Stability Analysis of Higher Order and Fractional Anti-Differences with Mixed Difference Operators

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***Abstract*—The objective of this article is to explore the anti-difference principle using mixed difference operators, deriving theorems for the m^{th} order anti-difference related to finite series. We establish higher order difference equations with factorial coefficients, extending to fractional orders and deriving a fractional anti-difference principle from its integer counterpart. Mixed gamma geometric factorials are introduced to formulate fundamental theorems for mixed fractional difference equations. We analyze the behavior of the ν^{th} order anti-difference principle, providing a solid theoretical foundation for applying mixed difference operators in discrete dynamics.**

***Index Terms*—Delta integrable function, Discrete Delta integration, Fractional sum, Closed form, Summation form, Numerical analysis.**

I. Introduction

DIFFERENCE equations play a pivotal role in mathematical modeling, offering a framework for analyzing systems that evolve in discrete steps. The development of mixed (q, h)-difference operators provides a powerful tool for studying such systems, particularly those defined on non-uniform time scales. This modern extension unifies concepts from difference and differential calculus into a more generalized discrete framework.

In [1], the authors examine the connection between the q-derivative operator D_q and divided differences within the scope of quantum calculus. Building on this, the (p, q)-calculus is used to derive various identities involving the (p, q)-derivative operator, generalizing classical results. The study in [2] extends these ideas by employing q, $q^{(\alpha)}$, and h-difference operators to derive fundamental theorems, offering both closed-form and summation solutions to higher-order difference equations.

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Operator theory continues to evolve in this domain. Two generalized q-exponential operators with three parameters are introduced in [3], leading to a range of operator identities and their application to extended q-Gauss summation formulas. Related developments in [4], yield q-exponential identities by solving q-difference equations, with applications to q-Mehler equations, generalized q-beta integrals, and generating functions for generalized Al-SalamCarlitz polynomials.

Several works investigate the construction and inversion of q-difference operators. In [5], a q-difference operator of arbitrary order is defined using generalized factorials and Stirling numbers, while [6] focus on existence and uniqueness results for nonlinear fractional q-difference equations under nonlocal conditions, employing fixed-point theorems and continuity principles. Operative symbol analysis and inverse operations are further explored in [7], and nonlinear fractional models with nonlocal boundary conditions are addressed in [8] using classical fixed-point results.

The framework of q-hypergeometric functions is extended in [9] through the development of new q-operators and identities such as q-binomial and q-ChuVandermonde summation formulas. These tools also support combinatorial identities and the structural theory of special functions.

The authors in [10] investigate fractional-order delta operators associated with the Fibonacci sequence, introducing an infinite series formulation. They derive explicit summation formulas for these series, enriching the framework of discrete fractional calculus. The work contributes to the theoretical understanding of fractional operators applied to special sequences. Further applications include the analysis of singular fractional q-integro-differential equations using Caputo derivatives in [11], where compactness arguments and convergence theorems underpin the theoretical framework. A related approach in [12] uses the q-Laplace transform to convert fractional q-differential equations into solvable integer-order forms, supported by illustrative examples.

Investigations into the algebraic structure of difference operators continue with studies in [13], which explore the interplay between divided differences and q-derivatives under (p, q)-calculus, and introduce symmetric difference operators to tackle higher-order problems. Discrete fractional calculus methods are developed in [14], applying composition rules and Laplace transform techniques to solve initial value problems.

A generalized theory of difference operators of the n^{th} kind is developed in [15], while [16] proposes a quantum algorithm based on block-encoded operator updates for solving linear systems using gradient descent techniques. But the theory of mixed difference operator is still unexplored.

Hence in this article, we extend these developments by introducing the (q,h)-mixed difference operator and its corresponding anti-difference principles, along with higher-order generalizations. Moreover we construct geometric and mixed gamma-geometric factorials to support applications in fractional-order settings. The effectiveness of our approach is demonstrated through its application to the SIR epidemic model, where the results highlight the operators generality and practical utility across discrete and fractional systems.

II. Preliminaries Related To (q,h) - Mixed Operator

Definition II.1. [10] For $\mathfrak{t} \in \mathbb{R}$ and $n \in \mathbb{N}$, the polynomial factorial function $\mathfrak{t}^{(n)}$ is defined by

$$\mathfrak{t}^{(n)} = \prod_{r=0}^{n-1} (\mathfrak{t} - r). \tag{1}$$

Also, for $\nu \in (-\infty, \infty)$, the ν^{th} factorial polynomial is given by

$$\mathfrak{t}^{(\nu)} = \frac{\Gamma(\mathfrak{t}+1)}{\Gamma(\mathfrak{t}-\nu+1)}, \tag{2}$$

where $\mathfrak{t} - \nu + 1, \mathfrak{t} + 1 \notin -\mathbb{N}_0 = \{0, -1, -2, \dots\}$.

Definition II.2. [10] The sum of the values of k^{th} factorial polynomials of the first $\mathfrak{t}$ natural numbers is given below

$$1^{(k)} + 2^{(k)} + 3^{(k)} + \cdots + \mathfrak{t}^{(k)} = \frac{(\mathfrak{t}+1)^{(k+1)}}{k+1}. \tag{3}$$

The ν^{th} fractional Taylor monomial, defined at s, is given by

$$h_\nu(s, \mathfrak{t}) = \frac{(\mathfrak{t}-s)^{(\nu)}}{\Gamma(1+\nu)}, \tag{4}$$

where $(\mathfrak{t}-s)^{(\nu)}$ is obtained by replacing $\mathfrak{t}$ with $\mathfrak{t}-s$ in (2).

Definition II.3. [15] Let $f : \mathscr{M}_h^q \to \mathbb{R}$ be a function. Then the (q,h)-difference operator, denoted by $\underset{(q,h)}{\Delta}$, is defined as

$$\underset{(q,h)}{\Delta} f(\mathfrak{t}) = f(\mathfrak{t}q + h) - f(\mathfrak{t}), \quad \mathfrak{t} \in \mathscr{M}_h^q. \tag{5}$$

Lemma II.4. *Let $f, g : \mathscr{M}_h^q \to \mathbb{R}$, $q \in \mathbb{R} - \{0, 1\}$ and $0 \neq h \in \mathbb{R}$. Then the product rule of (q,h) difference operator is obtained as* $\underset{(q,h)}{\overset{-1}{\Delta}} \{f(t)g(t)\}$

$$= f(t) \underset{(q,h)}{\overset{-1}{\Delta}} g(t) - \underset{(q,h)}{\overset{-1}{\Delta}} \{ \underset{(q,h)}{\overset{-1}{\Delta}} g(tq+h) \underset{(q,h)}{\Delta} f(t)\}. \tag{6}$$

Proof: *Applying the $\underset{(q,h)}{\Delta}$ operator on the function $f(t)g(t)$ and then adding and subtracting the term $f(t)w(tq+h)$, we obtain*

$$\underset{(q,h)}{\overset{-1}{\Delta}} \{f(t)g(t)\} = w(tq+h) \underset{(q,h)}{\Delta} f(t) + f(t) \underset{(q,h)}{\Delta} w(t).$$

Thus the proof completes by taking $\underset{(q,h)}{\Delta} w(t) = g(t)$ and $\underset{(q,h)}{\overset{-1}{\Delta}} g(t) = w(t)$.

III. Analysis of the (q,h)-Difference Operator

The (q,h)-difference operator generalizes the standard difference operator by incorporating both scaling and shifting. This paper aims to derive and analyze its fundamental properties, investigate its behavior under different transformations, and explores how it acts on several types of functions.

A. Linearity and Additivity

To establish that the operator is linear, we check the additivity and homogeneity properties. Let $f(t)$ and $g(t)$ be any two functions and α be any scalar. Then $\underset{(q,h)}{\Delta} (f(t) + g(t))$

$$= f(t+qh) - f(t) + g(t+qh) - g(t)$$
$$= \underset{(q,h)}{\Delta} f(t) + \underset{(q,h)}{\Delta} g(t).$$

Thus the operator satisfies additivity.

Now $\underset{(q,h)}{\Delta} (\alpha f(t)) = (\alpha f(t+qh)) - \alpha f(t)$

$$= \alpha (f(t+qh) - f(t)) = \alpha \underset{(q,h)}{\Delta} f(t).$$

Thus, the operator is linear.

B. Shift-invariance

For the shifted function $f(t+\delta)$, we can compute $\underset{(q,h)}{\Delta} f(t+\delta) = f(t+\delta+qh) - f(t+\delta)$.

This shows that the operator shifts the function's argument by both δ and h, preserving the form of the operator. Therefore, the operator shifts the argument and operates as expected on the shifted function.

C. Scaling-invariance

Now, consider the scaled function $f(qt)$. The operator acts as $\underset{(q,h)}{\Delta} f(qt) = f(q^2 t + h) - f(qt)$. Here, the operator incorporates both scaling by q and shifting by h. This indicates that the operator affects both the scale and the shift in the input variable.

D. Idempotence

If the operator is idempotent, the idempotence condition is $\underset{(q,h)}{\Delta} \underset{(q,h)}{\Delta} f(t) = \underset{(q,h)}{\Delta} f(t)$.

For simplicity, we consider the function $f(t) = t$.

$\underset{(q,h)}{\Delta} t = (qt+h) - t = (q-1)t + h$.

Now, apply the operator again to this result,

$\underset{(q,h)}{\Delta} ((q-1)t + h) = (q(q-1)t+h+h) - ((q-1)t + h)$.

Comparing this with $\underset{(q,h)}{\Delta} t = (q-1)t + h$ shows the operator is not idempotent for $f(t) = t$, but for some other types of functions (e.g., constant functions), idempotence may hold.

E. Commutativity with Other Operators

Next, we investigate whether the (q,h)-difference operator commutes with other standard operators, such as differentiation. Consider $\underset{(q,h)}{\Delta} \left(\frac{d}{dt} f(t)\right)$ .Then we write

$$\underset{(q,h)}{\Delta} \left(\tfrac{d}{dt} f(t)\right) = \frac{d}{dt} f(t+qh) - \tfrac{d}{dt} f(t).$$

But $\frac{d}{dt} f(t+qh) = \lim_{\epsilon\to 0} \dfrac{f(t+qh+\epsilon) - f(t+qh)}{\epsilon}$,
and $\dfrac{d}{dt} f(t) = \lim_{\epsilon\to 0} \dfrac{f(t+\epsilon) - f(t)}{\epsilon}$.
Thus, the two terms do not directly commute and this suggests that the operator does not commute with differentiation.

F. Taylor Series Expansion

Next, we investigate the effect of the (q,h)-difference operator on a Taylor series. For a function $f(t)$ with a Taylor expansion around $t = s$, $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(s)}{n!}(t-s)^n$, we apply the (q,h)-difference operator, that is

$$f(t+qh)f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(s)}{n!}\left[(t+qh-s)^n - (t-s)^n\right].$$

Using the binomial expansion for $(t+qh-s)^n$, we get $(t+qh-s)^n = \sum_{k=0}^{n} \binom{n}{k}(t-s)^{n-k}(qh)^k$. Thus, the difference becomes

$$(t+qh-s)^n - (t-s)^n = \sum_{k=1}^{n} \binom{n}{k}(t-s)^{n-k}(qh)^k.$$

$$\underset{(q,h)}{\Delta} f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(s)}{n!} \sum_{k=1}^{n} \binom{n}{k}(t-s)^{n-k}(qh)^k.$$

This provides the new form of the function under the action of the (q,h)-difference operator, showing how each term in the Taylor series is modified by the operator.

Here in this section, we derived the properties of additivity, homogeneity, shift-invariance, scaling-invariance, and examined its interaction with differentiation. We also expanded the effect of the operator on Taylor series.

IV. Anti-Difference Principle of Mixed Operator

In this section, we derive the theorems for the anti-difference principle using the mixed difference operator.

Theorem IV.1. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $\mathfrak{t} \in \mathbb{R}$, and $n \in \mathbb{N}$. Then the first-order anti-difference principle $(\mathfrak{q}, h)$ operator is given by*

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right) = \sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right). \quad (7)$$

Proof:

Consider
$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) = y(\mathfrak{t}). \quad (8)$$

Then we have
$$x(\mathfrak{t}) = \underset{(\mathfrak{q},h)}{\Delta} y(\mathfrak{t}). \quad (9)$$

Applying Definition II.3,in the above equation,

$$x(\mathfrak{t}) = y(\mathfrak{t}\mathfrak{q} + h) - y(\mathfrak{t}). \quad (10)$$

Next, substitute $\mathfrak{t}$ by $\mathfrak{t}/\mathfrak{q}$ in equation (10)*, we get*

$$y(\mathfrak{t} + h) = x\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right) + y\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right). \quad (11)$$

Now, substitute $\mathfrak{t}$ by $\mathfrak{t} - h$ in equation (11)*, and we get*

$$y(\mathfrak{t}) = x\left(\frac{\mathfrak{t}-h}{\mathfrak{q}}\right) + y\left(\frac{\mathfrak{t}-h}{\mathfrak{q}}\right). \quad (12)$$

Next, substituting $\mathfrak{t}$ by $\frac{\mathfrak{t}-h}{\mathfrak{q}}$ in equation (12) *gives*

$$x\left(\frac{\mathfrak{t}-h}{\mathfrak{q}}\right) = x\left(\frac{\mathfrak{t} - h\sum_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right) + y\left(\frac{\mathfrak{t} - h\sum_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right). \quad (13)$$

Substitute equation (13) *into equation* (12)*, yielding*

$$y(t) = x\left(\frac{\mathfrak{t}-h}{\mathfrak{q}}\right) + x\left(\frac{\mathfrak{t} - h\sum_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right) + y\left(\frac{\mathfrak{t} - h\sum_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right). \quad (14)$$

By continuing this iterative process, we arrive at the following generalization for n terms

$$y(t) = \sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right) + y\left(\frac{\mathfrak{t} - h\sum_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right). \quad (15)$$

Using (8)*, the above equation becomes*

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right) = \sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right), \quad (16)$$

which completes the proof.

Corollary IV.2. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, where $\mathfrak{q} \in \mathbb{R} - \{0, 1\}$, $n \in \mathbb{N}$, and $\mathfrak{t} \in \mathbb{R}$. If $h = 0$, then equation* (16) *reduces to the following form*

$$\overset{-1}{\underset{(\mathfrak{q},0)}{\Delta}} x(\mathfrak{t}) - \overset{-1}{\underset{(\mathfrak{q},0)}{\Delta}} x\left(\frac{\mathfrak{t}}{\mathfrak{q}^n}\right) = \sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t}}{\mathfrak{q}^{r+1}}\right). \quad (17)$$

Proof: *The proof is obvious when h=0 in* (16).

Corollary IV.3. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, where $h \in \mathbb{R} - \{0\}$, $\mathfrak{t} \in \mathbb{R}$ and $n \in \mathbb{N}$. If $\mathfrak{q} = 1$, then* (16) *simplifies as*

$$\overset{-1}{\underset{(1,h)}{\Delta}} x(\mathfrak{t}) - \overset{-1}{\underset{(1,h)}{\Delta}} x(\mathfrak{t} - nh) = \sum_{r=0}^{n-1} x(\mathfrak{t} - (r+1)h). \quad (18)$$

Proof: *The proof is trivial when q=1 in* (16)

Example IV.4. ***Case (i).*** *In equation* (18)*, let h=m=2 and $f(t) = 3^t$. We then have the following calculations:*

$$\underset{2}{\Delta}\left(\frac{3^t}{3^2-1}\right) = 3^t, \qquad \overset{2}{\underset{2}{\Delta}}\left(\frac{3^t}{(3^2-1)^2}\right) = 3^{t^2},$$

$$\overset{-2}{\underset{2}{\Delta}} 3^t = \frac{3^t}{(3^2-1)^2}, \qquad \overset{-2}{\underset{2}{\Delta}} 3^t\Big|_{-\infty} = 0,$$

and for $s = 5$, Using the formula in equation (18)*, we get*

$$\overset{-m}{\underset{h}{\Delta}} f(s) - \overset{-1}{\underset{-1/t}{\Delta}} f_t(m,s,h) = \sum_{r=0}^{t} f_r(m,s,h). \quad (19)$$

we use the result from equation (19) *to get*

$$\overset{-2}{\underset{2}{\Delta}} 3^s - \overset{-1}{\underset{-1/t}{\Delta}} f_t(2,s,2) = \sum_{r=0}^{t} f_r(2,s,2). \quad (20)$$

When $s = 5$, the first term in equation (20) simplifies to

$$\overset{-2}{\underset{2}{\Delta}} 3^s\Big|_{s=5} = \frac{3^s}{(3^2-1)^2}\Big|_5 = \frac{3^5}{8^2} = \frac{3^{10}}{3^5 \cdot 8^2}. \tag{21}$$

Now, by setting $k = 2$, we find

$$\overset{-1}{\underset{-1/t}{\Delta}} f_t(2,5,2) = \overset{-1}{\underset{-1/t}{\Delta}} \left\{(t+1)3^{5-(t+2)2}\right\}$$

$$= (t+1)\frac{3^{5-(t+2)2}}{3^2-1} + \frac{3^{5-(t+1)2}}{(3^2-1)^2}. \tag{22}$$

When $t = 3$, the above equation simplifies to

$$\overset{-1}{\underset{-1/t}{\Delta}} f_t(2,5,2)\Big|_{t=3} = \overset{-1}{\underset{-1/t}{\Delta}} \left\{(t+1)3^{5-(t+2)2}\right\}\Big|_{t=3}$$

$$= 4\frac{3^{-5}}{3^2-1} + \frac{3^{-3}}{(3^2-1)^2} = \frac{41}{3^5 \cdot 8^2}. \tag{23}$$

Expanding the right-hand side of equation (20), we get $\sum_{r=0}^{t} f_r(2,5,2) = \sum_{r=0}^{3}(r+1)3^{(1-2r)}$

$$= 3 + \frac{2}{3} + \frac{3}{3^3} + \frac{4}{3^5} = \frac{922}{3^5} = \frac{922 \cdot 8^2}{3^5 \cdot 8^2}. \tag{24}$$

Hence equation (20) is verified by equations (21), (23) and (24).

Case (ii). *Now, in equation (19), take $h = -2$, $m = 2$ and $f(t) = 3^{-t}$. We compute the following:*

$$\underset{-2}{\Delta}\left(\frac{3^{-t}}{3^2-1}\right) = -3^{-t}, \quad \overset{2}{\underset{-2}{\Delta}}\left(\frac{3^{-t}}{(3^2-1)^2}\right) = 3^{-t},$$

$$g(t) = \frac{3^{-t}}{(3^2-1)^2}, \quad \underset{t\to\infty}{g(t)} = 0,$$

and we have the equation

$$\overset{-2}{\underset{-2}{\Delta}} \frac{1}{3^s} - \overset{-1}{\underset{-1/t}{\Delta}} f_t(2,s,-2) = \sum_{r=0}^{t} f_r(2,s,-2). \tag{25}$$

When $s = 5$ and $t = 3$, as in case (i), we obtain

$$\overset{-2}{\underset{-2}{\Delta}} \frac{1}{3^s}\Big|_{s=5} = \frac{1}{3^s(3^2-1)^2}\Big|_5 = \frac{1}{3^5 \cdot 8^2} = \frac{3^{10}}{3^{15} \cdot 8^2}. \tag{26}$$

Next, using the definition of $f_t(m,s,-h)$, we find

$$\overset{-1}{\underset{-1/t}{\Delta}} f_t(2,5,-2)\Big|_{t=3} = \overset{-1}{\underset{-1/t}{\Delta}} \left\{\frac{(t+1)}{3^{5+(t+2)2}}\right\}\Big|_{t=3}$$

$$= 4\frac{1}{3^{15} \cdot 8} + \frac{1}{3^{13} \cdot 8^2} = \frac{41}{3^{15} \cdot 8^2}. \tag{27}$$

Expanding the RHS of the equation (25), we get

$$\sum_{r=0}^{t} f_r(2,5,-2) = \sum_{r=0}^{3} \frac{(r+1)}{3^{5+(r+2)2}} = \frac{922}{3^{15}}. \tag{28}$$

Therefore, equation (19) is verified by equations (26), (27) and (28).

V. Higher Order Anti-Difference Principle by $(\mathfrak{q}, h)$ Operator

In this section, we develop theorems and corollaries for m^{th} order anti-difference principle using the mixed difference operator.

Theorem V.1. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, where $m, n \in \mathbb{N}$ and $\mathfrak{t} \in \mathbb{R}$. Then the higher-order anti-difference principle of the mixed difference operator is given by*

$$\overset{-m}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \sum_{d=0}^{m-1} \frac{n^{(d)}}{d!} \overset{-(m-d)}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$= \sum_{r=m-1}^{n-1} \frac{r^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right). \tag{29}$$

Proof: Theorem IV.1 provides the proof for $m = 1$. Applying the operator $\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}}$ on both sides of equation (7), we obtain

$$\overset{-2}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \overset{-2}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$= \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}}\left[\sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right)\right]. \tag{30}$$

The right-hand side of equation (30) is calculated as follows:

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}}\left[\sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right)\right]$$

$$= \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{s=0}\mathfrak{q}^s}{\mathfrak{q}}\right) + \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{1}\mathfrak{q}^s}{\mathfrak{q}^2}\right)$$

$$+ \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{2}\mathfrak{q}^s}{\mathfrak{q}^3}\right) + \cdots + \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{n-1}\mathfrak{q}^s}{\mathfrak{q}^n}\right)$$

Simplifying further,

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}}\left[\sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right)\right] = \sum_{r=0}^{n-1} \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right) \tag{31}$$

Replacing $\mathfrak{t}$ by $(\mathfrak{t}-h)/\mathfrak{q}$, $(\mathfrak{t} - h\sum_{r=0}^{1}\mathfrak{q}^r)/\mathfrak{q}^2$, $(\mathfrak{t} - h\sum_{r=0}^{2}\mathfrak{q}^r)/\mathfrak{q}^3, \ldots$ in equation (15) and then substituting equation (15) into (31), we obtain

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h}{\mathfrak{q}}\right) = x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right) + \left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{2}\mathfrak{q}^r}{\mathfrak{q}^3}\right)$$

$$+\cdots+x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right) + \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right). \quad (32)$$

Similarly, we obtain

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right) = x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{2}\mathfrak{q}^r}{\mathfrak{q}^3}\right)$$

$$+\cdots+x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right) + \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right).$$

Repeating the same process, we arrive

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{2}\mathfrak{q}^r}{\mathfrak{q}^3}\right) = x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{3}\mathfrak{q}^r}{\mathfrak{q}^4}\right)$$

$$+\cdots+x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right) + \overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right). \quad (33)$$

Using these results in equation (31), we obtain

$$\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}}\left[\sum_{r=0}^{n-1} x\left(\frac{\mathfrak{t}-h\sum\limits_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right)\right] = x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right)$$

$$+2x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{2}\mathfrak{q}^r}{\mathfrak{q}^3}\right) + \cdots + (n-1)x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right)$$

$$+n\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{r=0}^{n-1}\mathfrak{q}^r}{\mathfrak{q}^n}\right).$$

Substituting this into equation (30), we get

$$\overset{-2}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \overset{-2}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$-n\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right) = \sum_{r=1}^{n-1} rx\left(\frac{\mathfrak{t}-h\sum\limits_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right). \quad (34)$$

Again, applying $\overset{-1}{\underset{(q,h)}{\Delta}}$ on both sides of (30) and using (15), we obtain

$$\overset{-3}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \overset{-3}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right) .-n\overset{-2}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$= p\cdot x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{2}\mathfrak{q}^j}{\mathfrak{q}^3}\right) + \sum_{p=1}^{2} p\cdot x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{3}\mathfrak{q}^j}{\mathfrak{q}^4}\right)$$

$$+\ldots+\sum_{p=1}^{n-1} p\cdot \underset{(\mathfrak{q},h)}{\Delta^{-1}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

Putting $k=1$ in (3) and then substituting in the above equation, we obtain

$$\overset{-3}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \overset{-3}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$-n\overset{-2}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right) - \frac{n^{(2)}}{2!}\overset{-1}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$= \sum_{r=2}^{n-1}\frac{r^{(2)}}{2!} x\left(\frac{\mathfrak{t}-h\sum\limits_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right).$$

Similarly, we obtain the fourth inverse as

$$\overset{-4}{\underset{(\mathfrak{q},h)}{\Delta}} x(\mathfrak{t}) - \overset{-4}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$-n\overset{-3}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$-\frac{n^{(2)}}{2!}\overset{-2}{\underset{(\mathfrak{q},h)}{\Delta}} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$= \sum_{p=2}^{n-1}\frac{p^{(2)}}{2!} x\left(\frac{\mathfrak{t}-h\sum\limits_{j=0}^{p+1}\mathfrak{q}^j}{\mathfrak{q}^{p+2}}\right) \quad (35)$$

Proceeding like this, we get the general form of m^{th} inverse

$$\underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x(\mathfrak{t}) - \underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$-\sum_{k=1}^{m-1}\frac{n^{(k)}}{k!}\underset{(\mathfrak{q},h)}{\overset{-(m-k)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$= \sum_{r=m-1}^{n-1}\frac{r^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right),$$

which completes the proof.

Corollary V.2. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $\mathfrak{q} \in \mathbb{R} - \{0\}$, $t \in \mathbb{R}$ and $\underset{(\mathfrak{q},0)}{\overset{-m}{\Delta}} = I_{\mathfrak{q}}^m$. If $h = 0$, then equation (29) becomes*

$$\underset{(\mathfrak{q},0)}{\overset{-m}{\Delta}} x(\mathfrak{t}) - \sum_{r=0}^{m-1}\frac{n^{(d)}}{d!}\underset{(\mathfrak{q},0)}{\overset{-(m-d)}{\Delta}} x\big(t/\mathfrak{q}^n\big)$$

$$= \sum_{r=m-1}^{n-1}\frac{r^{(m-1)}}{(m-1)!} x\big(t/\mathfrak{q}^{r+1}\big). \tag{36}$$

Corollary V.3. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $\mathfrak{t} \in \mathbb{R}$, $h \in \mathbb{R} - \{0\}$, $m, n \in \mathbb{N}$ and $\underset{(1,h)}{\overset{-m}{\Delta}} = \Delta_h^m$. If $\mathfrak{q} = 1$, then equation (29) simplifies to*

$$\underset{(1,h)}{\overset{-m}{\Delta}} x(\mathfrak{t}) - \sum_{d=0}^{m-1}\frac{n^{(d)}}{d!}\underset{(1,h)}{\overset{-(m-d)}{\Delta}} x(\mathfrak{t} - nh)$$

$$= \sum_{r=m-1}^{n-1}\frac{r^{(m-1)}}{(m-1)!} x(\mathfrak{t} - (r+1)h). \tag{37}$$

Corollary V.4. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $\mathfrak{t}, h \neq 0 \in \mathbb{R}$, $\mathfrak{q} \in \mathbb{R} - \{0, 1\}$, $n, m \in \mathbb{N}$ and $n > m$. Then the m^{th} order anti-difference principle related to mixed difference equation for finite series is given by*

$$\underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x(\mathfrak{t}) - \sum_{d=n-m}^{n-1}\frac{n^{(d-n+m)}}{(d-n+m)!}\underset{(\mathfrak{q},h)}{\overset{-(n-d)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$= \sum_{r=0}^{n-m}\frac{(m+r-1)^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{m+r-1}\mathfrak{q}^s}{\mathfrak{q}^{m+r}}\right). \tag{38}$$

Proof: *The proof follows by replacing*

$$\sum_{d=0}^{x-1}\frac{n^{(d)}}{d!}\underset{(\mathfrak{q},h)}{\overset{-(x-d)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right) \text{ with}$$

$$\sum_{d=n-m}^{n-1}\frac{n^{(d-n+m)}}{(d-n+m)!}\underset{(\mathfrak{q},h)}{\overset{-(n-d)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$

$$\text{and} \quad \sum_{r=m-1}^{n-1}\frac{r^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{r}\mathfrak{q}^s}{\mathfrak{q}^{r+1}}\right) \text{ with}$$

$$\sum_{r=0}^{n-m}\frac{(m+r-1)^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{m+r-1}\mathfrak{q}^s}{\mathfrak{q}^{m+r}}\right)$$

in equation (29).

Theorem V.5. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $\mathfrak{q}, h \in \mathbb{R} - \{0\}$ and $\underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x(0) = 0$. Then the m^{th} order anti-difference principle of the mixed operator for infinite series is given by*

$$\underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x(\mathfrak{t}) = \sum_{r=0}^{\infty}\frac{(m+r-1)^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{m+r-1}\mathfrak{q}^s}{\mathfrak{q}^{m+r}}\right), \tag{39}$$

where $m \in \mathbb{N}$ and $t \in \mathbb{R}$.

Proof: *The proof follows by taking $\lim_{n\to\infty}$ in equation* (38).

VI. Geometric Factorials in Mixed Difference Operator

In this section, we develop higher-order difference equations using the mixed difference operator and its inverse operator by incorporating factorial coefficient functions.

Definition VI.1. Let $m \in \mathbb{N}$, $s, \mathfrak{q}, h, t \in \mathbb{R}$ such that $s - h\sum_{j=0}^{\mathfrak{t}}\mathfrak{q}^j/\mathfrak{q}^{t+m} \in \mathscr{M}_h^{\mathfrak{q}}$ and let $x : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$ be a given function. Then the factorial coefficient of x at t on $(m, s, \mathfrak{q}/h)$ is defined as

$$x_{\mathfrak{t}}(m, s, \mathfrak{q}/h) = \frac{(t+m-1)^{(m-1)}}{(m-1)!} x\left(\frac{s - h\sum_{j=0}^{\mathfrak{t}}\mathfrak{q}^j}{\mathfrak{q}^{t+m}}\right). \tag{40}$$

Theorem VI.2. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $\mathfrak{q}, h \in \mathbb{R} - \{0, 1\}$, $t \in \mathbb{N}$, $s \in \mathbb{R}$ and if the series $\sum_{r=t+1}^{\infty} x(s - h\sum_{j=0}^{r}\mathfrak{q}^j)/\mathfrak{q}^{r+1}$ is convergent and $x_{\mathfrak{t}}(m, s, \mathfrak{q}/h)$ is given by* (40)*, then*

$$\underset{(\mathfrak{q},h)}{\overset{-1}{\Delta}} x(s) - \sum_{r=t+1}^{\infty} x_r(1, s, \mathfrak{q}/h) = \sum_{r=0}^{\mathfrak{t}} x_r(1, s, \mathfrak{q}/h). \tag{41}$$

Proof: *Taking $\lim_{n\to\infty}$ in equation* (15) *and assuming $\underset{(\mathfrak{q},h)}{\overset{-1}{\Delta}} x(0) = 0$, we get*

$$\underset{(\mathfrak{q},h)}{\overset{-1}{\Delta}} x(\mathfrak{t}) = x\left(\frac{\mathfrak{t}-h}{\mathfrak{q}}\right) + x\left(\frac{\mathfrak{t}-h\sum_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right) + \dots$$

$$+ x\left(\frac{\mathfrak{t} - h\sum_{p=0}^{r}\mathfrak{q}^p}{\mathfrak{q}^{r+1}}\right) + x\left(\frac{\mathfrak{t} - h\sum_{p=0}^{r+1}\mathfrak{q}^p}{\mathfrak{q}^{r+2}}\right) + \dots \tag{42}$$

Replacing t by s and r by t in (42)*, we obtain*

$$\underset{(\mathfrak{q},h)}{\overset{-1}{\Delta}} x(s) = x\left(\frac{s-h}{\mathfrak{q}}\right) + x\left(\frac{s-h\sum_{r=0}^{1}\mathfrak{q}^r}{\mathfrak{q}^2}\right) + \dots + x\left(\frac{s-h\sum_{r=0}^{\mathfrak{t}}\mathfrak{q}^r}{\mathfrak{q}^{t+1}}\right) + x\left(\frac{s-h\sum_{r=0}^{t+1}\mathfrak{q}^r}{\mathfrak{q}^{t+2}}\right) + \dots.$$

Rewriting in terms of summation, we obtain

$$\underset{(\mathfrak{q},h)}{\overset{-1}{\Delta}} x(s) = \sum_{r=0}^{\mathfrak{t}} x\left(\frac{s-h\sum_{j=0}^{r}\mathfrak{q}^j}{\mathfrak{q}^{r+1}}\right) + \sum_{r=t+1}^{\infty} x\left(\frac{s-h\sum_{j=0}^{r}\mathfrak{q}^j}{\mathfrak{q}^{r+1}}\right).$$

Finally, applying equation (40) *in* (42) *for $m=1$, the proof is completed.*

Lemma VI.3. *Let $s\in\mathbb{R}, \mathfrak{t}\in\mathbb{N}$, $h,\mathfrak{q}\in\mathbb{R}-\{0\}$ and assume that the series $\sum_{r=t+1}^{\infty} x(s-h\sum_{j=0}^{r}\mathfrak{q}^j)/\mathfrak{q}^{r+1}$ is convergent. Then, we have*

$$\sum_{r=\mathfrak{t}+1}^{\infty} x_r(m,s,\mathfrak{q}/h) = \frac{\left[x_{\mathfrak{t}+1}(m,s,\mathfrak{q}/h)\right]^2}{x_{\mathfrak{t}+1}(m,s,\mathfrak{q}/h) - x_{\mathfrak{t}+2}(m,s,\mathfrak{q}/h)}. \tag{43}$$

Proof: *By the property of the sum of a geometric series, we obtain* (43)*.*

The following theorem provides a higher-order finite series formula for the $(\mathfrak{q},h)$ difference operator.

Theorem VI.4. *Consider the conditions given in Theorem VI.2. Then the first-order anti-difference principle of the $(\mathfrak{q},h)$ difference operator is given by*

$$\underset{(\mathfrak{q},h)}{\overset{-1}{\Delta}} x(s) - \frac{[x_{\mathfrak{t}+1}(1,s,\mathfrak{q}/h)]^2}{x_{\mathfrak{t}+1}(1,s,\mathfrak{q}/h) - x_{\mathfrak{t}+2}(1,s,\mathfrak{q}/h)} = \sum_{r=0}^{\mathfrak{t}} x_r(1,s,\mathfrak{q}/h) \tag{44}$$

Proof: *The proof follows by substituting* (43) *into* (41) *for $m=1$.*

Theorem VI.5. *Let $x,y:\mathscr{M}_h^{\mathfrak{q}}\to\mathbb{R}$, $s\in\mathbb{R}$, $m,t\in\mathbb{N}$, $h>0$ and $\mathfrak{q}>1$. Then, the higher-order $(\mathfrak{q},h)$ difference operator is given by*

$$\underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x(s) - \frac{[x_{t+1}(m,s,\mathfrak{q}/h)]^2}{x_{t+1}(m,s,\mathfrak{q}/h) - x_{t+2}(m,s,\mathfrak{q}/h)} = \sum_{r=0}^{\mathfrak{t}} x_r(m,s,\mathfrak{q}/h).$$

Proof: *From equation* (39)*, we have*

$$\underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x(\mathfrak{t}) = \frac{(m-1)^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t}-h\sum_{j=0}^{m-1}\mathfrak{q}^j}{\mathfrak{q}^m}\right) + \frac{m^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t}-h\sum_{j=0}^{m}\mathfrak{q}^j}{\mathfrak{q}^{m+1}}\right) + \dots + \frac{(m-(r-1))^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t}-h\sum_{j=0}^{m-(r-1)}\mathfrak{q}^j}{\mathfrak{q}^{m+r}}\right) + \frac{(m-r)^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t}-h\sum_{j=0}^{m-r}\mathfrak{q}^j}{\mathfrak{q}^{m+r+1}}\right) + \dots$$

Replacing t,r by s,t in the above equation gives

$$\underset{(\mathfrak{q},h)}{\overset{-m}{\Delta}} x(s) = \sum_{y=m-1}^{m+t-1} \frac{y^{(m-1)}}{(m-1)!} x\left(\frac{s-h\sum_{j=0}^{y}\mathfrak{q}^j}{\mathfrak{q}^{y+1}}\right) + \sum_{y=m+t}^{\infty} \frac{y^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t}-h\sum_{j=0}^{y}\mathfrak{q}^j}{\mathfrak{q}^{y+1}}\right).$$

Interchanging the terms

$$\sum_{y=m-1}^{m+t-1} \frac{y^{(m-1)}}{(m-1)!} x\left(s-h\sum_{j=0}^{y}\mathfrak{q}^j\right) \quad \text{by}$$

$$\sum_{r=0}^{\mathfrak{t}} \frac{(m+r-1)^{(m-1)}}{(m-1)!} x\left(\frac{s-h\sum_{j=0}^{m+r-1}\mathfrak{q}^j}{\mathfrak{q}^{m+r}}\right)$$

$$\text{and} \sum_{y=m+t}^{\infty} \frac{y^{(m-1)}}{(m-1)!} x\left(\frac{\mathfrak{t}-h\sum_{j=0}^{y}\mathfrak{q}^j}{\mathfrak{q}^{y+1}}\right) \quad \text{by}$$

$$\sum_{r=\mathfrak{t}+1}^{\infty} \frac{(m+r-1)^{(m-1)}}{(m-1)!} x\left(\frac{s-h\sum_{j=0}^{m+r-1}\mathfrak{q}^j}{\mathfrak{q}^{m+r}}\right).$$

and then applying equation (43) *completes the proof.*

VII. Mixed Gamma Geometric Factorials in Fractional Order Difference Equations

In this section, we develop the fractional-order anti-difference principle from its integer-order equation given in Lemma VI.3, by which we derive fundamental theorems of mixed fractional difference equations involving mixed gamma geometric factorials.

Definition VII.1. Let $s,\mathfrak{q},t,\nu\in\mathbb{R}$ such that $(s-h\sum_{j=0}^{\mathfrak{t}}\mathfrak{q}^j)/\mathfrak{q}^{\mathfrak{t}+\nu}\in\mathscr{M}_h^{\mathfrak{q}}$ and let $x:\mathscr{M}_h^{\mathfrak{q}}\to\mathbb{R}$ be a function. Then, the gamma factorial-coefficient of x at t on $(\nu,s,\mathfrak{q}/h)$ is defined as

$$x_{\mathfrak{t}}(\nu,s,\mathfrak{q}/h) = \frac{\Gamma(\mathfrak{t}+\nu)}{\Gamma(\mathfrak{t}+1)\Gamma(\nu)} x\frac{\left(s-h\sum_{j=0}^{\mathfrak{t}}\mathfrak{q}^j\right)}{\mathfrak{q}^{\mathfrak{t}+\nu}}. \tag{45}$$

Definition VII.2. Let $s, \mathfrak{q}, \mathfrak{t}, \nu \in \mathbb{R}$ such that $(s - h \sum_{j=0}^{\mathfrak{t}} \mathfrak{q}^j)$ $\mathfrak{q}^{t+\nu} \in \mathscr{M}_h^{\mathfrak{q}}$ and let the function $x_{\mathfrak{t}}(\nu, s, \mathfrak{q}/h)$ be given in equation (45). Then the $\mathfrak{q}/h$-Geometric factorial function is defined as

$$\sum_{r=\mathfrak{t}+1}^{\infty} x_r(\nu, s, \mathfrak{q}/h) = \frac{[x_{\mathfrak{t}+1}(\nu, s, \mathfrak{q}/h)]^2}{x_{\mathfrak{t}+1}(\nu, s, \mathfrak{q}/h) - x_{\mathfrak{t}+2}(\nu, s, \mathfrak{q}/h)}. \tag{46}$$

Theorem VII.3. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $\mathfrak{q}, h \in \mathbb{R} - \{0, 1\}$, $\mathfrak{t}, \nu, n \in \mathbb{R}$ and $n - \nu \in \mathbb{N}$. Then, the ν^{th} order anti-difference principle of the $(\mathfrak{q}, h)$ difference equation is given by*

$$\underset{(\mathfrak{q},h)}{\overset{-\nu}{\Delta}} x(\mathfrak{t}) - \sum_{r=n-\nu}^{n-1} \frac{\Gamma(n+1)}{\Gamma(2n-r-\nu+1)\Gamma(r-n+\nu-1)} \times \underset{(\mathfrak{q},h)}{\overset{-(n-r)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$
$$= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{n-\nu} \frac{x\Gamma(\nu+r)}{\Gamma(r+1)} \left(\frac{\mathfrak{t} - h\sum_{s=0}^{\nu+r-1}\mathfrak{q}^s}{\mathfrak{q}^{\nu+r}}\right). \tag{47}$$

Proof: *By generalizing the integer order to real order $(m > 0 \in \mathbb{R} = \nu)$ in equation (38), we obtain*

$$\underset{(\mathfrak{q},h)}{\overset{-\nu}{\Delta}} x(\mathfrak{t}) - \sum_{r=n-\nu}^{n-1} \frac{n^{(r-n+\nu)}}{(r-n+\nu)!} \underset{(\mathfrak{q},h)}{\overset{-(n-r)}{\Delta}} x\left(\frac{\mathfrak{t} - h\sum_{j=0}^{n-1}\mathfrak{q}^j}{\mathfrak{q}^n}\right)$$
$$= \sum_{r=0}^{n-\nu} \frac{(\nu+r-1)^{(\nu-1)}}{(\nu-1)!} x\left(\frac{\mathfrak{t} - h\sum_{s=0}^{\nu+r-1}\mathfrak{q}^s}{\mathfrak{q}^{\nu+r}}\right). \tag{48}$$

Now from (2), *we obtain*

$$n^{(r-n+\nu)} = \frac{\Gamma(n+1)}{\Gamma(2n-r-\nu+1)} \quad \textit{and} \tag{49}$$

$$(\nu+r-1)^{(\nu-1)} = \frac{\Gamma(\nu+r)}{\Gamma(r+1)}. \tag{50}$$

Hence the proof is completed by applying (49) *&* (50) *in* (48).

Theorem VII.4. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $h \in \mathbb{R} - \{0\}$, $t \in \mathbb{N}$, $\mathfrak{q} \in \mathbb{R} - \{0, 1\}$ and $s, \nu \in \mathbb{R}$. Then the ν^{th} order (fractional or real order) anti-difference principle of the $(\mathfrak{q}, h)$ difference equation in terms of $f_{\mathfrak{t}}(\nu, s, \mathfrak{q}/h)$ is given by*

$$\underset{(\mathfrak{q},h)}{\overset{-\nu}{\Delta}} x(s) - \frac{[x_{t+1}(\nu, s, \mathfrak{q}/h)]^2}{x_{t+1}(\nu, s, \mathfrak{q}/h) - x_{t+2}(\nu, s, \mathfrak{q}/h)} = f_{\mathfrak{t}}(\nu, s, \mathfrak{q}/h). \tag{51}$$

Proof: *From equation* (2), *we have*

$$(t+\nu)^{(\nu-1)} = \frac{\Gamma(t+\nu+1)}{\Gamma(t+2)} \tag{52}$$

and

$$(t+\nu+1)^{(\nu-1)} = \frac{\Gamma(t+\nu+2)}{\Gamma(t+3)} \tag{53}$$

To generalize the integer order of equation in Theorem VI.5 to real $\nu > 0$, we proceed as follows
1. ***Gamma Function Properties***
The Gamma function has the property $\Gamma(z+1) = z\Gamma(z)$ This can be useful when deriving relations for integer arguments.
2. ***Transformation of Equations***
By utilizing properties of the Gamma function and the shifts in ν, we can equate both expressions derived from (52) *and* (53). *This leads us to focus on the limit forms of fractional orders as they converge into integer sequences.*
3. ***Fractional Order Derivation*** *Each term of the ν^{th} order can be expanded and simplified leading us directly into the implications of the difference equation. Given the forms derived from the transformation, focusing on powers and ratios within the terms leads to the result:*

$$\underset{(\mathfrak{q},h)}{\overset{-\nu}{\Delta}} x(s) = f_{\mathfrak{t}}(\nu, s, \mathfrak{q}/h) + \frac{[x_{t+1}(\nu, s, \mathfrak{q}/h)]^2}{x_{t+1}(\nu, s, \mathfrak{q}/h) - x_{t+2}(\nu, s, \mathfrak{q}/h)} \tag{54}$$

In conclusion, after transforming the appropriate Gamma function identities and applying fractional calculus techniques, we arrive at the ν^{th} anti-difference principle, completing the proof.

Result VII.5. For finding the fractional difference equation using mixed difference operator for an infinite series, we should analyze the behavior of the series $\sum_{j=0}^{s} \mathfrak{q}^j$.

1) If $\mathfrak{q} \in \mathbb{R}$ and s is odd, then

$$\left(1+\mathfrak{q}^2+\mathfrak{q}^4+\cdots+\mathfrak{q}^s\right)(1+\mathfrak{q}) = \sum_{j=0}^{s/2} \mathfrak{q}^{2j}(1+\mathfrak{q}). \tag{55}$$

2) If $\mathfrak{q} \in \mathbb{R}$ and s is even, then

$$\left(1+\mathfrak{q}^2+\cdots+\mathfrak{q}^s\right)(1+\mathfrak{q}) = \sum_{j=0}^{(s-2)/2} \mathfrak{q}^{2j}(1+\mathfrak{q})+\mathfrak{q}^s. \tag{56}$$

Theorem VII.6. *Let $x, y : \mathscr{M}_h^{\mathfrak{q}} \to \mathbb{R}$, $\nu, t \in \mathbb{R}$, $\mathfrak{q} > 1$, $h > 0$, $(\nu + r - 1)/2 \in \mathbb{N}$, and $(\nu + r - 3)/2 \in \mathbb{N}$. Then the ν-th order of the $(\mathfrak{q}, h)$ difference operator for an infinite series is given by*

$$\underset{(\mathfrak{q},h)}{\overset{-\nu}{\Delta}} x(\mathfrak{t}) = \sum_{r=0}^{\infty} h_\nu(t+\nu-1) \;\; x\left(\frac{\mathfrak{t} - hS_r}{\mathfrak{q}^{\nu+r}}\right), \tag{57}$$

where $S_r = \sum_{s=0}^{(\nu+r-1)/2} (1+\mathfrak{q})\mathfrak{q}^{2r}$.
Also, it can be expressed as

$$\underset{(\mathfrak{q},h)}{\overset{-\nu}{\Delta}} x(\mathfrak{t}) = \sum_{r=0}^{\infty} h_\nu(t+\nu-1) \;\; x\left(\frac{\mathfrak{t} - hS_r + \mathfrak{q}^{\nu+r-1}}{\mathfrak{q}^{\nu+r}}\right), \tag{58}$$

where $S_r = \sum_{s=0}^{(\nu+r-3)/2} \mathfrak{q}^{2r}(1+\mathfrak{q})$.

Proof: *To show that the relations hold, we begin by recalling the properties of the mixed difference operator and its inverse.*

1. Understanding the components: Each term in the summation represents the application of the mixed operator on a shifted argument of x. By using the definition of S_r, we can interpret its role as adjusting the exponent based on the order of the difference operator and its fractional nature determined by ν.

2. Convergence: The sums S_r converge due to the conditions placed on q and the fact that $(\nu + r - 1)/2$ and $(\nu + r - 3)/2$ are integers. Thus, for each r, the component $x\left(\frac{\mathsf{t}-hS_r}{\mathsf{q}^{\nu+r}}\right)$ is well-defined and contributes to the summation.
3. Substituting with manipulations
By substituting the definitions of S_r into the equations and manipulating the sums appropriately, we can derive that

$$\underset{(\mathsf{q},h)}{\overset{-\nu}{\Delta}} x(\mathsf{t}) = \sum_{r=0}^{\infty} h_\nu(t+\nu-1) x\left(\mathsf{t} - h \sum_{s=0}^{(\nu+r-1)/2} (1+\mathsf{q})\mathsf{q}^{2r}\right). \tag{59}$$

4. Equality establishment: By generalizing the operator as stated in Theorem V.5, we demonstrate that both expressions in equations (1) and (2) yield the same infinite series representation of the ν-th order anti-difference principle, thus completing the proof. Thus, the formulation is proven, culminating in the established results of the theorem.

The following example examines the behavior of the ν-th order anti-difference principle, as expressed in Theorem VII.4.

Example VII.7. *By fixing the values $s = 8.3$ and $t = 50$, Figure 1 shows that for any $\nu > 0 \in \mathbb{R}$, the values of the (q, h) difference equation decrease over time, indicating convergence.*

Figure 1 presents the general solution for Theorem VII.4 for any real q and h values. As a result, we can predict the stability behavior of (q, h) difference operators.

Hence in this section, we utilized factorial-coefficient and gamma geometric factorial methods to develop integer and fractional order theorems for the mixed difference operator. Finally, we analyzed the value stability of the mixed difference operator.

VIII. Application of the Mixed Difference Operator to the SIR Epidemic Model

A. Classical SIR Model

The classical SIR epidemic model is given by the system of differential equations

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I, \quad \frac{dR}{dt} = \gamma I$$

We now replace the derivatives with the mixed difference operator $\Delta_{(q,h)}$, leading to the following discrete-time equations

$$-\underset{(q,h)}{\Delta} S(t) = -\beta S(t)I(t)$$
$$-\underset{(q,h)}{\Delta} I(t) = \beta S(t)I(t) - \gamma I(t)$$
$$-\underset{(q,h)}{\Delta} R(t) = \gamma I(t)$$

B. Higher-Order Terms in the Mixed Difference Operator

By using Theorem V.1, we generalize the difference operator as

$$-m \underset{(q,h)}{\Delta} x(t) = \sum_{r=m-1}^{n-1} \frac{r^{(m-1)}}{(m-1)!} x\left(t - h\sum_{s=0}^{r} q^s\right)$$

This leads to a more complex model where higher-order interactions between the compartments are captured.

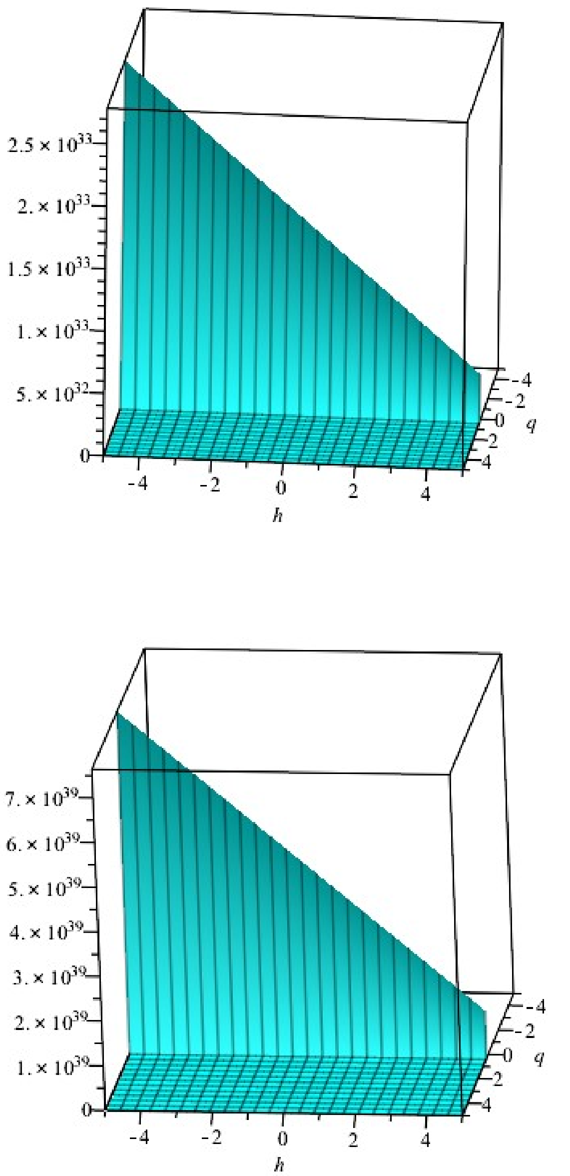

Fig. 1. Solution for Theorem VII.4 with ν values 0.3, 1.4, 2.8, and 3.8, where q and h vary as $-4, -2, 0, 2, 4$.

C. Time-Dependent Parameters and Interventions

We introduce time-dependent parameters for the infection and recovery rates

$$\beta(t) = \beta_0\left(1 - \frac{t}{T}\right)$$
$$\gamma(t) = \gamma_0 + \delta \sin\left(\frac{2\pi t}{T}\right)$$

The equations become $-\underset{(q,h)}{\Delta} S(t) = -\beta(t)S(t)I(t)$

$$-\underset{(q,h)}{\Delta} I(t) = \beta(t)S(t)I(t) - \gamma(t)I(t)$$

and $-\underset{(q,h)}{\Delta} R(t) = \gamma(t)I(t)$

D. Numerical Example

Consider an epidemic with $S(0) = 1000$, $I(0) = 10$, and $R(0) = 0$, $\beta = 0.3$, $\gamma = 0.1$ and $h = 0.1$. The population

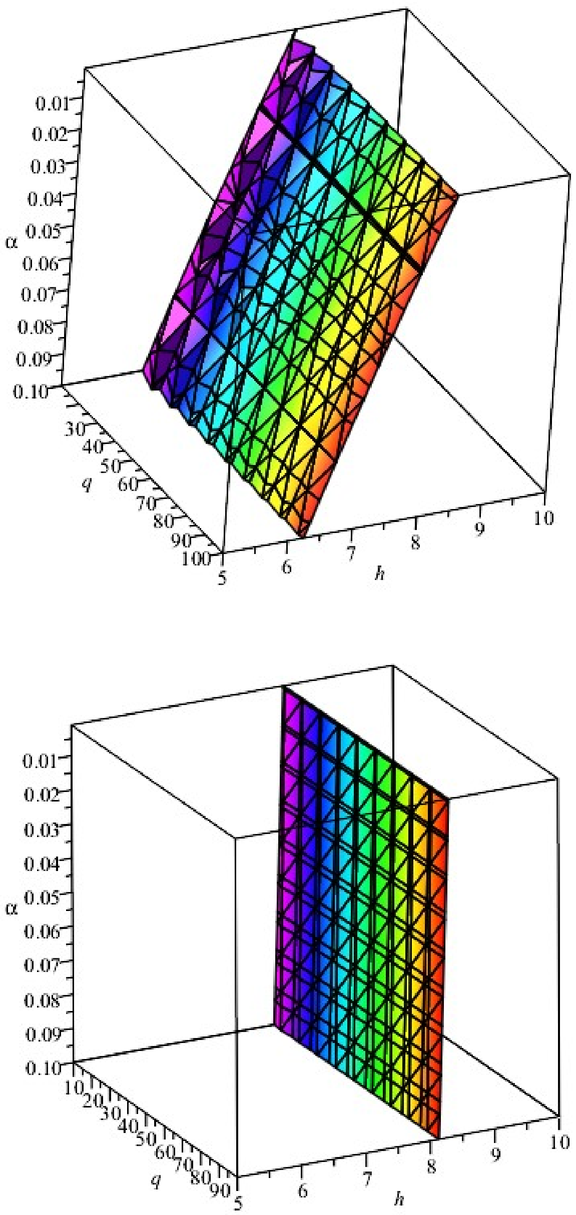

Fig. 2. Additional solution plots for different (q, h) values.

dynamics can be simulated iteratively as

$$S(t+1) = S(t) - h\beta S(t)I(t)$$

$$I(t+1) = I(t) + h(\beta S(t)I(t) - \gamma I(t))$$

$$R(t+1) = R(t) + h\gamma I(t).$$

The results can be visualized through numerical plots for $S(t)$, $I(t)$ and $R(t)$.

E. Incorporating Higher-Order Anti-Difference Principles

We apply the first-order anti-difference principle from Theorem IV.1 to extend the SIR epidemic model

$$\underset{(\mathsf{q},h)}{\overset{-1}{\Delta}} S(t) = \sum_{r=0}^{n-1} S\left(\frac{t - h\sum_{s=0}^{r} \mathsf{q}^s}{\mathsf{q}^{r+1}}\right)$$

This principle introduces more complexity by capturing delayed interactions, where the state of the disease at time t depends on past values of $S(t)$, $I(t)$ and $R(t)$.

F. Modeling Delayed Effects in Disease Spread

The parameter q controls the amount of memory in the epidemic, and adjusting this value allows us to model different scenarios with short or long-term memory effects in the disease dynamics.

G. Higher-Order Anti-Difference Principle

We apply the higher-order anti-difference principle from Theorem V.1 to model interactions over multiple time steps

$$\underset{(\mathsf{q},h)}{\overset{-m}{\Delta}} S(t) = \sum_{r=0}^{\infty} \frac{(m+r-1)^{(m-1)}}{(m-1)!} S\left(\frac{t - h\sum_{s=0}^{m+r-1} \mathsf{q}^s}{\mathsf{q}^{m+r}}\right)$$

. This accounts for higher-order effects in the spread of the epidemic.

H. Time-Dependent Parameters and Interventions

We introduce time-dependent parameters $\beta(t)$ and $\gamma(t)$ where $\beta(t) = \beta_0\left(1 - \frac{t}{T}\right)$ and $\gamma(t) = \gamma_0 + \delta \sin\left(\frac{2\pi t}{T}\right)$. These parameters allow us to simulate the effects of changing intervention strategies.

I. Numerical Simulation

Given initial conditions $S(0) = 1000$, $I(0) = 10$, $R(0) = 0$ and the parameters $\beta = 0.3$, $\gamma = 0.1$, and $h = 0.1$. Then we simulate the epidemic dynamics iteratively as

$$S(t+1) = S(t) - h \cdot \beta(t)S(t)I(t)$$

$$I(t+1) = I(t) + h \cdot (\beta(t)S(t)I(t) - \gamma(t)I(t))$$

$$R(t+1) = R(t) + h \cdot \gamma(t)I(t).$$

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IX. Results and Discussion

A. Key Findings

Our research has led to several critical discoveries regarding the application of mixed difference operators in the framework of anti-difference principles, particularly

- **Higher-Order Anti-Difference Principles:** We successfully derived new formulations for higher-order and fractional anti-difference principles using mixed difference operators. These principles extend existing theories in discrete mathematics. Notably, we explored their applications in various systems such as epidemic modeling and population dynamics.
- **Mixed Gamma Geometric Factorials:** The integration of mixed gamma geometric factorials allowed us to formulate key theorems relevant to the analysis of mixed fractional difference equations, opening up avenues for advanced discrete models with more memory and delayed responses.
- **Application to Epidemic Models:** We demonstrated the application of these advanced principles to refine the SIR epidemic model, capturing more complex interactions, memory effects, and varying

time-dependent parameters for infection and recovery rates. This approach allows for a more granular understanding of disease dynamics under varying intervention strategies.

B. Presentation of Results

The results obtained from our study are presented through various formats that enhance comprehension

- **Equations and Theorems:** Significant findings are documented in mathematical theorems and equations. For example, Theorem V.5 provides a detailed formulation related to mixed difference operators:

$$\underset{q}{\overset{m}{\Delta}} f(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k f(x + kq),$$

  where $\underset{q}{\Delta}$ denotes the q-difference operator. This equation allows us to capture higher-order behavior in dynamic systems.
- **Figures and Graphs:** The behavior of the ν-th order anti-difference principle and its applications are illustrated through figures and examples (e.g., Figures I and II), which represent distinct behavior across defined intervals. These visualizations provide a concrete understanding of how the mixed difference operator influences system evolution over time.

C. Discussion

1) Implications of the Results: The implications from our findings highlight essential contributions to our research question

- **Theoretical Advancements:** The extension of the anti-difference principle invites new methodologies for evaluating discrete dynamic systems, affirming our original hypothesis surrounding the potential applications of mixed operators. By applying higher-order principles, we have shown that discrete models can better capture real-world phenomena that involve delays and memory.
- **Broader Applications:** The results significantly bolster the theoretical foundation established within previous studies, indicating practical relevance in various mathematical modeling scenarios, including population dynamics, epidemic modeling, and financial systems. Our framework could be used to enhance predictive models in epidemiology, considering the complexities of real-world interventions and immunity effects.

2) Alignment with Existing Literature:

- Our findings show a strong alignment with previous research efforts focused on mixed difference equations. Specifically, our advancements support extensive literature discussing polynomial relationships in discrete calculus, validating many of the theoretical assertions made in earlier studies. However, our work extends these relationships to handle non-linear dynamics, such as epidemic models where feedback loops and memory effects play a significant role.

3) Theoretical and Practical Implications:

- **Real-World Applications** The frameworks developed may be adapted for use in numerical simulations, predictive modeling, and other quantitative analyses, emphasizing their applicability in fields that depend on discrete processes. The SIR model application serves as a key example where these principles provide more accurate predictions than traditional models by incorporating memory and delayed responses.

4) Results on q-difference equations and mixed operators:

- **Consistency** Many correlations identified are consistent with findings from prior studies, reinforcing established knowledge. However, our derived relationships extend beyond traditional applications by introducing the concept of fractional anti-differences and memory effects.
- **Differences** Certain divergences, especially observed in extreme behaviors of anti-differences for large inputs, merit further exploration. These discrepancies may arise from novel methodologies employed in our research, such as the introduction of time-varying parameters in the SIR epidemic model.

5) Limitations of the Study:

- **Scope and Sample Constraints:** The theoretical constructs may not have been exhaustively tested across all variable states. Future investigations should strive for a broader sample size to enhance generalizability, particularly in real-world epidemic simulations.
- **Measurement Variability:** Potential inaccuracies in measuring certain polynomial outputs could affect result consistency. Addressing these through refined methods would be beneficial in future research avenues. Additionally, the sensitivity of mixed difference models to initial conditions warrants further exploration.

6) Suggested Future Research: Building upon our findings, future research directions could include:

- **Exploring New Applications** Further investigations could apply mixed difference operators in non-linear or multi-variable contexts, enhancing the empirical foundation of discrete calculus approaches. Potential applications in climate modeling, disease forecasting, and complex systems could provide valuable insights.
- **Methodological Innovations:** Enhancements in analytical methodologies are encouraged, potentially allowing for the incorporation of advanced statistical measures that could yield more robust validations of the anti-difference principles established in this study. A deeper exploration of numerical stability and error propagation in higher-order difference models could also be pursued.

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