Pricing of Digital Exchange Option under Mixed Fractional Jump Diffusion Environment

Mingxuan Shen, Qingqian Shi, Xue Gong, Chunhui Mei

Abstract—**In this paper, the digital exchange option pricing under mixed fractional jump diffusion enviroment is discussed. Under risk neutral measure, a closed form solution for the price of digital exchange option is established by quasi conditional expectation.**

Index Terms—**Mixed fraction Brownian motion; Quasi conditional expectation; Exchange option; Jump diffusion process.**

I. Introduction

BLACK and Scholes [1] are the first to solve the European option pricing model based on the hypothesis that underlying asset price is governed by Brownian motion (BM). Thereafter the theories of option pricing under Brownian motion have been greatly developed. Garman [2] proposed the assumptions of the foreign currency option, and gave a pricing model for standard European currency options. Garman [2] presented a pricing model specifically for standard European currency options. Subsequently, Carr [3] derived an alternative representation of McKee's equation and effectively demonstrated the consistency of the results by decomposing the value of an American put option into the corresponding European put price and an early exercise premium. Bakshi [4] further expanded the scope by introducing an alternative option and conducting a comprehensive study of alternative models from multiple angles, particularly when interest rates, volatility, and jumps were permitted to be stochastic. Andreasen [5] contributed significantly by providing pricing models for both Asian options and lookback options.

However, a substantial body of empirical research has revealed that the stock market price does not adhere to the traditional geometric Brownian motion. Instead, it exhibits several distinct characteristics, such as non-independence, non-normality, long range dependence, and self-similarity. Fama [6] highlighted that the underlying asset price process deviates from geometric Brownian motion and displays a characteristic of sharp peaks and heavy tails. Duan [7] further pointed out that the Black-Scholes (BS) model, which is based on geometric Brownian motion, fails to adequately explain two key phenomena in the stock market: asymmetric peak characteristics and the volatility smile. Given these observations, fractional Brownian motion (FBM), which possesses the properties of self-similarity and long range dependence, emerged as a suitable tool in mathematical finance. FBM was first introduced by Kolmogorov [8] to describe the fluctuation of asset prices. Under the risk neutral measure, Necul [9] utilized the Fourier transform and the Girsanov measure transform to derive the European option pricing formula under FBM. Kalantari [10] subsequently employed the finite difference method to investigate the pricing model of American put options under the fractional Brownian motion model.

It is important to note that fractional Brownian motion has been proposed as a model for logarithmic stock price motion, which allows for long range dependence between returns. This implies the potential for arbitrage opportunities. Rogers [11] indirectly proved the existence of such arbitrage by constructing an example. Rostek [12] also emphasized that while the favorable time series properties of fractional Brownian motion, such as long range dependence, are advantageous, they are accompanied by a seemingly insurmountable disadvantage: the presence of arbitrage. In fact, FBM is not a semimartingale, which poses a significant obstacle to its application in finance. To address this issue, the mixed fractional Brownian motion (MFBM) was introduced. MFBM, which is a semimartingale, represents a linear combination of Brownian motion and fractional Brownian motion. This ensures market completeness and eliminates arbitrage opportunities. Cheridito [13] was the first to establish and prove this. Consequently, MFBM is more suitable for capturing the fluctuations in financial markets. In this context, Ghasemalipour [14] conducted research on financial forecasting models in a mixed fractional Brownian motion environment and estimated European call options. Shokrollahi [15] provided the pricing formula for geometric mean Asian options under the MFBM environment. Additionally, Sun [16] established a currency option pricing model in the MFBM environment.

The classic BS formula hinges on the key assumption that the price of the underlying asset can be described by a continuous stochastic process. However, this assumption has been widely challenged. Merton [17] pointed out that continuous trading is impractical in reality and that no empirical time series exhibits a truly continuous sample path. These observations suggest that the classic BS formula is inadequate for capturing drastic price changes caused by events such as natural disasters, wars, and other special circumstances that can lead to discontinuous or even violent fluctuations in the price of the underlying

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asset [18]. To address these limitations, researchers have explored alternative models that incorporate jumps. Eraker [19] investigated the performance of the dynamic jump diffusion model of stock prices using joint option and stock market data and found that complex jump models offer superior performance in fitting both options and return data simultaneously. Lee [20] proposed that models for individual stocks and all market indices should incorporate Lévy jumps to better capture the empirical characteristics of asset prices. Amin [21] constructed a simple discrete time model of the underlying asset price following a jump diffusion process to value options and provided early exercise boundaries for American call and put options. In 2002, Kou [22] proposed a double exponential jump diffusion model and derived option pricing formulas under this framework. Recently, scholars have further extended these models to incorporate fractional jump diffusion. For example, [23], [24] have explored option pricing under fractional jump diffusion models, aiming to provide more accurate and robust pricing mechanisms that account for both the continuous diffusion and discontinuous jump components of asset price movements.

The exchange option is a unique type of exotic option that grants the option holder the right to exchange two risky assets at the maturity date. Margrabe [25] was the pioneer in this field, proposing a closed form solution for exchange options and thereby laying the foundation for their pricing. Building on this work, Blenman [26] subsequently addressed the pricing of European power exchange options under the risk neutral measure. Kim [27] further extended the research by studying the pricing of exchange options with default risk, utilizing the Klein model to incorporate credit risk and deriving a closed form pricing formula that accounts for this additional risk. In this paper, we introduce a novel type of exchange option known as the digital power exchange option. This innovative option incorporates an indicator function based on the ratio of the prices of the two underlying assets [28]. We begin by establishing the relevant pricing models and definitions for the digital exchange option under the risk neutral probability measure. Subsequently, we derive the pricing formulas for the digital exchange option, leveraging the necessary pricing knowledge and methodologies.

II. Preliminaries

Let (Ω, F, P) be a complete filtered probability space, where P is a probability measure. $B_H(t)$ is an FBM with Hurst parameter $H \in (\frac{3}{4}, 1)$, $B(t)$ is a BM, and $N_i(t) = 1, 2, \cdots, N$ is a Poisson process with respect to P.

The riskless asset $P(t)$ and the two risky assets $S_i(t)(i = 1, 2)$ are governed by the following equations:

$$dP(t) = r(t)P(t)dt, P(0) = 1, 0 \leq t \leq T, \tag{1}$$

where $P(t)$ is the price of the riskless asset at time t and $r(t)$ is the riskless interest rate.

$$\begin{aligned} dS_i(t) =& S_i(t)[(\mu(t) - \lambda\mu_{J_i(t)}dt + \varepsilon_i(t)dB_H(t) \\ &+ \sigma_i(t)dB(t) + (e^{J_i(t)} - 1)dN_i(t)], i = 1, 2 \end{aligned} \tag{2}$$

where $B_H(t)$ is a FBM, which Hurst parameter satisfies $H \in (\frac{3}{4}, 1)$ and $B(t)$ is a standard BM, $\mu(t)$, $\varepsilon_i(t)$, $\sigma_i(t)$ are deterministic functions. $N_i(t)$ are Poisson process with same rate λ representing the number of jumps between $[0, t]$, $J_i(t)$ are the jump size at the time t which are sequence of independent identically distributed random variables with $(e^{J_i(t)} - 1) \sim N(\mu_{J_i(t)}, \sigma^2_{J_i(t)})$. In addition, $B(t)$, $B_H(t)$ and $N_i(t)$ are assumed to be independent each other.

The risk neutral probability measure is a fundamental concept in arbitrage pricing theory. In this measure, the current price of each security in the economy is determined by discounting the expected value of its future payoffs using a risk-free interest rate. The basic theorem of asset pricing shows that the risk neutral probability measure is guaranteed to exist under certain conditions no arbitrage assumption.

Let

$$\hat{B}(t) = B(t) + \int_0^t \frac{\mu(s) - r(s)}{\sigma_i(s)} ds, \quad \hat{B}_H(t) = B_H(t),$$

a risk nuetral measure $\hat{P}$ is defined as

$$\frac{d\hat{P}}{dP} = \exp(-\int_R \varphi(s)dB(s) - \frac{1}{2}\int_R \varphi^2(s)ds),$$

where $\varphi(t) = \frac{\mu(t)-r(t)}{\sigma_i(t)}, i = 1, 2$. The $\hat{B}(t) + \hat{B}_H(t)$ is a new MFBM.

Under the risk neutral measure $\hat{P}$, the equation (2) can be discribed as:

$$\begin{aligned} dS_1(t) =& S_1(t)[(r(t) - \lambda\mu_{J_1(t)}dt + \varepsilon_1(t)d\hat{B}_H(t) \\ &+ \sigma_1(t)d\hat{B}(t) + (e^{J_1(t)} - 1)dN_1(t)], \\ dS_2(t) =& S_2(t)[(r(t) - \lambda\mu_{J_2(t)}dt + \varepsilon_2(t)d\hat{B}_H(t) \\ &+ \sigma_2(t)d\hat{B}(t) + (e^{J_2(t)} - 1)dN_2(t)]. \end{aligned} \tag{3}$$

The exchange option is a contract in which the holder of an option has the right at maturity but does not have to exchange one asset for another, so exchange option is one option of multiple risky assets. The payoff of the exchange option at maturity T is

$$\max\{\beta_1 S_1(T) - \beta_2 S(T), 0\},$$

where $\beta_i \geq 0$ are constants for $i = 1, 2$.

Digital exchange option is an option by adding an indicator function about the ratio range of the two underlying assets on the basis of ordinary exchange option [28].

Definition 1. Let $\beta_i \geq 0$ and $K_i > 0$ are constants for $i = 1, 2$. If the payoff of an option satisfies at maturity T

$$C(T) = \max\{\beta_1 S_1(T) - \beta_2 S_2(T), 0\} I_{\{K_1 \leq \frac{S_1(T)}{S_2(T)} \leq K_2\}},$$

where $\beta_i \geq 0$, $I_{\{\cdot\}}$ is an indicator function and $[K_1, K_2]$ is the execution price interval, this option is named digital exchange option.

III. Some lemmas of the quasi-conditional expectation

In this section, some assumptions and lemmas are introduced on quasi-conditional expectations that are required for the rest of the paper. These lemmas can be found in the fundamental papers concerning the fractional Itô integral [16] and [24].

Assumption 1. Let $T > 0$, supposing $\xi(t) \in (0 \leq t \leq T)$ be a continuous function and $L(t)$ be a function that

$$\xi(t) = \int_R L(u)\phi(u, t)du, 0 \leq t \leq T,$$

where $\phi(u,t) = H(2H-1)|u-t|^{2H-2}$.

Lemma 1. The price at every $t \in [0,T]$ of a bounded F_T^H-measureable claim $F \in L^2$ is given by

$$F(t) = e^{-\int_t^T r(s)ds} E_{\hat{P}}[F|F_t^H],$$

where $E_{\hat{P}}[\cdot]$ denotes the quasi-conditional expectation with respect to the risk-neutral measure.

Lemma 2. Let Assumption 1 holds, let f be a function such that $E_{\hat{P}}[f(\hat{B}_H(T),\hat{B}(T))] < \infty$. Then, for every $0 \le t \le T$ and deterministic functions $\varepsilon(t)$, $\sigma(t)$, we have

$$\begin{aligned} &E_{\hat{P}}[f(\int_0^T \varepsilon(s)d\hat{B}_H(s) + \int_0^T \sigma(s)d\hat{B}(s))] \\ &= \int_R \frac{1}{\sqrt{2\pi\hat{\sigma}_T}} \\ &\times \exp(-\frac{\left(x - \int_0^t \varepsilon(s)d\hat{B}_H(s) - \int_0^t \sigma(s)d\hat{B}(s)\right)^2}{2\hat{\sigma}_T})f(x)dx, \end{aligned}$$

where $\hat{\sigma}_T = \int_t^T \int_t^T \varepsilon(u)\varepsilon(v)\phi(u,v)dudv + \int_t^T \sigma^2(s)ds$.

We consider the process

$$\begin{aligned} Z_t^* &= B^*(t) + B_H^*(t) \\ &= \hat{B}(t) + \hat{B}_H(t) - \int_0^t \sigma(s)ds - \int_0^t \xi(s)ds, 0 \le t \le T, \\ \xi(s) &= \int_t^T \varepsilon(u)\phi(u,s)du. \end{aligned}$$

Define a probability measure P^* by

$$\begin{aligned} \frac{dP^*}{d\hat{P}} &= X_T = \exp[\int_t^T \varepsilon_i(s)d\hat{B}_H(s) + \int_t^T \sigma_i(s)d\hat{B}(s) \\ &- \frac{1}{2}\int_t^T \sigma_i^2(s)ds - \frac{1}{2}\int_t^T\int_t^T \varepsilon_i(u)\varepsilon_i(v)\phi(u,v)dudv]. \end{aligned}$$

Then Z_t^* is a new MFBM under probability measure P^*.

Lemma 3. Let f be a function such that $E_{\hat{p}}[f(\hat{B}_H(T),\hat{B}(T))] < \infty$. Then for every $t \le T$,

$$\begin{aligned} &E_{P^*}[f(\int_0^T \varepsilon(s)dB_H^*(s) + \int_0^T \sigma(s)dB^*(s))] \\ &= \frac{1}{X(t)}E_{\hat{P}}[f(\int_0^T \varepsilon(s)d\hat{B}_H(s) + \int_0^T \sigma(s)d\hat{B}(s))]X(T). \end{aligned}$$

We denote $E_{P^*}[\cdot]$ the quasi-conditional expectation with respect to P^*.

IV. Pricing for digital exchange option

In this section, the pricing formula of exchange option are given in JMFBM.

Theorem 1. The pricing formula at every $t \in [0,T]$ of digital exchange option with maturity T and execution price interval $[K_1, K_2]$ is given as follows:

(i) when $\frac{\beta_2}{\beta_1} > K_2,\ C(t,T) = 0$.

(ii) when $K_1 \le \frac{\beta_2}{\beta_1} \le K_2$,

$$\begin{aligned} C(t,T) &= E_p[e^{-\int_t^T r(s)ds}C(T)|F_t] \\ &= e^{-\int_t^T r(s)ds}\sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}[\Phi_1 - \Phi_2], \end{aligned}$$

where

$$\begin{aligned} \Phi_1 =& \beta_1 S_1(t)\exp[\int_t^T (r(s) - \lambda\mu_{J_1(t)})ds + \sum_{i=1}^n J_1(T_i)] \\ &\times [N(d_1) - N(d_3)], \\ \Phi_1 =& \beta_2 S_2(t)\exp[\int_t^T (r(s) - \lambda\mu_{J_2(t)})ds + \sum_{i=1}^n J_2(T_i)] \\ &\times [N(d_2) - N(d_4)]. \end{aligned}$$

$$d_1 = \frac{\ln\frac{K_2}{\frac{S_1(t)}{S_2(t)}} + \int_t^T (\lambda\mu_{J_1(t)} - \lambda\mu_{J_2(t)})ds - B}{\sqrt{b}},$$

$$d_2 = \frac{\ln\frac{K_2}{\frac{S_1(t)}{S_2(t)}} + \int_t^T (\lambda\mu_{J_1(t)} - \lambda\mu_{J_2(t)})ds - D}{\sqrt{b}},$$

$$d_3 = \frac{\ln\frac{S_1(t)}{S_2(t)} - \ln(\frac{\beta_2}{\beta_1}) - \int_t^T (\lambda\mu_{J_1(t)} - \lambda\mu_{J_2(t)})ds + B}{\sqrt{b}},$$

$$d_4 = \frac{\ln\frac{S_1(t)}{S_2(t)} - \ln(\frac{\beta_2}{\beta_1}) - \int_t^T (\lambda\mu_{J_1(t)} - \lambda\mu_{J_2(t)})ds + D}{\sqrt{b}},$$

$$B = \sum_{i=1}^n [J_1(T_i) - J_2(T_i)] + \tfrac{1}{2b},$$

$$D = \sum_{i=1}^n [J_1(T_i) - J_2(T_i)] - \tfrac{1}{2b},$$

$$\begin{aligned} b =& \int_t^T [\sigma_1(s) - \sigma_2(s)]^2 ds \\ &+ \int_t^T\int_t^T [\varepsilon_1(u) - \varepsilon_2(u)][\varepsilon_1(v) - \varepsilon_2(v)]\phi(u,v)dudv. \end{aligned}$$

(iii) when $\frac{\beta_2}{\beta_1} \le K_1$,

$$\begin{aligned} C(t,T) &= E_{\hat{P}}[e^{-\int_t^T r(s)ds}C(T)|F_t] \\ &= e^{-\int_t^T r(s)ds}\sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}[\Phi_3 - \Phi_4], \end{aligned}$$

where

$$\begin{aligned} \Phi_3 =& \beta_1 S_1(t)\exp[\int_t^T (r(s) - \lambda\mu_{J_1(t)})ds + \sum_{i=1}^n J_1(T_i)] \\ &\times [N(d_1) - N(d_7)], \end{aligned}$$

$$\begin{aligned} \Phi_4 =& \beta_2 S_2(t)\exp[\int_t^T (r(s) - \lambda\mu_{J_2(t)})ds + \sum_{i=1}^n J_2(T_i)] \\ &\times [N(d_2) - N(d_8)], \end{aligned}$$

$$d_7 = \frac{\ln\frac{\frac{S_1(t)}{S_2(t)}}{K_1} - \int_t^T (\lambda\mu_{J_1(t)} - \lambda\mu_{J_2(t)})ds + B}{\sqrt{b}},$$

$$d_8 = \frac{\ln\frac{\frac{S_1(t)}{S_2(t)}}{K_1} - \int_t^T (\lambda\mu_{J_1(t)} - \lambda\mu_{J_2(t)})ds + D}{\sqrt{b}}.$$

Proof.

Letting $N(T-t)=n$, we have

$$S_{in}(T)=S_i(t)\exp[\int_t^T r(s)-\lambda\mu_{J_i(t)})ds+\sum_{k=1}^{n}J_i(T_k)$$
$$+\int_t^T \varepsilon_i(s)d\hat{B}_H(s)+\int_t^T \sigma_i(s)d\hat{B}(s)-\frac{1}{2}\int_t^T\sigma_i^2(s)ds$$
$$-\frac{1}{2}\int_t^T\int_t^T \varepsilon_i(u)\varepsilon_i(v)\phi(u,v)dudv], i=1,2.$$

By Lemma 1, we have

$$C(t,T)=E_{\hat{P}}[e^{-\int_t^T r(s)ds}C(T)|F_t]$$
$$=e^{-\int_t^T r(s)ds}\sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}$$
$$\times\{\beta_1 E_{\hat{P}}[S_{2n}(T)\cdot(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{\{K_1\le\frac{S_{1n}}{S_{2n}}\le K_2\}}|F_t].$$

By Lemma 2 and Lemma 3, we have

$$E_{\hat{P}}[S_{2n}(T)\cdot(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{\{K_1\le\frac{S_{1n}}{S_{2n}}\le K_2\}}|F_t]$$
$$=E_{\hat{P}}[S_2(t)\exp[\int_t^T (r(s)-\lambda\mu_{J_2(t)})ds+\sum_{i=1}^{n}J_2(T_i)]$$
$$\times\frac{X(T)}{X(t)}(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[K_1\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t]$$
$$=S_2(t)\exp[\int_t^T (r(s)-\lambda\mu_{J_2(t)})ds+\sum_{i=1}^{n}J_2(T_i)]$$
$$\times E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[K_1\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t].$$

Then some outcomes can be given in follow:

(i) When $\frac{\beta_2}{\beta_1}>K_2$,

$$E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[K_1\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t]=0.$$

(ii) When $K_1\le\frac{\beta_2}{\beta_1}\le K_2$,

$$E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[K_1\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t]$$
$$=E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[\frac{\beta_2}{\beta_1}\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t].$$

(iii) When $\frac{\beta_2}{\beta_1}\le K_1$,

$$E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[K_1\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t]$$
$$=E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[K_1\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t].$$

Due to the proof of (iii) is similar to that of (ii), we only give the proof of (ii).

$$E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[\frac{\beta_2}{\beta_1}\le\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t]$$
$$=E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t]$$
$$-E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[\frac{S_{1n}}{S_{2n}}\ge\frac{\beta_2}{\beta_1}]}|F_t]$$
$$=\Pi_1-\Pi_2.$$

Let

$$x=\int_0^T[\varepsilon_1(s)-\varepsilon_2(s)]dB_H^*(s)$$
$$+\int_0^T[\sigma_1(s)-\sigma_2(s)]dB^*(s)],$$

$$c=\int_0^t[\varepsilon_1(s)-\varepsilon_2(s)]dB_H^*(s)$$
$$+\int_0^t[\sigma_1(s)-\sigma_2(s)]dB^*(s)].$$

By Lemma 2, we have

$$\Pi_1=E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[\frac{S_{1n}}{S_{2n}}\le K_2]}|F_t]$$
$$=\int_{-\infty}^{d^*}\frac{S_1(t)}{S_2(t)}\exp\{-\int_t^T(\lambda\mu_{J_1(t)}-\lambda\mu_{J_2(t)})ds$$
$$+\sum_{i=1}^{n}[J_1(T_i)-J_2(T_i)+x-c-\frac{1}{2}b]\}$$
$$\times\frac{1}{\sqrt{2\pi b}}\exp^{-\frac{(x-c)^2}{2b}}dx-\frac{\beta_2}{\beta_1}\int_{-\infty}^{d^*}\frac{1}{\sqrt{2\pi b}}\exp^{-\frac{(x-c)^2}{2b}}dx$$
$$=\frac{S_1(t)}{S_2(t)}\exp\{-\int_t^T(\lambda\mu_{J_1(t)}-\lambda\mu_{J_2(t)})ds$$
$$+\sum_{i=1}^{n}[J_1(T_i)-J_2(T_i)]\}N(d_1)-\frac{\beta_2}{\beta_1}N(d_2),$$

where

$$d^*=\ln\frac{K_2}{\frac{S_1(t)}{S_2(t)}}+\int_t^T(\lambda\mu_{J_1(t)}-\lambda\mu_{J_2(t)})ds-D+c.$$

The same procedure may be easily adapted to obtain:

$$\Pi_2=E_{P^*}[(\frac{S_{1n}(T)}{S_{2n}(T)}-\frac{\beta_2}{\beta_1})^+ I_{[\frac{S_{1n}}{S_{2n}}\ge\frac{\beta_2}{\beta_1}]}|F_t]$$
$$=\frac{S_1(t)}{S_2(t)}\exp\{-\int_t^T(\lambda\mu_{J_1(t)}-\lambda\mu_{J_2(t)})ds$$
$$+\sum_{i=1}^{n}[J_1(T_i)-J_2(T_i)]\}N(d_3)-\frac{\beta_2}{\beta_1}N(d_4).$$

Thus, we obtain the result of Theorem 1.

V. CONCLUSIONS

In this paper, we introduced Poisson jumps to the mixed fractional Brownian motion, under risk neutral measure, a closed form solution for the price of digital exchange option is established by means of quasi-conditional expectation. As one of the financial derivative instruments, options have certain functions of hedging and hedging risks. Our research work on option pricing under mixed fractional Brownian motion enriches the theory of option pricing and has certain practical significance for risk management.

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