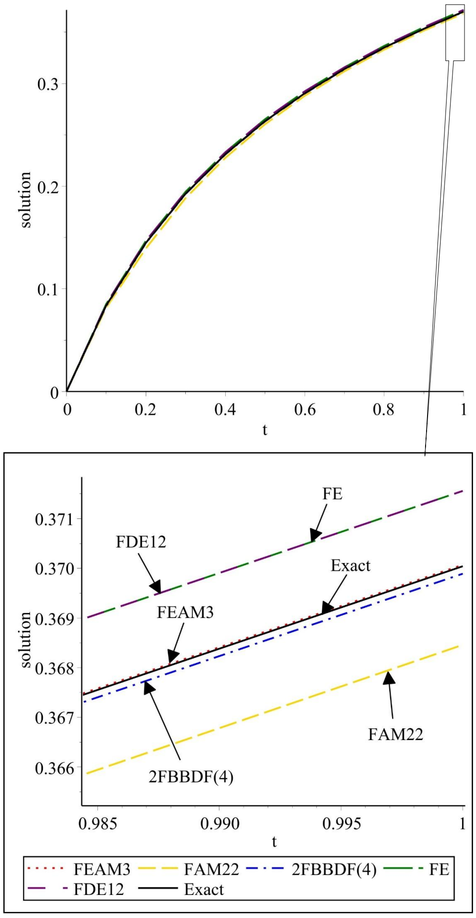

Fig. 10. Plots of the exact and approximate solutions for Problem 3 with $h = 0.01$

to the unique properties of the system being modelled. The gap between theoretical advancements and practical implementation could be bridged by conducting additional case studies that illustrate the practicality of these methods in real-world scenarios. Not only that, the findings indicate that practitioners ought to favour non-block approaches such as FEAM3 for high-accuracy issues that require small step sizes, but block methods like 2FBBDF(4) may be more suitable for parallel computing contexts. Future research should include adaptive step-size implementations and hybrid models customised for specific applications.

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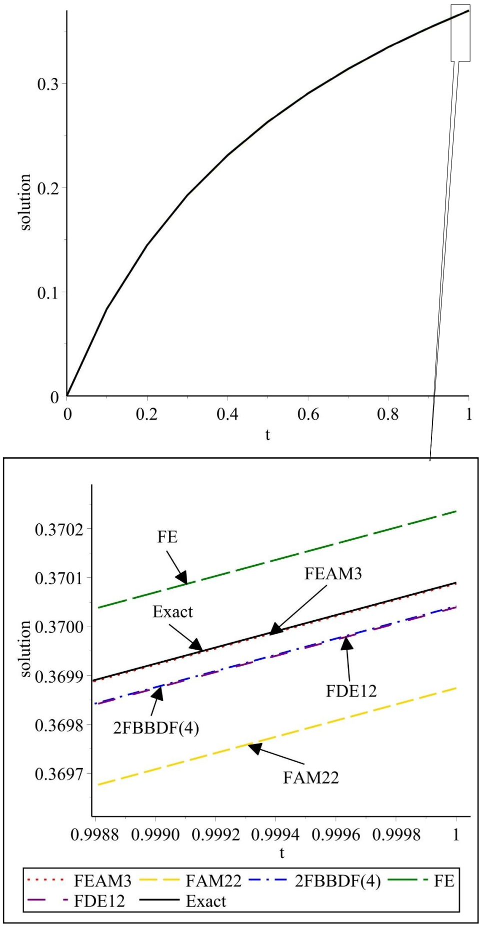

Fig. 11. Plots of the exact and approximate solutions for Problem 3 with $h = 0.001$

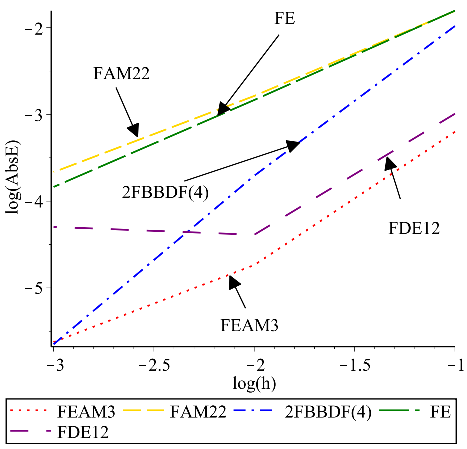

Fig. 12. Efficiency curves for Problem 3

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