Adjacent Vertex Reducible Edge Coloring of Splitting Graphs

N. Yathavan, N. Paramaguru

***Abstract*—In graph theory, a graph $G = (V, E)$ is labeled as having an adjacent vertex reducible edge coloring if there exists a mapping $f : E(G) \to 1, 2, 3, 4 \ldots K$, such that adjacent vertices u and v with the same degree generate sets $s(u)$ and $s(v)$ respectively, where $S(u) = \{f(uv)/uv \in E(g)\}$ and $S(u) = S(v)$. This study focuses on calculating the chromatic number for adjacent vertex reducible edge coloring of splitting graphs.**

***Index Terms*—Edge coloring, Chromatic number, Avrec, Splitting graphs.**

I. Introduction

Graph theory, a field rich with diverse structures and applications, continues to unveil intricate relationships and solve complex problems across disciplines [1]. One of the fascinating areas within graph theory is the study of adjacent vertex reducible edge coloring, particularly in the context of split graphs. Split graphs, for each vertex of v take a new vertex $v^{'}$. Joining $v^{'}$ to all vertices in G adjacent to v provide a compelling backdrop for exploring coloring techniques [2]. The notion of reducible edge coloring for neighboring vertex extends traditional graph coloring by ensuring that vertices with adjacent connections share identical color sets, thereby introducing a nuanced approach to graph representation and analysis. The idea of reducible edge coloring for neighboring vertex was introduced by Zhong Fu Zong et al., building on the idea of distinguishable edge coloring [3], [4], [5], [6]. The key distinction between these two concepts lies in their approach to edge coloring in simple graphs: distinguishable edge coloring involves assigning distinct positive integer colors to edges incident to different vertices v_1 and v_2, whereas reducible edge coloring ensures that the colors assigned to the edges incident to vertices of the same degree are similar. When the color sets of two vertices are identical, the sum of the color sets are also the same, a topic explored in the sum reducible edge coloring introduced by Jing Wen Li et al. [7], [8]. Additionally, Distance of (2)-vertex reducible edge coloring, where the color sets of vertices with the same degree and within a distance of no more than 2 are same [9]. Further research has focused on adjacent vertex reducible edge total coloring algorithms applied to various joint graph, and this concept has been extended to labeling adjacent vertex reducible total labeling of corona graphs [10], [11]. In the complex world of transportation networks, understanding the interplay between edge and node capacities is crucial for optimizing efficiency and ensuring seamless connectivity. Edge weights in these networks symbolize the transportation capacity of routes, while the node's capacity is determined by the cumulative weight of its connecting edges. An essential challenge in network design is to balance these capacities, especially when dealing with nodes of similar adjacent degrees. The Adjacent Vertex Reducible Edge Coloring (AVREC) model provides a framework for addressing this challenge. By striving to equalize the transportation capacities of adjacent nodes, AVREC offers a structured approach to optimizing network performance and enhancing overall system robustness. This article delves into the principles of AVREC and explores its application in achieving balanced and efficient transportation network. Here, we introduce new theories on split graphs, namely, split graphs of path (P_m), cycle (C_n), complete graph (K_r), fan graph (F_n), tadpole graph $T_{(m,n)}$, and wheel graph (W_n) graph using well-known coloring principles. This paper is organized as follows in the ongoing sections: Definitions of AVREC and split graphs are included in the preliminary section. Theorems about the chromatic number of split graphs are then presented, along with the proofs for each. Finally, a comprehensive discussion of the conclusions drawn from our findings is provided.

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N. Yathavan is a PhD candidate in the Department of Mathematics, Annamalalai University, Chidambaram 608002, India (Corresponding author to provide e-mail: yathavan1508@gmail.com).

N. Paramaguru is an Assistant Professor in the Department of Mathematics, Government Arts College for Women, Krishnagiri 635002, India (e-mail: npguru@gmail.com).

II. Results and Discussion

Definition 1. [12] Let the graph $G(V, E)$ is a simple graph. If there is a positive integer $k(1 \leq k \leq E)$ and a mapping $f : E(G) \to 1, 2, 3, 4 \ldots k$. For any two vertices with distance 1 are $u, v \in V(G)$, when degree of u is equal to degree of v, then $S(u) = S(v)$. Here $S(u) = \cup \ uw \in E(G) \{\ f(uw)\}$. Then f is adjacent vertex reducible edge coloring, referred to as AVREC and $\chi^{'}(G) = \max\{k|k - \text{AVREC of } G\}$ is adjacent vertex reducible edge coloring chromatic number.

Definition 2. [13] To construct the splitting graph $S(G)$ of a graph G, follow these steps: For every vertex v in G, create a new vertex $v^{'}$. Connect $v^{'}$ to all vertices in G that are adjacent to v. The resulting graph $S(G)$ is known as the splitting graph of G.

Theorem 1. If the graph G is a splitting graph, then $\chi'_{\text{avrec}} S(P_m) = \chi'_{\text{avrec}} \overline{S(P_m)} + 1$.

Theorem 2. For splitting graph of P_m, the

$$\chi'_{\text{avrec}} S(P_m) = \begin{cases} 2, m = & 2 \\ 4, m = & 3 \\ 6, m \geq & 4 \end{cases}$$

Proof: Taking a vertex set of S(P_m) is $u_1, u_2, u_3, \ldots u_m$ and $u'_1, u'_2, u'_3, \ldots u'_m$ satisfying f coloring rule.

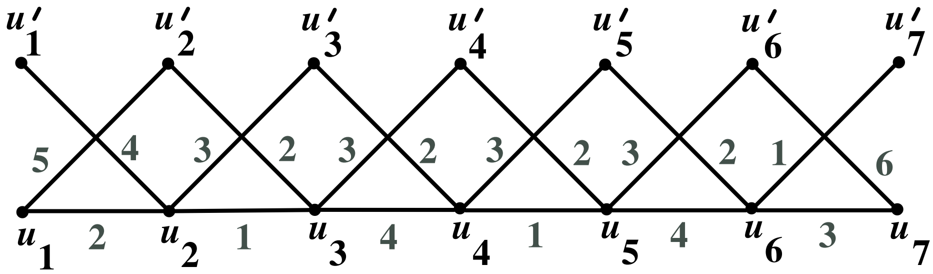

Fig. 1: $\chi'_{avrec}S(P_7) = 6$

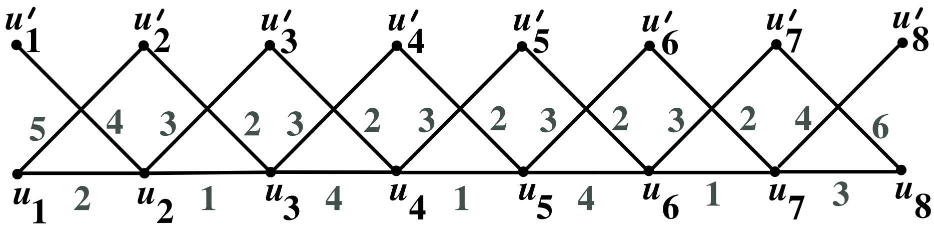

Fig. 2: $\chi'_{avrec}S(P_8) = 6$

For $m = 2$, two 2-degree vertices are adjacent and two 1-degree vertex are not adjacent. The color sets of 2-degree adjacent vertex are the same.

For $m = 3$, three 2-degree vertices are adjacent. This vertex color sets are the same.

For $m \geq 4$, We divide this proof into odd n and even n because at$(m-1)^{th}$ vertex set colors are not same for any value of n.

For even m:

$f(u_i u_{i+1}) = 1,\ i = 2, 4, 6, ..., m-2$

$$f\left(u_i u'_{i-1}\right) = 2,\ i = 3, 4, 5, \ldots, m-1$$

$$f\left(u_i u'_{i+1}\right) = 3,\ i = 2, 3, 4, \ldots, m-2$$

$$f(u_{n-1} u_m) = 3,\ n = 4, 6, 8, \ldots$$

$$f(u_i u_{i+1}) = 4,\ i = 3, 5, 7, ..., m-3 \text{ except } m \neq 4$$

$$f(u_{n-1} u'_m) = 4,\ m = 4, 6, 8, ...$$

For odd m:

$f(u_i u_{i+1}) = 1,\ i = 2, 4, 6, ..., m-3$

$$f(u_{m-1} u'_m) = 1,\ m = 5, 7, 9, \ldots$$

$$f\left(u_i u'_{i-1}\right) = 2,\ i = 3, 4, 5, \ldots, m-1$$

$$f\left(u_i u'_{i+1}\right) = 3,\ i = 2, 3, 4, \ldots, m-2$$

$$f(u_i u_{i+1}) = 4,\ i = 3, 5, 7, ..., m-2$$

The following are common for all m

$$f(u_1 u_2) = 2$$

$$f(u_2 u'_1) = 4$$

$$f(u_{m-1} u_m) = 3,\ m \geq 4$$

$$f(u_1 u'_2) = 5$$

$$f\left(u_n u'_{n-1}\right) = 6, m \geq 4$$

In general, we ensure that adjacent same degree vertex are assigned same color sets. The splitting graph of P_m from $u_2, u_3, \ldots u_{m-1}$ vertex degree is 4 and adjacent to each other. In context of AVREC, four colors 1,2,3,4 are enough to make all these vertex color sets to achieved similarly. Apart from 4-degree vertex there are 2-degree vertices$f(u_1 u'_2)$ and $f(u_m\ u'_{m-1})$ that are adjacent and the same degree vertex for these edges introduce 2 different new colors that is $f(u_1 u'_2)$ and $f\left(u_m u'_{m-1}\right)$ are 5 and 6 respectively. So the chromatic number will be maximum and also in graph 1-degree u'_1,u'_m and 2 degree vertices $u'_3, \ldots u'_{m-2}$ are not adjacent and colors are already assigned since these vertices are adjacent with 4-degree vertices. This type of graph can be colored using up to 6 different colors, following the basic rules of graph coloring. If the edge chromatic number is 7, adjacent vertices of the same degree will have distinct sets of colors, resulting in a discontinuous chromatic number. For example, $f(u_3 u_4) = 7 \Rightarrow\ u_3$ color sets = u_4 color sets, but u_3, u_4 color sets are not equal to other adjacent 4-degree vertices. Therefore, we cannot take more than six colors to make color sets equal, therefore $\chi'_{avrec}(P_m)$ = 6, $(m \geq 4)$. ■

Theorem 3. For the splitting graph of a cycle C_n (where $n \geq 3$), we have: $\chi'_{\text{avrec}}S(C_n) = 4$.

Proof: Let the vertex sets of $S(C_n)$ be $u_1, u_2, u_3, \ldots, u_n$ and $u'_1, u'_2, u'_3, \ldots, u'_n$. We can assign colors to the edges according to the f-coloring rule, depending on whether n is even or odd.

For even n:

$$f(u_i u'_{i+1}) = 1, \quad i = 1, 2, 3, \ldots, n-1$$

$$f(u_n u'_1) = 1, \quad n = 4, 6, 8, \ldots$$

$$f(u_i u_{i+1}) = 2, \quad i = 1, 3, 5, \ldots, n-1$$

$$f(u_i u_{i+1}) = 3, \quad i = 2, 4, 6, \ldots, n-2$$

$$f(u_n u_1) = 3, \quad n = 4, 6, 8, \ldots$$

$$f(u_i u'_{i-1}) = 4, \quad i = 2, 3, 4, \ldots, n$$

$$f(u_1 u'_n) = 4, \quad n = 4, 6, 8, \ldots$$

For odd n:

$$f(u_iu'_{i+1}) = 1, \quad i = 1, 2, 3, \ldots, n-1$$

$$f(u_nu'_1) = 1, \quad n = 3, 5, 7, \ldots$$

$$f(u_iu_{i+1}) = 2, \quad i = 1, 3, 5, \ldots, n-2$$

$$f(u_nu'_{n-1}) = 2, \quad n = 3, 5, 7, \ldots$$

$$f(u_iu_{i+1}) = 3, \quad i = 2, 4, 6, \ldots, n-1$$

$$f(u_1u'_n) = 3, \quad n = 3, 5, 7, \ldots$$

$$f(u_iu'_{i-1}) = 4, \quad i = 2, 3, 4, \ldots, n-1$$

$$f(u_nu_1) = 4, \quad n = 3, 5, 7, \ldots$$

The splitting graph of C_n has n maximum degree vertices with a degree value of four. For these 4-degree adjacent vertices, four colors are sufficient, resulting in uniform color sets. Besides the n maximum degree vertices, there are n two-degree vertices in the graph that are not adjacent. These two-degree vertices connect to the already colored 4-degree vertices. Interestingly, the colors for the 2-degree vertices have already been assigned. We used only four colors for the entire graph to ensure uniform color sets for adjacent vertices with the same degree. Therefore, $\chi'_{\text{avrec}}S(C_n) = 4$.

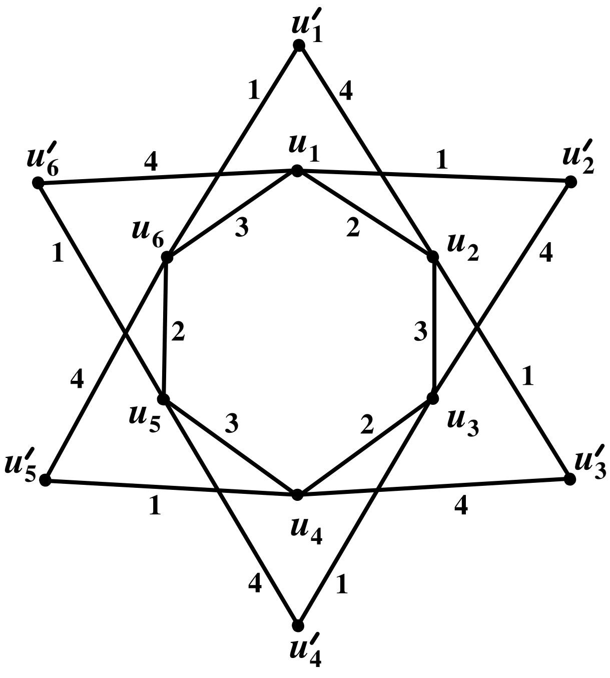

Fig. 3: $\chi'_{avrec}S(C_6) = 4$

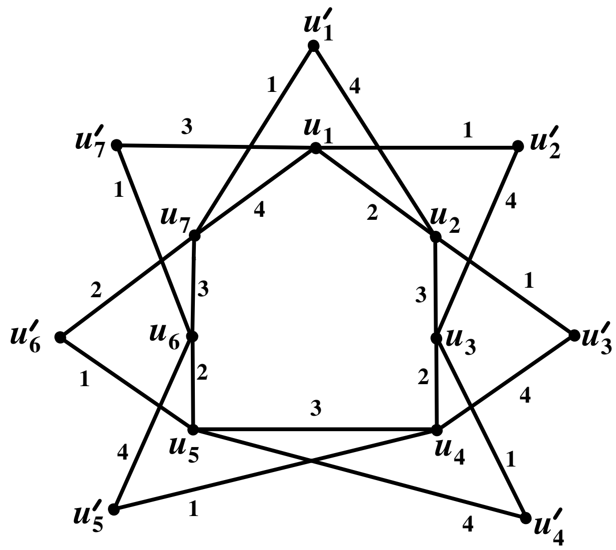

Fig. 4: $\chi'_{avrec}S(C_7) = 4$

■

Theorem 4. For the splitting graph of the complete graph K_r, $\chi'_{\text{avrec}}S(K_r) = \Delta$, where $r \geq 2$.

Proof: Let the vertex sets of the splitting graph of K_r be $u_1, u_2, u_3, \ldots, u_r$ and $u'_1, u'_2, u'_3, \ldots, u'_r$, satisfying the f-coloring rule: When $r = 2$, there are two maximum-degree vertices with degree 2 that are adjacent, and two 1-degree vertices that are not adjacent. To satisfy the coloring condition, only two colors are sufficient. When $r = 3$, there are three maximum-degree vertices with degree 4 that are adjacent, and three 2-degree vertices that are not adjacent. To meet the coloring requirement, four colors are sufficient. When $r = 4$, there are four maximum-degree vertices with degree 6 that are adjacent, and four 3-degree vertices that are not adjacent. Here, six colors are sufficient. For $r = 5$, $\Delta = 8$, thus $\chi'_{\text{avrec}}(K_5) = 8$. For $r = 6$, $\Delta = 10$, so $\chi'_{\text{avrec}}(K_6) = 10$, and this pattern continues. In all these graphs, colors are assigned for adjacent vertices with the same degree. In the splitting graph of K_r, the maximum-degree vertices are similar and adjacent to each other. The other degree vertices, which are not adjacent, each have $r-1$ degrees. Additionally, colors are already assigned to these vertices since they are adjacent to existing maximum-degree vertices. Therefore, $\chi'_{\text{avrec}}(K_r) = \Delta$, for $r \geq 2$. ■

Observation 1. For the tadpole graph, we have: $\chi'_{\text{avrec}}(T_{n,1}) = 3, \quad n \geq 3$.

Proof: Let the vertex set of the tadpole graph $T_{n,1}$ be $u_1, u_2, u_3, \ldots, u_n$ and v_1, satisfying the f-coloring rule. Now, we assign adjacent vertex-reducible edge coloring to the edges as follows:

$$f(u_nv_1) = 3, \quad n \geq 3$$

$$f(u_1u_n) = 2, \quad n \geq 3$$

$$f(u_iu_{i+1}) = \begin{cases} 1, & i \equiv 1 \pmod 2 \\ 2, & i \equiv 0 \pmod 2 \end{cases}, \quad i = 1, 2, 3, \ldots$$

The tadpole graph $T_{n,1}$ has $n-1$ same 2-degree vertices that are adjacent for all values of n. All 2-degree vertices have color sets assigned with $\{1, 2\}$. Apart from these, in all graphs, there are two different categories of degree vertices: u_n and v_1. Therefore, the total colors used for all graphs is 3, and the chromatic number of the tadpole graph is $\chi'_{\text{avrec}}(T_{n,1}) = 3$, for $n \geq 3$. ■

Observation 2. For the tadpole graph, we have: $\chi'_{\text{avrec}}(T_{n,m}) = 4, \quad n \geq 3,\ m \geq 2$.

Proof: Let the vertices of the cycle C_n be $u_1, u_2, u_3, \dots, u_n$ and the vertices of the path P_m be $v_1, v_2, v_3, \dots, v_m$. By connecting these two graphs with the bridge $e = u_n v_1$, we obtain the tadpole graph $T_{n,m}$. Now, we assign adjacent vertex-reducible edge coloring to the edges as follows:

$$f(u_n v_1) = 3, \quad n \geq 3$$

$$f(u_n u_1) = 2, \quad n \geq 3$$

$$f(u_i u_{i+1}) = \begin{cases} 1, & i = 1, 3, 5, \dots \\ 2, & i = 2, 4, 6, \dots \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} 4, & i = 1, 3, 5, \dots \\ 3, & i = 2, 4, 6, \dots \end{cases}$$

Observation 1 follows similarly from $T_{n,1}$, regarding the coloring for C_n and the bridge $u_n v_1 = 3$. Additionally, in the path P_m, there are $m-1$ vertices of degree 2, and the m-th vertex is pendent. Here, the color set for 2-degree vertices is $\{3, 4\}$. Thus, the maximum number of colors assigned is 4. Therefore, the AVREC chromatic number of $T_{n,m} = 4$, for $n \geq 3$ and $m \geq 2$. ■

Theorem 5. For the splitting graph of the tadpole graph of $T_{(n,m)}$, the $\chi'_{\text{avrec}} S(T_{(n,m)}) = 12, \quad n \geq 3,\ m \geq 3$.

Proof: Taking the vertices of the splitting graph of the tadpole graph, we have $u_1, u_2, u_3, \dots, u_n$, $u'_1, u'_2, u'_3, \dots, u'_n$, and $v_1, v_2, v_3, \dots, v_m$, $v'_1, v'_2, v'_3, \dots, v'_m$. Now AVREC is assigned to the edges as follows:

$$f(u_i u_{i+1}) = \begin{cases} 1, & i = 1, 3, 5, \dots \\ 2, & i = 2, 4, 6, \dots \end{cases}$$

$$f(u_n u_1) = 2, \quad n \geq 3$$

$$f(u_i u'_{i+1}) = 3, \quad i = 1, 2, 3, \dots, n-1$$

$$f(u_i u'_{i-1}) = 4, \quad i = 2, 3, 4, \dots, n-1$$

$$f(u_n u'_{n-1}) = 5, \quad n \geq 3$$

$$f(u_n u'_1) = 6, \quad n \geq 3$$

$$f(u_n v'_1) = 7, \quad n \geq 3$$

$$f(u_n v_1) = 8, \quad n \geq 3$$

$$f(v_i v_{i+1}) = \begin{cases} 8, & i = 2, 4, 6, \dots, m-2 & \text{if } m \text{ is even} \\ 8, & i = 2, 4, 6, \dots, m-3 & \text{if } m \text{ is odd} \\ 9, & i = 1, 3, 5, \dots, m-2 & \text{if } m \text{ is odd} \\ 9, & i = 1, 3, 5, \dots, m-3 & \text{if } m \text{ is even} \end{cases}$$

$$f(v_{n-1} v'_m) = \begin{cases} 8, & \text{if } m \text{ is odd} \\ 9, & \text{if } m \text{ is even} \end{cases}$$

$$f(u'_n v_1) = 10, \quad n \geq 3$$

$$f(v_{m-1} v_m) = 11, \quad m \geq 3$$

$$f(v_m v'_{m-1}) = 12, \quad m \geq 3$$

Following the above pattern for $n \geq 3, m \geq 3$, the obtained chromatic number of the splitting graph of the tadpole graph $S(T_{(n,m)})$ is 12. Refer figure 5. ■

Theorem 6. For the splitting graph of the tadpole graph of $\chi'_{\text{avrec}} S(T_{(n,1)}) = 9$, $n \geq 3$.

Proof: A cycle C_n graph is joined to a singleton graph k_1 by a bridge to form the $(n, 1)$-tadpole graph, also known as the n-pan graph. This means that the $(n, 1)$-tadpole graph and the n-pan graph are isomorphic. Two particular examples of this graph are the $(4, 1)$-tadpole graph, also called the banner graph, and the $(3, 1)$-tadpole graph, also called the paw graph. The vertices of the splitting graph of the tadpole graph $T_{(n,1)}$ are $u_1, u_2, u_3, \dots, u_n$, $u'_1, u'_2, u'_3, \dots, u'_n$, and v_1, v'_1. Now, by assign AVREC to the edges we get:

$$f(u_n u_1) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 4, & \text{if } n \text{ is even} \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 1, & i = 1, 3, 5, \dots, n-1 \\ 4, & i = 2, 4, 6, \dots, n-1 \end{cases}$$

$$f(u_i u'_{i-1}) = 2, \quad i = 2, 4, 6, \dots$$

$$f(u_i u'_{i+1}) = 2, \quad i = 3, 5, 7, \dots$$

$$f(u_n u'_1) = \begin{cases} 2, & n = 3, 5, 7, \dots \\ 3, & n = 4, 6, 8, \dots \end{cases}$$

$$f(u_1 u'_n) = 5, \quad n \geq 3$$

$$f(u_1 u'_2) = 6$$

$$f(u_1 v'_1) = 7$$

$$f(u_1 v_1) = 8$$

$$f(u'_1 v_1) = 9$$

As the maximum value of k used is 9, for the $(n, 1)$ tadpole graph, the chromatic number based on AVREC is 9. ■

Theorem 7. For the tadpole graph's splitting graph, $\chi'_{\text{avrec}} S(T_{(n,2)}) = 11$, $n \geq 3$.

Proof: Assume that the vertices of the tadpole graph's splitting graph are $u_1, u_2, u_3, \dots, u_n, u'_1, u'_2, u'_3, \dots, u'_n$ and v_1, v_2, v'_1, v'_2. This is referred to with a forbidden theorem. ■

Theorem 8. For the wheel graph's splitting graph, $\chi'_{\text{avrec}} S(W_n) = n + 5$, $n \geq 5$.

Proof: Assume that the vertices of the wheel graph's splitting graph are $u_1, u_2, u_3, \dots, u_m, u'_1, u'_2, u'_3, \dots, u'_m$. To prove this theorem, we will discuss two cases: odd and even.

Case (i): If n is odd

$$f(u_i u_n) = 1, \quad i = 1, 2, 3, \dots, n-1, \quad n = 5, 7, 9, \dots$$

$$f(u_i u'_n) = 6, \quad i = 1, 2, 3, \dots, n-1, \quad n = 5, 7, 9, \dots$$

$$f(u_i u_{i+1}) = \begin{cases} 2, & i \equiv 1 \pmod 2 \\ 3, & i \equiv 0 \pmod 2 \end{cases}, \quad i = 1, 2, 3, \dots, n$$

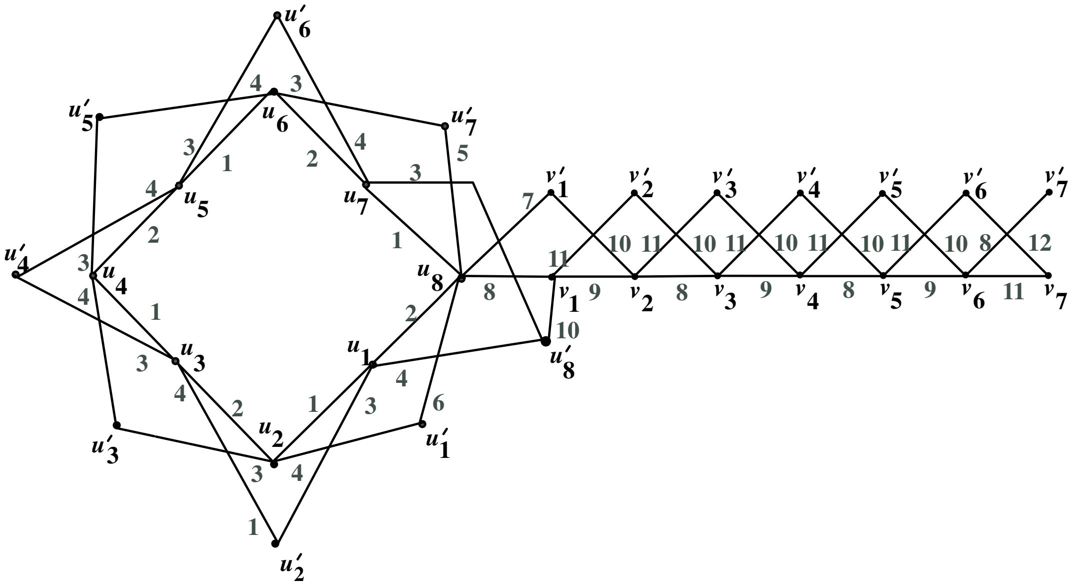

Fig. 5: $\chi'_{avrec}S(T_{8,7}) = 12$

$$f(u_iu'_{i+1}) = 4, \quad i = 1, 2, 3, \ldots, n-2$$

$$f(u_{n-1}u'_1) = 4, \quad n = 5, 7, 9, \ldots$$

$$f(u_{i+1}u'_i) = 5, \quad i = 1, 2, 3, \ldots, n-1$$

$$f(u_1u'_{n-1}) = 5, \quad n = 5, 7, 9, \ldots$$

$$f(u_nu'_i) = i + n - 1, \quad i = 1, 2, 3, \ldots, n, \quad n = 5, 7, 9, \ldots$$

Case (ii): If n is even

$$f(u_iu'_{i+1}) = 1, \quad i = 1, 2, 3, \ldots, n-2$$

$$f(u_{n-1}u'_i) = 1, \quad i = 6, 8, 10, \ldots$$

$$f(u_iu_n) = 2, \quad i = 1, 2, 3, \ldots, n-1, \quad n = 6, 7, 8, \ldots$$

$$f(u_iu_{i+1}) = \begin{cases} 3, & i \equiv 1 \pmod 2 \\ 4, & i \equiv 0 \pmod 2 \end{cases}, \quad i = 1, 2, 3, \ldots, n-2$$

$$f(u_{n-1}u'_{n-2}) = 3, \quad n = 6, 8, 10, \ldots$$

$$f(u_1u'_{n-1}) = 4, \quad n = 6, 8, 10, \ldots$$

$$f(u_{n-1}u_1) = 5, \quad n = 6, 8, 10, \ldots$$

$$f(u_iu'_{i-1}) = 5, \quad i = 2, 3, 4, \ldots, n-2$$

$$f(u_iu'_n) = 6, \quad i = 1, 2, 3, \ldots, n-1, \quad n = 6, 8, 10, \ldots$$

$$f(u_nu'_i) = i + n - 2, \quad i = 1, 2, 3, \ldots, n, \quad n = 6, 8, 10, \ldots$$

The splitting graph of the wheel graph W_n has one maximum degree vertex that is $2(n-1)$, denoted u_n, $(n-1)$ 6-degree vertices $u_1, u_2, u_3, \ldots, u_{n-1}$, one $(n-1)$-degree vertex that is u'_0, and $(n-1)$ 3-degree vertices $u'_1, u'_2, u'_3, \ldots, u'_{n-1}$. There are $(n-1)$ 6-degree vertices that are only adjacent, so this vertex color set must be the same. The remaining vertices, which are not adjacent, also need to maximize the chromatic number by allocating different colors. Therefore, the chromatic number of avrec$S(W_n) = n + 5$, $n \geq 5$. ■

Observation 3. For the complete bipartite graph, $\chi'_{\text{avrec}}(K_{n,n}) = n$, $n \geq 3$.

Theorem 9. For the splitting graph of the complete bipartite graph, $\chi'_{\text{avrec}}\ S(K_{n,n}) = 2n$, $n \geq 3$.

Proof: Let X, Y be vertex sets of complete bipartite graph $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ respectively and to create a splitting graph of complete bipartite graph taking new vertices $u'_1, u'_2, u'_3, \ldots, u'_n$ and $v'_1, v'_2, v'_3, \ldots, v'_n$. In this graph $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_n$ are mutually adjacent to each other. Consequently, edge incident 2n maximum degree vertices are assigned with 2n color sets. That is for corresponding n vertices, the maximum assigned colors are $2n$. Therefore, the chromatic number of the splitting graph of the complete bipartite graph is $2n$. Also, we have $2n$ number of 3-degree vertices which are not adjacent and these vertices are incident to already colored maximum degree vertices. ■

Theorem 10. For the splitting graph of fan graph $\chi'_{\text{avrec}}S(F_n) = n + 12$, $n \geq 5$.

Proof: Let the splitting graph of the fan graph vertices be $u_0, u_1, u_2, u_3, \ldots, u_n$ and $u'_0, u'_1, u'_2, u'_3, \ldots, u'_n$. Now assign AVREC to the edges as follows.

$$f(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i = 1, 3, 5, \ldots, n-1, \text{ if } n \text{ is even}, \\ 1, & \text{for odd } n, i = 1, 3, 5, \ldots, n-2, \\ 2, & \text{if } i = 2, 4, 6, \ldots, n-2, \text{ if } n \text{ is even}, \\ 2, & \text{for odd } n, i = 2, 4, 6, \ldots, n-1. \end{cases}$$

$$f(u_iu'_1) = 3, \quad i = 2, 3, 4, \ldots, n-1$$

$$f(u_iu'_0) = 4, \quad i = 2, 3, 4, \ldots, n-1$$

$$f(u_iu'_{i+1}) = 5, \quad i = 2, 3, 4, \ldots, n-1$$

$$f(u_iu_0) = 6, \quad i = 2, 3, 4, \ldots, n-1$$

$$f(u'_iu_0) = i + 11, \quad i = 1, 2, 3, \ldots, n$$

$$f(u_1u'_2) = 7$$

$$f(u_nu'_{n-1}) = 8, \quad n \geq 5$$

$$f(u_1u'_0) = 9$$

$$f(u_n u_0') = 10, \quad n \geq 5$$
$$f(u_1 u_0) = 11$$
$$f(u_n u_0) = n + 12, \quad n \geq 5$$

As per the f-coloring rule, by following the above pattern, the chromatic numbers are obtained. In this graph, there are $n-2$ same 6-degree adjacent vertices. There are six colors which are enough to make color sets the same. These six colors are incident to various different degree vertices, such as $n-2$ 3-degree independent vertices, two 2-degree vertices for all n, one n-degree vertex, two 4-degree vertices, and one 2n-degree vertex. However, these six colors do not cover all the edges, so we assign different colors to the remaining edges that maximize the chromatic number. Therefore, the chromatic number of the splitting graphs of the fan graph is $n + 12$, $n \geq 5$. ■

III. Conclusion

We have studied the splitting graphs of path, cycles, complete graphs, complete bipartite graphs, tadpole graphs, fan graph and wheel graphs in our investigation of adjacent vertex reducible edge coloring of splitting graphs. Our research shows that the structural characteristics of the original graphs drive the application of edge coloring techniques in these splitting graphs, revealing distinct and intriguing patterns.

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N. Yathavan is currently a Ph.D. Research Scholar in the Department of Mathematics at Annamalai University, Chidambaram, India. He also serves as an Assistant Professor in the Department of Mathematics at Panimalar Engineering College, Chennai, India. His primary area of research is graph theory, with a specific focus on graph coloring and its applications.

N. Paramaguru is an Assistant Professor in the Department of Mathematics at Annamalai University, Chidambaram, India, currently deputed to Government Arts College for Women, Krishnagiri, India. His research interests lie in the field of graph theory, with a particular emphasis on graph coloring.