Transit System with Rectangular Service Zone

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***Abstract*—We study a deterministic model of a transit system with a rectangular service area to develop a new solution approach. We first point out that sometimes the iterative algorithm may be convergent to a point that is not the wanted solution. We construct an upper bound and a lower bound to run the bisection method that will converge to the optimal solution. The same numerical example used in two previously published papers was adopted by us to demonstrate that our solution approach is effective and efficient. Our paper will help researchers develop their solution algorithms.**

***Index Terms*—Bisection method, Iterative algorithm, Optimal solution, Bus transit model**

I. Introduction

Bus transit systems with a rectangular service area had been studied by Kocur and Hendrickson [1], Chang and Schonfeld [2, 3], Imam [4], Yang et al. [5], Hung and Julianne [6], Lin and Julian [7], Tung et al. [8], Yang et al. [9], Lin and Hopscotch [10], Luo [11], Chen and Julian [12], and Wang et al. [13] to indicate that is a hot research topic.

This paper will focus on Yang et al. [5] to provide a further discussion. Yang et al. [5] published a paper in the Journal of Transportation Engineering to revise the formulated solution proposed by Chang and Schonfeld [3] for bus service zones of a traffic model. Moreover, Yang et al. [5] provided an iterative method to construct a sequence that will converge to the optimal formulated solution. In this paper, we will first point out that sometimes the sequence generated by the iterative method is almost impossible to converge to the optimal solution. Second, we offer our approach with the bisection method to locate the optimal solution. Our findings will help those researchers who are not familiar with the algebraic approach of Yang et al. [5] and Chang and Schonfeld [3]. On the other hand, our sequence is derived by the bisection method that can be applied to any solution problem with continuous objective function which will be applied to many research areas.

Several related articles with traffic models such as Furth [14], Kuah and Perl [15, 16], Yang and Bell [17], Yan and Chen [18], Ceder and Israeli [19], Tom and Mohan [20], Jara-Díaz and Gschwender [21], Agrawal and Mathew [22], Gao et al. [23], Kepaptsoglou and Karlaftis [24], Mauttone and Urquhart [25], Bagloee and Ceder [26], Ranjbari et al. [27], Tirachini and Hensher [28], Cipriani et al. [29], Roca-Riu et al. [30], Xiong et al. [31], and Fielbaum et al. [32] that are worthy to be mentioned.

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II. Notation and Assumptions

We use the same notation and assumptions as Chang and Schonfeld [3] and Yang et al. [5].

z_x = geometric factor for determining access time;

z_w = ratio of wait time/headway;

y = express speed/local speed = express ratio;

x = value of access time ($/hr);

w = value of wait time ($/hr);

V_t = bus speed during period t (miles/hr);

v = value of in-vehicle time ($/hr);

T_t = duration of period t (hrs);

s = stop spacing (miles);

r = route spacing (miles);

q_t = potential demand density in period t (trips/sq. mile/hr);

L = length of local bus route (miles);

J = express distance (miles);

h_t = headway in period t (hr);

g = access speed (miles/hr);

C = total system cost ($); and

B_t = bus operating cost in period t ($/vehicle hr).

III. Review of Chang and Schonfeld [3]

We will directly cite the results of Chang and Schonfeld [3] for the background explanation. They constructed a traffic model for bus service zones with the following total cost per trip $C = C(r, h_1, ..., h_m, L)$:

$$C = \frac{2}{r\sum_{t=1}^{m} q_t T_t}\sum_{t=1}^{m}\frac{B_t T_t}{h_t V_t} + \frac{2J}{Lry\sum_{t=1}^{m} q_t T_t}\sum_{t=1}^{m}\frac{B_t T_t}{h_t V_t} + wz_w\sum_{t=1}^{m} q_t T_t h_t \Big/ \sum_{t=1}^{m} q_t T_t + \frac{xz_x}{g} r + \frac{v\sum_{t=1}^{m}\frac{q_t T_t}{V_t}}{2\sum_{t=1}^{m} q_t T_t} L + \frac{xz_x}{g} s + \frac{Jv\sum_{t=1}^{m}\frac{q_t T_t}{V_t}}{y\sum_{t=1}^{m} q_t T_t}. \quad (3.1)$$

They computed the first partial derivatives to imply that

$$\frac{\partial C}{\partial r}=\frac{-2}{r^2\sum_{t=1}^{m}q_tT_t}\sum_{t=1}^{m}\frac{B_tT_t}{V_t}\frac{1}{h_t}$$

$$+\frac{-2J}{r^2Ly\sum_{t=1}^{m}q_tT_t}\sum_{t=1}^{m}\frac{B_tT_t}{V_t}\frac{1}{h_t}+\frac{xz_x}{g}, \quad (3.2)$$

$$\frac{\partial C}{\partial h_t}=\frac{-2}{h_t^2r\sum_{t=1}^{m}q_tT_t}\sum_{t=1}^{m}\frac{B_tT_t}{V_t}$$

$$+\frac{-2J}{Lryh_t^2\sum_{t=1}^{m}q_tT_t}\sum_{t=1}^{m}\frac{B_tT_t}{V_t}+\frac{wz_wq_tT_t}{\sum_{t=1}^{m}q_tT_t}, \quad (3.3)$$

for $t=1,2,...,m$, and

$$\frac{\partial C}{\partial L}=\frac{-2J}{ryL^2\sum_{t=1}^{m}q_tT_t}\sum_{t=1}^{m}\frac{B_tT_t}{V_t}\frac{1}{h_t}+\frac{v\sum_{t=1}^{m}\frac{q_tT_t}{V_t}}{2\sum_{t=1}^{m}q_tT_t}. \quad (3.4)$$

They solved the zeros for the system of first partial derivatives, $\frac{\partial C}{\partial r}=0$, $\frac{\partial C}{\partial h_t}=0$ $t=1,2,...,m$, and $\frac{\partial C}{\partial L}=0$, to yield a family of three relations that satisfy the first partial derivatives equaling to zero conditions,

$$r=\left[\frac{2(yL+J)g^2wX^2z_w}{x^2yLz_x^2}\right]^{\frac{1}{3}}, \quad (3.5)$$

$$h_t=\left(\frac{B_t}{q_tV_t}\right)^{\frac{1}{2}}\left[\frac{2(yL+J)xz_x}{gw^2yLXz_w^2}\right]^{\frac{1}{3}}, \quad (3.6)$$

for $t=1,2,...,m$, and

$$L=\sqrt{4J\sum_{t=1}^{m}\frac{B_tT_t}{h_tV_t}\bigg/ rvy\sum_{t=1}^{m}\frac{q_tT_t}{V_t}}, \quad (3.7)$$

where X is defined as

$$X=\sum_{t=1}^{m}T_t\left(q_tB_t/V_t\right)^{0.5}\bigg/\sum_{t=1}^{m}q_tT_t. \quad (3.8)$$

Next, they tried to combine equations in the above system to reduce into an equation of one variable, the length of a local bus route, L. They derived that

$$L^3+(J/y)L^2-MJ^{1.5}=0, \quad (3.9)$$

in which

$$M=4X\left(\sqrt{wxP^3z_wz_x/gv^3y^3}\right), \quad (3.10)$$

and

$$P=\sum_{t=1}^{m}q_tT_t\bigg/\sum_{t=1}^{m}\frac{q_tT_t}{V_t}. \quad (3.11)$$

The above result of Equation (3.9) provides a formulated solution for the optimal length with respect to the bus transit model with a rectangular service area.

IV. Review of Yang et al. [5]

Yang et al. [5] first revised the questionable in Chang and Schonfeld [3] for the algebraic method of a cubic polynomial, and then they offered an analytical approach to rewrite Equation (3.9) as

$$L^2=\frac{yM}{yL+J}J^{\frac{3}{2}}. \quad (4.1)$$

Based on Equation (4.1), an iterative algorithm was generated with the initial point, $L_0=0$ to imply that

$$L_{n+1}=\sqrt{\frac{yM}{yL_n+J}J^{\frac{3}{2}}}. \quad (4.2)$$

Yang et al. [5] proved that their sequence derived by Equation (4.2) is an alternative sequence that will converge to the optimal solution of the area length.

V. Our Challenge for the Iterative Method

Before we apply the bisection method to find the optimal solution, we present a motivation to explain under some extreme cases, unless the researchers knew the desired solution in advance, to execute an iterative algorithm cannot converge to the desired solution.

Our goal is to find local positive minimum points of the objective function, f(x), with

$$f(x)=x^3-21x^2+144x+168, \quad (5.1)$$

for $-\infty<x<\infty$. Researchers computed the first derivetive to imply that

$$df(x)/dx=3x^2-42x^2+144, \quad (5.2)$$

for $-\infty<x<\infty$. Researchers tried to solve $df(x)/dx=0$ by an iterative algorithm, and then there are two possible procedures,

$$x=(14x-48)/x, \quad (5.3)$$

and

$$x=(x^2+48)/14. \quad (5.4)$$

Based on Equation (5.3), the researchers will run

$$x_{k+1}=(14x_k-48)/x_k, \quad (5.5)$$

According to Equation (5.4), the researchers will execute

$$x_{k+1}=(x_k^2+48)/14. \quad (5.6)$$

We recall the restriction of a local positive minimum point, the iterative algorithm proposed by Equation (5.5) cannot search for $0\le x\le 24/7$ such that researchers will select Equation (5.6) to operate their iterative algorithm.

Researchers considered the monotonic property of the generated sequence (x_k) to computed $x_{k+1}-x_k\ge 0$ is equivalent to $x_k^2+48\ge 14x_k$ to imply that

$$(x_k-8)(x_k-6)\ge 0. \quad (5.7)$$

Based on Equation (5.7), researchers has five different selections for the initial point, (i) $x_1<6$, (ii) $x_1=6$, (iii) $6<x_1<8$, (iv) $x_1=8$, and (v) $x_1>8$.

For the case (i), with $x_1<6$, we already derive that $x_2>x_1$. We recall Equation (5.6) to derive that if $x_k<6$, then $x_{k+1}<6$ such that (x_k) is a bounded above and an increasing sequence such that the limit of the (x_k) exists.

We will begin to prove that the sequence (x_k) will converge to 6. We compute that

$$\begin{aligned}6-x_{k+1}&=(36-x_k^2)/14,\\&=(6-x_k)[(6+x_k)/14],\\&<6(6-x_k)/7. \quad (5.8)\end{aligned}$$

Referring to Equation (5.8), we obtain that

$$6 - x_{k+2} < 6(6 - x_{k+1})/7, \\ < (6/7)^2(6 - x_k). \quad (5.9)$$

We can abstractly express the finding of Equation (5.9) as follows,

$$6 - x_{1+k} < (6/7)^k(6 - x_1). \quad (5.10)$$

We take the limit on the both sides of Equation (5.10) to imply that

$$6 - \lim_{k\to\infty} x_k = 0. \quad (5.11)$$

For the case (ii), with $x_1 = 6$, we obtain that every $x_k = 6$, such that (x_k) is a constant which will converge to 6.

For the case (iii), with $6 < x_1 < 8$, our previous discussion only imply that $x_2 < x_1$. We recall Equation (5.6), based on $6 < x_1$, we derive that $6 < x_2$ such that we show that $6 < x_k < 8$, for $k = 1,2, \ldots$, and (x_k) is a deccreasing sequence.

We will begin to prove that the sequence (x_k) will converge to 6. We compute that

$$x_{k+1} - 6 = (x_k^2 - 36)/14, \\ = (x_k - 6)[(x_k + 6)/14], \\ < (x_k - 6)[(x_1 + 6)/14]. \quad (5.12)$$

Referring to Equation (5.12), we obtain that

$$x_{k+2} - 6 < (x_{k+1} - 6)[(x_1 + 6)/14], \\ < (x_k - 6)[(x_1 + 6)/14]^2. \quad (5.13)$$

We can abstractly express the finding of Equation (5.13) as follows,

$$x_{k+1} - 6 < [(x_1 + 6)/14]^k(x_1 - 6). \quad (5.14)$$

We take the limit on the both sides of Equation (5.14) to imply that

$$(\lim_{k\to\infty} x_k) - 6 = 0. \quad (5.15)$$

For the case (iv), with $x_1 = 8$, we obtain that every $x_k = 8$, such that (x_k) is a constant which will converge to 8.

For the case (v), with $8 < x_1$, we already derive that $x_2 > x_1$, and then directly we know that $8 < x_2$. Consequently, for $k = 1,2, \ldots$, we obtain that $8 < x_k$.

We begin to show that (x_k) will deverge to ∞. We compute

$$x_{k+1} - 8 = (x_k^2 - 64)/14, \\ = (x_k - 8)[(x_k + 8)/14], \\ > 8(x_k - 8)/7. \quad (5.16)$$

Based on Equation (5.16), we obtain that

$$x_{k+2} - 8 = (x_{k+1}^2 - 64)/14, \\ = (x_{k+1} - 8)[(x_{k+1} + 8)/14], \\ > (8/7)^2(x_k - 8). \quad (5.17)$$

We can abstractly express the finding of Equation (5.17) as follows,

$$x_{1+k} - 8 > (8/7)^k(x_1 - 8). \quad (5.18)$$

We take the limit on the both sides of Equation (5.18) to imply that

$$(\lim_{k\to\infty} x_k) - 8 = \infty. \quad (5.19)$$

Based on our above discussion, for cases (i), (ii), (iii), the sequences will converge to 6. Under the case (v), the sequences will diverge to infinite. Only for case (iv), the constant sequence will converge to 8.

For the objective function, f(x), of Equation (5.1), it have a local maximum point at x = 6, and a local minimum point at x = 8. If researchers tried to apply the iterative algorithm to find a local minimum point, unless in the initial setting to assume $x_1 = 8$ (the desired result), the other initial point will not converge to the desired result.

The above discussion provides a solid motivation for our study to apply the bisection method to locate the optimal solution.

Hence, in the following, we will provide another approach to solving the positive solutions of equation (3.9).

VI. Our proposed approach

Yang et al. [5] already showed that $f(L)$ is an increasing function where $f(L) = L^3 + (J/y)L^2 - MJ^{1.5}$.

From $f(0) < 0$ and $\lim_{L\to\infty} f(L) = \infty$ so that $f(L) = 0$ has a unique positive solution, say L^*. In the following, we will construct a lower bound, L_l and an upper bound, L^u with the following condition, $0 < L_l < L^* < L^u < \infty$.

Under a reasonable condition of $L \ge 1$, we find an auxiliary function,

$$h(L) = L^2 + (J/y)L^2 - MJ^{1.5}, \quad (6.1)$$

with $h(L) \le f(L)$. The auxiliary function, $h(L)$ also is an increasing function with $h(0) < 0$ and $\lim_{L\to\infty} h(L) = \infty$ so that $h(L) = 0$ has a unique positive solution, say $L^\#$ such that $L^\#$ is an upper bound for L^*. Hence, we will take $L^\#$ where the assumed upper bound is denoted as

$$L^u = \sqrt{yMJ^{1.5}/(J+y)}. \quad (6.2)$$

Next, we try to solve $f(L) = 0$ with

$$f(L) = L^3 + (J/y)L^2 - MJ^{1.5}. \quad (6.3)$$

We rewrite Equation (6.3) as

$$f(L) = L^3(1 + (J/yL)) - MJ^{1.5}, \quad (6.4)$$

and then plugged $L = 1$ into Equation (12) to obtain a new equation to denote it as $g(L)$, and then

$$g(L) = L^3(1 + (J/y)) - MJ^{1.5}. \quad (6.5)$$

We solved the positive root, say $L^\wedge$, for $g(L) = 0$ to yield that

$$L^\wedge = \sqrt[3]{yMJ^{1.5}/(J+y)}. \quad (6.6)$$

In the following, we will prove that $L^\wedge$ is a lower bound for the solution of $f(L) = 0$.

Owing to Yang et al. [5], we already knew that $f(0) < 0$ and $f(L)$ is an increasing function. Hence, we will show that $f(L^\wedge) < 0$. That is,

$$yM + (y^{-1/3}M^{2/3}J^{0.5})(y+J)^{1/3} < M(y+J). \quad (6.7)$$

Therefore, we derive that

$$J(y^{2/3}M^{2/3}J)(y+J)^{1/3} < MJ^{2.5}y. \quad (6.8)$$

We simplify Equation (6.8) as follows

$$(y+J)^{1/3} < M^{1/3}J^{1/2}y^{1/3}. \quad (6.9)$$

We can simplify the result of equation (6.9) in a compact expression,

$$y + J < MyJ^{1.5}. \quad (6.10)$$

For further discussion, we rewrite equation (6.10) as follows,

$$(1/y) + (1/J) < MJ^{0.5}. \quad (6.11)$$

Table 1. Results from bisection method.

iteration	1	2	3	4	5	6	7	8	9	10
L	5.1226	5.9080	5.5153	5.7117	5.6135	5.6626	5.6380	5.6258	5.6196	5.6166
L-bound	3.5517	5.1226	5.1226	5.5153	5.5153	5.6135	5.6135	5.6135	5.6135	5.6135
U-bound	6.6935	6.6935	5.9080	5.9080	5.7117	5.7117	5.6626	5.6380	5.6258	5.6196
IL	3.1418	1.5709	0.7854	0.3927	0.1964	0.0982	0.0491	0.0245	0.0123	0.0061

L: Bus route length; L-bound: Low bound; U-bound: upper bound; IL: Interval length.

From the examples in Chang and Schonfeld [3] and Yang et al. [5], with $M = 5.51$, $Y = 2$, and $J = 20$, so that we claim that the following three conditions,

$$1 < y, \tag{6.12}$$

$$1 < J, \tag{6.13}$$

and

$$2 < M, \tag{6.14}$$

satisfied.

Based on the conditions of equations (6.12-6.14), we show that

$$(1/y) + (1/J) < 2 < M < MJ^{0.5}. \tag{6.15}$$

We provide analytical proof to show that $f(L^{\wedge}) < 0$ that was supported by the conditions of equations (6.12-6.14), such that $L^{\wedge}$ is a lower bound.

Hence we will take the lower bound as $L^{\wedge}$. Therefore, the lower bound is denoted as

$$L_l = \sqrt[3]{yMJ^{1.5}/(J+y)}. \tag{6.16}$$

Consequently, we will use the derivations of equations (6.2) and (6.16) as the upper bound and lower bound to run the bisection method.

VII. Numerical examples

Under the same parameter setting as Chang and Schonfeld [3] and Yang et al. [5] with the following data: $B_1 = 50$, $B_2 = 25$, $B_3 = 25$, $g = 2.5$, $J = 20$, $q_1 = 120$, $q_2 = 48$, $q_3 = 12$, $T_1 = 3$, $T_2 = 3$, $T_3 = 4$, $v = 5$, $v_1 = 10$, $v_2 = 12$, $v_3 = 15$, $w = 10$, $x = 10$, $y = 2$, $z_w = 0.5$, and $z_x = 0.25$ to execute our numerical example.

We stop our iterative process when the difference between two consecutive terms in the derived length is less than 0.01.

Based on the findings of Table 1, our increasing sequence converges to the optimal solution, 5.617, which is the same optimal solution for the service area length which was proposed by Yang et al. [5] to show the effectiveness and efficiency of our proposed solution approach.

VIII. Direction for future research

There are several papers, for example, Wang et al. [13], Luo [11], Yang et al. [9], Lin and Julian [7], and Yang et al. [5], that have worked on iterative approaches to find the optimal solution. After we point out the possible dilemma that may cause by the iterative algorithm. Researchers may reconsider those above-mentioned papers to apply our proposed solution method to find the optimal solution.

Moreover, Chen and Julian [12], Lin and Hopscotch [10], Tung et al. [8], Hung and Julianne [6], Imam [4], Chang and Schonfeld [2, 3], and Kocur and Hendrickson [1] also studied transit bus models with a rectangular service zone. Our study will provide a new solution procedure for those models.

There are some related articles that are recently published to indicate the hot research directions such that we list them in the following for practitioners: Ismail, and Al-Gounmeein [33], Octarina et al. [34], Fang et al. [35], Yayah et al. [36], Liu et al. [37], and Yendri et al. [38]. Ismail and Al-Gounmeein [33] provided a review for financial and economic time series models with related time series and long memory. Octarina et al. [34] developed heuristic algorithms and models to deal with discrete place issues in Palembang City with temporary disposal locations. Fang et al. [35] considered single species logistic models through bifurcation and qualitative analysis with feedback control and Allee effect. Yayah et al. [36] examined classification methods to carry out Telco customer dataset for trouble tickets. According to second-hand university platform, Liu et al. [37] developed E-sporas models to learn reputation mechanism. Yendri et al. [38] worked out rectangle partitioning problems by dynamic programming approach. Moreover, we recall that Yu et al. [39] dealt with data classification problems through support vector machine by Archimedes optimization algorithm and Henry gas solubility optimization algorithm. By quality factor and center frequency, Wai et al. [40] examined voltage mode biquad filter with electronic and independent control. With an inclined non-uniform channel, Gudekote et al. [41] considered Eyring Powell fluid with peristaltic transport under mass and heat transfer effects. Based on those referred papers, we provide several research directions for future development.

IX. A Related Problem

In this section, we study fuzzy controller design proposed by Guan and Zhao [42] to apply similarity measures among vague sets that had been examined by Li et al. [43], Hong and Choi [44], and Gau and Buwhrer [45]. They worked to analyze among vague sets the reasoning of similarity measures.

Gau and Buwhrer [45] claimed that the fuzzy value along with the evidence of item in the family and the evidence of item not in the family is not sufficient to show that how much is the accuracy of those values, and then Gau and Buwhrer [45] construct a new approach to solve this dilemma. Gau and Buwhrer [45] developed vague sets to provide an alternative approach besides fuzzy sets that arouse attentions among researchers. Vague sets has the distinct character than fuzzy sets owing to the development of the membership function is easier than that proposed by fuzzy sets.

We recall the fuzzy set,

$$[t_A, 1 - f_A] = [0.5, 0.8], \tag{9.1}$$

to indicate the among ten votes, five votes is favor the resolution, two votes are against the resolution, and the votes do not express their attention.

Hence, the membership function is denoted as the lower bound in the favor of the proposal, and one minus the non-membership function is expressed as the lower bound if the against the proposal.

Guan and Zhao [42] constructed the vague set,

$$[t_A(u_i), 1 - f_A(u_i)], \tag{9.2}$$

to show the exact grade, under the condition that sometimes the membership function is unknown. Guan and Zhao [42] assumed that

$$t_A(u_i) \le \mu_A(u_i) \le 1 - f_A(u_i), \tag{9.3}$$

with the restriction,

$$t_A(u_i) + f_A(u_i) \le 1. \tag{9.4}$$

The i[th] rule is defined as a column vector, its transport row vector is expressed in the following,

$$P_i = [[t_1^i, 1 - f_1^i], \ldots, [t_m^i, 1 - f_m^i]], \tag{9.5}$$

The "fact", denoted as "R" is expressed as

$$R = [[t_1, 1 - f_1], \ldots, [t_m, 1 - f_m]]. \tag{9.6}$$

The main issue is to construct the procedure to defuzzy a vague set to a crisp set such that research can compare among vague sets.

Guan and Zhao [42] considered the universe of discourse, denoted as E to represent the error,

$$E = \{e_1, \ldots, e_n\}, \tag{9.7}$$

where "a" is defined as the maximum among then, then

$$a = \max\{e_1, \ldots, e_n\}. \tag{9.8}$$

The vague set with domain E is to express the largest error, then

$$t_A(e_i) = \frac{1}{1 + \frac{(e_i - a)^2}{a}}, \tag{9.9}$$

and

$$f_A(e_i) = \frac{\frac{(e_i - a)^2}{a+1}}{1 + \frac{(e_i - a)^2}{a+1}}. \tag{9.10}$$

X. Our Revisions

In this section, we will provide our improvements. We can rewrite Equations (9.9) and (9.10) as follows,

$$t_A(e_i) = a / ((a - e_i)^2 + a), \tag{10.1}$$

and

$$f_A(e_i) = (a - e_i)^2 / ((a - e_i)^2 + a + 1). \tag{10.2}$$

Based on our expressions of Equation (10.1) and (10.2), we begin to compute

$$t_A(e_i) + f_A(e_i) <,$$

$$\frac{a}{a + (e_i - a)^2} + \frac{(e_i - a)^2}{a + (e_i - a)^2} = 1. \tag{10.3}$$

However, we need that an extra restriction,

$$a \ge 0, \tag{10.4}$$

to guarantee that

$$t_A(e_i) \ge 0. \tag{10.5}$$

In the following, we provide an example,

$$E = \{-2, -1, -0.5, -0.2, -0.1 \\ -0.05, 0, 0.05, 0.1, 0.2, 0.5, 1, 2\}, \tag{10.6}$$

to imply that

$$e_1 = -2, \tag{10.7}$$

then we obtain

$$t_A(e_1) = \frac{2}{2 + (e_1 - 2)^2} = \frac{1}{9}, \tag{10.8}$$

and

$$f_A(e_1) = \frac{(e_1 - 2)^2}{2 + 1 + (e_1 - 2)^2} = \frac{16}{19}. \tag{10.9}$$

The cardinal number of E is 13.

The entries of $Mat_{13\times13}$ are

$$\{-2, -2, -2, -1, -1, -1, -0.5, -0.2, \\ -0.1, -0.1, -0.1, -0.05, 0, 0.05, \\ 0.1, 0.1, 0.1, 0.2, 0.5, 1, 1, 1, 2, 2, 2\}. \tag{10.10}$$

Hence, we know that in first row

$$\{-2, -2, -2, -1, -1, -1, \\ -0.5, -0.2, -0.1, -0.1, -0.1, -0.05, 0\}, \tag{10.11}$$

and in the 13[th] row,

$$\{0, 0.05, 0.1, 0.1, 0.1, 0.2, \\ 0.5, 1, 1, 1, 2, 2, 2\}. \tag{10.12}$$

We may conclude that the membership value of this type of controller is more simpler to guarantee than traditional fuzzy controller. Consequently, more information can be preserved in the vague sets.

XI. Examination of Interior Optimal Solution

After an inventory model is constructed, how to derive the optimal solution becomes the important issue to solve the replenishment policy. We recall the following two research trends.

First, Ho [46] developed an inventory system with constant rate defective items and stochastic crashable lead time. on the other hand, she could not decide how many solutions for the first partial derivative system. Lin et al. [47] found a pair of bounds to prove that the first partial derivative system has a unique solution. Nevertheless, Lin et al. [47] overlooked the local minimum on the boundary when the safety factor reduces to zero. Tung and Deng [48] completed the solution procedure proposed by Lin et al. [47] to prove the interior minimum value, if it exists, is less than the boundary minimum value.

Second, Wu and Ouyang [49] constructed an inventory model with two holding costs and a probability distribution for defective items, but Wu and Ouyang [49] only provide an iterative procedure to search for the optimal solution. Tung et al. [50] pointed out the iterative procedure proposed by Wu and Ouyang [49] did not work and then they presented revision. Moreover, Tung et al. [50] obtained two reasonable conditions to prove the interior optimal solution exists and is unique. However, Tung et al. [50] ignored to check the two boundary local minimums. Lin et al. [51] provided a patch work for Tung et al. [50] to compare (i) An interior local minimum, and (ii) Two boundary local minimums to verify the interior minimum is the global minimum.

Based on our above discussion, to compare interior and boundary minimums is a significant problem in inventory systems.

XII. Information Entropy

Information entropy is to discuss the evaluate of vagueness. If there is only one possible result then the matter is completely decided and then the evaluation of hesitation is defined as zero. On the other hand, if there are many possible different results, and then the uncertainty should be bigger. We suppose that there are n possible events with the probability p_k that will occur, with the restriction,

$$\sum_{k=1}^{n} p_k = 1. \tag{12.1}$$

Shannon [52] assumed the information entropy as follows,

$$g(p_1, p_2, \ldots, p_n) = -\sum_{k=1}^{n} p_k \ln p_k, \tag{12.2}$$

which will attain its maximum when those n events have the identical probability, with

$$p_1 = p_2 = \cdots = p_n = 1/n. \tag{12.3}$$

We plug the restriction of Equation (12.1) into the objective mapping of Equation (12.2) to convert from $g(p_1, p_2, \ldots, p_n)$ to $g(p_1, p_2, \ldots, p_{n-1})$ as follows,

$$g(p_1, p_2, \ldots, p_{n-1}) = -\sum_{k=1}^{n-1} p_k \ln p_k, \\ -\left(1-\sum_{k=1}^{n-1} p_k\right)\ln\left(1-\sum_{k=1}^{n-1} p_k\right). \tag{12.4}$$

Based on Equation (12.4), we compute the partial derivative of $g(p_1, p_2, \ldots, p_{n-1})$ in the following, for $s = 1,2,\ldots, n-1$,

$$\frac{\partial}{\partial p_s} g(p_1, p_2, \ldots, p_{n-1}) = -\ln p_s - p_s\left(\frac{1}{p_s}\right), \\ \ln\left(1-\sum_{k=1}^{n-1} p_k\right) + 1. \tag{12.5}$$

According to Equation (12.5), we solve $\frac{\partial}{\partial p_s} g(p_1, p_2, \ldots, p_{n-1}) = 0$, to imply that

$$\ln\left(1-\sum_{k=1}^{n-1} p_k\right) = \ln p_s, \tag{12.6}$$

for $s = 1,2,\ldots, n-1$.

Referring to Equation (12.6), we derive that

$$\ln p_1 = \ln p_2 = \cdots = \ln p_{n-1} = \ln\left(1-\sum_{k=1}^{n-1} p_k\right). \tag{12.6}$$

We recall that the logarithmic mapping is a one to one function, such that we obtain that

$$p_1 = p_2 = \cdots = p_{n-1} = \left(1-\sum_{k=1}^{n-1} p_k\right). \tag{12.7}$$

Owing to Equation (12.7), we show that

$$p_1 = p_2 = \cdots = p_{n-1} = 1/n. \tag{12.8}$$

We plug our findings of Equation (12.8) into Equation (12.1) to yield that

$$p_n = 1/n. \tag{12.9}$$

Based on Equations (12.8) and (12.9), we show that the assertion of Equation (12.3) is valid.

XIII. Reviewing of an Inexact Inventory Model

We examine the inventory model developed by Covert and Philip [53], and extended by Misra [54], and then generalized by Jalan et al. [55] to find the exact holding cost to construct an exact inventory model. We will prove the optimal solution exists and is unique. Jalan et al. [55] assumed that the inventory holding cost might not be computed exactly as adopted by Covert and Philip [53] and Misra [54]. However, in this section, we will try to compute the exact holding cost and then derive the optimal solution. We recall the Equations (1) and (2) of Jalan et al. [55] so the system governing the inventory level, $q(t)$, satisfying the following two equations:

$$\frac{d}{dt}q(t) + \alpha\beta\, t^{\beta-1} q(t) = -(a+bt), \tag{13.1}$$

for $0 < t < t_1$, and

$$\frac{d}{dt}q(t) = -(a+bt), \tag{13.2}$$

for $t_1 < t < T$, with $q(t_1) = 0$.

Jalan et al. [55] derived that

$$e^{\alpha t^{\beta}} q(t) - q_0 = -\int_0^t (a+bx) e^{\alpha x^{\beta}} dx, \tag{13.3}$$

for $0 \le t \le t_1$.

$q_0 = q(0)$ is the beginning inventory level. Based on Equation (13.3) with $q(t_1) = 0$, Jalan et al. [55] implied that

$$q_0 = \int_0^{t_1} (a+bx) e^{\alpha x^{\beta}} dx. \tag{13.4}$$

Jalan et al. [55] combined Equations (13.3) and (13.4) to obtain that

$$q(t) = e^{-\alpha t^{\beta}} \int_t^{t_1} (a+bx) e^{\alpha x^{\beta}} dx, \tag{13.5}$$

for $0 \le t \le t_1$, and it is trivial from Equation (13.2) to get

$$q(t) = a(t_1 - t) + \left[b\left(t_1^2 - t^2\right)/2\right], \tag{13.6}$$

for $t_1 \le t \le T$. Jalan et al. [55] knew that the deteriorated items were

$$q_0 - \int_0^{t_1} (a+bx) dx. \tag{13.7}$$

However, we may rewrite the deteriorated items in a compact expression as follows,

$$\int_t^{t_1} (a+bx)\left(e^{\alpha x^{\beta}} - 1\right) dx. \tag{13.8}$$

For the inventory carrying cost, they followed the approximated method as Covert and Philip [53] and Misra [54] such that they did not compute the exact carrying cost. Instead, they only considered the inventory level as a straight line so they used the average of the beginning inventory level, $q(0)$, and the ending inventory level, $q(t_1)$, as

$$[q(0) + q(t_1)]/2 = [q(0)/2], \tag{13.9}$$

to represent the inventory level during $0 \le t \le t_1$. Hence, the carrying cost is $C_1 t_1 (q(0)/2)$. We can claim that their model is not accurate when the deterioration rate is significant, the relative error for the carrying cost may considerable influence the optimal solution. Therefore, in this section, we will derive the exact carrying cost such that our average carrying cost per unit time will indicate the exact cost.

XIV. Solution of Our Exact Inventory Model

The average carrying cost per unit time is $\frac{C_1}{T}\int_0^{t_1} q(t) dt$.

We change the order of integration to rewrite it as

$$\frac{C_1}{T}\int_0^{t_1} q(t)dt = \frac{C_1}{T}\int_0^{t_1} e^{-\alpha t^\beta}\int_t^{t_1}(a+bx)e^{\alpha x^\beta}dxdt,$$

$$=\frac{C_1}{T}\int_0^{t_1}(a+bx)e^{\alpha \beta^x}\int_0^x e^{-\alpha t^\beta}dtdx. \quad 14.1)$$

The average system cost, $C(t_1,T)$, is composed of ordering cost (set-up cost), carrying cost, shortage cost, and deterioration cost such that

$$C(t_1,T)=\frac{C_1}{T}\int_0^{t_1}(a+bx)e^{\alpha x^\beta}\int_0^x e^{-\alpha t^\beta}dtdx,$$

$$+\frac{C_2}{6T}(T-t_1)^2(3a+bT+2bt_1)$$

$$+\frac{C_3}{T}+\frac{C_4}{T}\int_0^{t_1}(a+bx)\left(e^{\alpha x^\beta}-1\right)dx. \quad (14.2)$$

We found that

$$\frac{\partial}{\partial T}C(t_1,T)=\frac{-C_1}{T^2}\int_0^{t_1}(a+bx)e^{\alpha \beta^x}\int_0^x e^{-\alpha t^\beta}dtdx$$

$$+\frac{C_2a}{2T^2}\left(T^2-t_1^2\right)+\frac{C_2b}{3T^2}\left(T^3-t_1^3\right)$$

$$-\frac{C_3}{T^2}-\frac{C_4}{T^2}\int_0^{t_1}(a+bx)\left(e^{\alpha \beta^x}-1\right)dx, \quad (14.3)$$

and

$$\frac{\partial}{\partial t_1}C(t_1,T)=\frac{(a+bt_1)}{T}C_1e^{\alpha t_1^\beta}\int_0^{t_1}e^{-\alpha t^\beta}dt,$$

$$+\frac{(a+bt_1)}{T}\left[-C_2(T-t_1)+C_4\left(e^{\alpha t_1^\beta}-1\right)\right]. \quad (14.4)$$

Based on Equation (14.3), if we solve $\frac{\partial}{\partial T}C(t_1,T)=0$ then it implies that

$$C_1\int_0^{t_1}(a+bx)e^{\alpha \beta^x}\int_0^x e^{-\alpha t^\beta}dtdx+C_3$$

$$+C_4\int_0^{t_1}(a+bx)\left(e^{\alpha \beta^x}-1\right)dx$$

$$=\frac{C_2a}{2}\left(T^2-t_1^2\right)+\frac{C_2b}{3}\left(T^3-t_1^3\right). \quad (14.5)$$

On the other hand, referring to Equation (14.4), if we consider $\frac{\partial}{\partial t_1}C(t_1,T)=0$, then it yields

$$C_1e^{\alpha t_1^\beta}\int_0^{t_1}e^{-\alpha t^\beta}dt+C_4\left(e^{\alpha t_1^\beta}-1\right)=C_2(T-t_1). \quad (14.6)$$

Motivated by Equation (14.5), we assume an auxiliary function, say $g(t_1)$, as follows

$$g(t_1)=t_1+\frac{C_1}{C_2}e^{\alpha t_1^\beta}\int_0^{t_1}e^{-\alpha t^\beta}dt+\frac{C_4}{C_2}\left(e^{\alpha t_1^\beta}-1\right), \quad (14.7)$$

such that $T=g(t_1)$ satisfies the first partial derivative condition $\frac{\partial}{\partial t_1}C(t_1,T)=0$. With the help of $g(t_1)$, if we combine Equations (14.5) and (14.6), and then it yields a function in only one variable t_1. Hence, we assume a second auxiliary function, say $h(t_1)$, as follows

$$h(t_1)=\frac{C_2a}{2}\left((g(t_1))^2-t_1^2\right)+\frac{C_2b}{3}\left((g(t_1))^3-t_1^3\right),$$

$$-C_1\int_0^{t_1}(a+bx)e^{\alpha \beta^x}\int_0^x e^{-\alpha t^\beta}dtdx$$

$$-C_3-C_4\int_0^{t_1}(a+bx)\left(e^{\alpha \beta^x}-1\right)dx. \quad (14.8)$$

To solve the system of equations for the first partial derivatives of $C(t_1,T)$ is equivalent to solve $h(t_1)=0$. In the following, we provide a possible solution approach to prove that

(a) $h(0)<0$,

(b) $h(t_1)$ is an increasing function,

and

(c) $\lim_{t_1\to\infty}h(t_1)=\infty$.

Consequently, following our proposed solution approach, researchers will obtain that there is a unique value, say t^*, that satisfies $h(t^*)=0$ and then researchers will prove that t^* is the optimal point.

XV. A Related Problem of Imam

Imam [4] also studied bus transit system with a rectangular service zone which is motivated by Chang and Schonfeld [2,3]. however, in Chang and Schonfeld [2,3], the combination of cost and profit is the weighted arithmetic mean. On the other hand, in Imam [4], the combination of cost and profit is the weighted geometric mean that will result in the calculation of the optimal solution for the maximum profit problem becomes a severely difficult challenge.

We study the optimal solution for a traffic model under an exponential relation and rectangular service zones. There are two papers; Yang et al. [9] and Lin and Hopscotch [10] both have tried to solve the same problem. However, their solution procedure contained questionable results that will be explained in this paper. The same numerical example in Yang et al. [9] and Lin and Hopscotch [10] will be used to demonstrate their doubtful findings.

Yang et al. [9] pointed the formulated solution of Imam [4] is not workable and then Yang et al. [9] provided their formulated optimal solution for the service route length. However, Lin and Hopscotch [10] showed that Yang et al. [9] committed mathematical derivation mistakes such that their findings need revisions. In this paper, we will point out that Imam [4], Yang et al. [9] and Lin and Hopscotch [10] overlooked a fundamental criterion in the second model of Imam [4]. We will provide a detailed explanation for their negligence. The same numerical example in Yang et al. [9]

and Lin and Hopscotch [10] is adopted to demonstrate their negligence. We adopt the same notation and assumptions of Imam [4], Yang et al. [9] and Lin and Hopscotch [10], in the following notation.

W stands for the width of bus service zone measured in miles.

T stands for the time interval with bus service.

P stands for the local transit travelers conceded over area LW and time T.

L stands for the length of examined zone measured in miles.

F stands for the charge for the local bus service.

$a_{(\ )}$ stands for the coefficient of each mode choice.

v stands for the average bus velocity measured in miles.

t stands for the transit type divide.

s stands for the distance between two adjacent bus routes measured in miles.

q stand for the bus capacity.

p stands for journey density with all other modes.

$\ln$ stands for the mapping of natural logarithms.

k stands for the proportion of expected user waiting period to the headway.

j stands for the average on foot velocity per mile for per minute.

h stands for the headway on bus course measure in minutes.

d stands for the average passenger journey length measured in miles.

c stands for the bus operating fee for each hour.

b stands for the distance between bus stops along each route measured in miles.

We directly quote the findings of Lin and Hopscotch [10] as the starting point for a review of previous results. Lin and Hopscotch [10] obtained the objective function in two variables: headway, h and service route width s,

$$mP(h,s)=,$$

$$\left(\frac{B_2}{hs}-\frac{B_3}{h^2s^2\left(kh+\frac{s}{4j}+\frac{b}{4j}\right)^{a_2}}\right)\frac{q}{B_1} \quad (15.1)$$

with the following abbreviations,

$$B_2 = TpWa_1(d/v)^{a_3}F^{a_4+1}d^{a_5}, \quad (15.2)$$

$$B_3 = 2WTc/v, \quad (15.3)$$

and

$$B_1 = pa_1(d/v)^{a_3}F^{a_4}d^{a_5}. \quad (15.4)$$

XVI. Our Improvements

In Imam [4], Yang et al. [9], and Lin and Hopscotch [10], they considered traffic models for a rectangular service zone with length L and width W. Researchers uniformly partition width W into N parts such that we know the relationship,

$$\frac{W}{N}=s. \quad (16.1)$$

Hence, we derive that

$$\frac{W}{s}=N, \quad (16.2)$$

must be a natural number.

We use the numerical example of Yang et al. [9] and Lin and Hopscotch [10] that is cited from Kocur and Hendrickson [1] with the following data: $a_1=0.38$, $a_2=-0.0081$, $a_3=-0.0033$, $a_4=-0.0014$, $a_5=0.0328$, $b=0.2$, $c=0.5$, $d=3$, $j=0.05$, $k=0.4$, $p=3.59$, $q=45$, $T=60$, $v=0.167$, $F=0.74$ and $W=4$. In Yang et al. [9], they obtained that

$$h^*=9.142, \quad (16.3)$$

$$s^*=0.731, \quad (16.4)$$

and

$$L^*=4.885. \quad (16.5)$$

In Lin and Hopscotch [10], they derived that

$$s^*=0.9695, \quad (16.6)$$

$$h^*=12.1187, \quad (16.7)$$

$$L^*=2.7859, \quad (16.8)$$

and

$$mP(s^*,h^*,L^*)=339.4508. \quad (16.9)$$

We recall the results in Yang et al. [9] with $W=4$ and $s^*=0.731$ such that

$$\frac{W}{s^*}=5.47, \quad (16.10)$$

which is not a natural number. When $W = 4$, in Lin and Hopscotch [10], they found that $s^* = 0.9695$ so that

$$\frac{W}{s^*} = 4.13. \tag{16.11}$$

Both of the results in Equations (14.7) and (14.8) are not a natural number which is violated the restriction for the partition number must be a natural number.

We may point out several possible directions for the future study that is motivated by Yepes and Medina [56].

The most possible research direction is to solve $\frac{d}{dN} mP\left(h(N), N\right)$ by Equation (16.4) with $\frac{d}{dN} h(N)$ derived by Equation (16.3), and then we expect that

$$\frac{d}{dN} mP\left(h(N), N\right) = 0, \tag{16.12}$$

has a unique positive solution, which was denoted as N^{Δ}. We predict that

$$\frac{d}{dN} mP\left(h(N), N\right) > 0, \tag{16.13}$$

for $0 < N < N^{\Delta}$ and

$$\frac{d}{dN} mP\left(h(N), N\right) < 0, \tag{16.14}$$

for $N^{\Delta} < N < \infty$, such that N^{Δ} is the optimal solution for continuous restricted domain as mentioned in Spiegel [57]. For the discrete restriction, then $\lfloor N^{\Delta} \rfloor$ and $\lfloor N^{\Delta} \rfloor + 1$ are two candidates for the maximum solution for the natural number.

XVII. Conclusion

In this paper, we first provide an example to illustrate that sometimes the convergent sequence generated by an iterative method will converge to a solution that is not the desired goal. Moreover, we illustrate a new solution method that will use the bisection method to search for the optimal solution. We provide a pair of upper bound and lower bound to execute the convergent process. Our approach is examined by the same example used in Chang and Schonfeld [3] and Yang et al. [5] to indicate our approach can derive the optimal solution.

We also examined Guan and Zhao [42] to derive the matrix representation by Equation (10.10-10.12). We derive the exact carrying cost to revise Jalan et al. [55], and then for our proposed model, we prove the existence and uniqueness of the optimal solution.

We studied the bus transit system proposed by Imam [4], Yang et al. [9], and Lin and Hopscotch [10], to point out their negligence to check whether or not the rectangular width divide by their derived the route width will result in a natural number.

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