A New Fixed Point Iterative Method for Weak Contraction Mapping and Its Application to Integral Equations

Ahmad Ansar, Meryta Febrilian Fatimah, and Darma Ekawati.

Abstract—**This study aims to develop a novel approximation algorithm for estimating fixed points of weak contraction mappings in Banach spaces. We propose an iterative method that demonstrates strong convergence and stability for weak contraction mappings in closed and convex subsets of Banach spaces. Theoretical analysis proves that the proposed algorithm achieves a faster convergence rate compared to existing iterative schemes, thereby improving upon previous results. Additionally, we apply our results to estimate solutions for integral equations as its practical utility. Finally, numerical simulations are provided to validate the theoretical findings and highlight the efficiency of the algorithm.**

Index Terms—**fixed point, weak contraction, iterative method, convergence rate, integral equation.**

I. Introduction

FIXED point theory is a fundamental and useful mathematical concept that provides useful techniques to solve various problems in many different fields. A deep understanding of fixed points allows to make good prediction of complex behavior in real problems such as economics [1], [2], game theory [3], [4], [5], engineering [6], [7], and applied mathematics [8], [9], [10], [11], [12], etc. In fixed point theory, research focuses on showing the existence and then proving the uniqueness of fixed points of various kinds of contractive mappings and modification metric spaces. Afterward, once the existence of fixed points has been proved, we have to determine the fixed point of related mapping. However, finding a fixed point is a difficult task, especially for nonlinear operators. Hence, approximation theory appears to deal with this problem.

The first iterative method was introduced by Picard to find the fixed point of Banach contraction mapping. Since the rapid development and complexity of contraction mapping, many researchers have proposed some new iteration processes, such as Mann iteration [13], two-step iteration by Ishikawa [14], and Agarwal methods [15]. In 2000, Noor [16] proposed a three-step iteration process with a better convergence rate than previous methods. Afterward, the authors continue to compete to find better iterative schemes with fast convergence rates, so that fixed points can be obtained quickly. For more iteration processes can be seen in [17], [18], [19], [20], [21].

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Over the last few years, some researchers have proposed new iteration processes with high-rate convergence to approximate fixed points of weak contraction mapping [8]. Some authors are interested in weak contraction mapping in closed convex Banach spaces because it provides concepts to approximate the solution of integral equations. Integral equations can be formulated in the form of an operator or mapping in metric space whose solutions are in the form of fixed points. Hammad and Kattan [22] use fixed point of weak contraction to study the solution Volterra integral equations with delay. Also, Saif et.al. [23] used the Thakur algorithm scheme to obtain the approximate solution to boundary value problems. The use of integral equations in various fields encourages many researchers to study, develop, and prove new types of fixed points and estimate the solution [24], [25].

Based on the description, it is important to define a new iteration method that can be used to estimate the fixed point of weak contraction mapping. The strong convergence property and stability theorems will be investigated and studied. In addition, we compare the convergence rate with some other iteration schemes. The results obtained were applied to estimate the solution of integral equation problems. Also, some illustrative numerical simulations are given to support our main results.

II. Preliminaries

As the foundational knowledge, we give some definitions and theorems related to iterative fixed points of weak contractions mapping. Also, we assume that $\mathcal{A}$ is closed and convex Banach space and $\mathcal{B}$ is a subset of $\mathcal{A}$ and not empty. Consider $\mathcal{J}$ be self-mapping defined on $\mathcal{B}$ and $Fix(\mathcal{J})$ is the set that contains all of fixed points from mapping $\mathcal{J}$.

Berinde [26] proposed a new definition of a new type of contraction mapping called weak contraction mapping. The concept of weak contraction mapping is given as follows.

Definition II.1. [26] Suppose that $\mathcal{A}$ is Banach spaces. A mapping $\mathcal{J} : \mathcal{A} \to \mathcal{A}$ is defined as weak contraction mapping if there exist number $k \in (0, 1)$ and $K \geq 0$ satisfied the following conditions

$$\|\mathcal{J}r - \mathcal{J}s\| \leq k\|r - s\| + K\|s - \mathcal{J}r\| \tag{1}$$

for all $r, s \in \mathcal{A}$.

Some authors called weak contraction mapping with almost contraction mapping. This mapping type does not have a single fixed point unless the condition of the following theorems is satisfied. The following theorem shows that almost contraction mapping just has a single fixed point with particular conditions.

Theorem II.1. *[26] Consider that $\mathcal{A}$ is Banach spaces and $\mathcal{J} : \mathcal{A} \to \mathcal{A}$ is defined as weak contraction mapping that satisfied (1). If mapping $\mathcal{J}$ satisfying*

$$\|\mathcal{J}r - \mathcal{J}s\| \leq k\|r - s\| + K\|r - \mathcal{J}s\| \tag{2}$$

for any $r, s \in \mathcal{A}$, then weak contraction $\mathcal{J}$ has fixed point in $\mathcal{A}$. Furthermore, the self-mapping $\mathcal{J}$ just has single fixed point.

Example II.1. Given $\mathbb{R}$ equipped with usual norm. The pair $(\mathbb{R}, |.|)$ is a Banach spaces. Supposed $\mathcal{B} = [0, 100]$ and $\mathcal{J} : Y \to Y$ be a mapping defined by $\mathcal{J}a = a - 1 + e^{-a}$. For any $a, b \in \mathcal{B}$, we obtain

$$|\mathcal{J}a - \mathcal{J}b| \leq (1 - e^{-a})|a - b| + 0|b - (a - 1 + e^{-a})| \tag{3}$$

Hence, $\mathcal{J}a$ is weak contraction mapping on Y. Furthermore, we get

$$|\mathcal{J}a - \mathcal{J}b| \leq \frac{1}{2}|a - b| + |a - (a - 1 + e^{-b})| \tag{4}$$

So, $\mathcal{J}$ has fixed point on $\mathcal{B}$.

Fixed points of weak contraction mapping can usually be gained numerically using an iterative process. Some researchers have introduced and investigated some iterative algorithms to find the approximate value of fixed points of weak contraction mapping. Suppose that given three positive real sequences $\{\tau_n\}, \{\sigma_n\}, \{\rho_n\}$ on the interval $(0, 1)$ and $u_1 \in \mathcal{B}$ as initial value. Thakur, at. al. [27] proposed a new three-step iteration algorithm with definition below:

$$\begin{cases} u_1 \in \mathcal{B} \\ w_n = (1 - \tau_n)u_n + \tau_n \mathcal{J}u_m \\ v_n = \mathcal{J}((1 - \sigma_n)u_n + \sigma_n w_n) \\ u_{n+1} = \mathcal{J}v_n. \end{cases} \tag{5}$$

Afterward, Ali, et. al. [28] defined a two-step iteration algorithm and called F^* iteration process with sequence $\{u_n\}$ defined by

$$\begin{cases} u_1 \in \mathcal{B} \\ v_n = \mathcal{J}(1 - \tau_n)u_n + \tau_n \mathcal{J}u_n) \\ u_{n+1} = \mathcal{J}v_n. \end{cases} \tag{6}$$

Jubair, et. al. [29] also defined a new kind of estimation process to obtained fixed point of almost contraction mapping. The algorithm was developed by following sequences

$$\begin{cases} u_1 \in \mathcal{B} \\ w_n = (1 - \tau_n)u_n + \tau_n \mathcal{J}u_n \\ v_n = (1 - \sigma_n)\mathcal{J}u_n + \sigma_n w_n \\ u_{n+1} = \mathcal{J}((1 - \rho_n)\mathcal{J}v_n + \rho_n \mathcal{J}v_n). \end{cases} \tag{7}$$

Piri, et. al. [30] introduced another iteration process as follows

$$\begin{cases} u_1 \in \mathcal{B} \\ w_n = \mathcal{J}(1 - \tau_n)u_n + \tau_n \mathcal{J}u_n \\ v_n = \mathcal{J}w_n \\ u_{n+1} = (1 - \rho_n)\mathcal{J}w_n + \rho_n \mathcal{J}v_n. \end{cases} \tag{8}$$

In 2021, Hussain, et. al. [31] proposed D iteration process according to sequences defined by

$$\begin{cases} u_1 \in \mathcal{B} \\ w_n = \mathcal{J}((1 - \tau_n)u_n + \tau_n \mathcal{J}u_n) \\ v_n = \mathcal{J}((1 - \sigma_n)\mathcal{J}u_n + \sigma_n \mathcal{J}w_n) \\ u_{n+1} = \mathcal{J}v_n. \end{cases} \tag{9}$$

The convergence rate of the iteration process can be investigated using the definition below.

Definition II.2. [32] Let sequences $\{\tau_n\}, \{\sigma_n\} \subset \mathbb{R}$ are sequences that converges to τ and σ, respectively. Consider that

$$\lim_{n\to\infty} \frac{|\tau_n - \tau|}{|\sigma_n - \sigma|} = L \tag{10}$$

1) The number $L = 0$, implies that sequence $\{\tau_n\}$ converges to τ faster than $\{\sigma_n\}$ to σ;
2) The number $0 < L < \infty$, implies that sequence $\{\tau_n\}$ and $\{\sigma_n\}$ have the same rate of convergence.

Definition II.3. [32] Supposed $\{q_n\}$ and $\{p_n\}$ are two estimation process that converges to the similar point p and satisfied

$$|q_n - p| \leq \tau_n \text{ and } |p_n - p| \leq \sigma_n$$

If $\lim_{n\to\infty} \frac{\tau_n}{\sigma_n} = 0$, then $\{q_n\}$ converges to point p faster than $\{p_n\}$ to point p.

The stability of the iteration process can be investigated using a definition by Ostrowoski that states as follows:

Definition II.4. [33] Supposed $\mathcal{B}$ be a nonempty subset of Banach spaces $\mathcal{A}$ and $\mathcal{J} : \mathcal{B} \to \mathcal{B}$ is a mapping that has fixed point p. Consider that $p_0 \in \mathcal{B}$ and $u_{n+1} = \mathcal{H}(T, u_n)$ is an algorithmic process related to $\mathcal{H}$. Consider $\{x_n\}$ be a sequence that estimated $\{u_n\}$. If $\mu_n = \|x_{n+1} - \mathcal{H}(T, x_n)\|$, then $u_{n+1} = \mathcal{H}(T, u_n)$ stable in respect to T (T-stable) if satisfied

$$\lim_{n\to\infty} \mu_n = 0 \Longleftrightarrow \lim_{n\to\infty} x_n = p.$$

Afterward, Osilike proposed the notion of almost stable of iteration methods which generalised the notion of stability from Ostrowski. The definition is as follows.

Definition II.5. [34] Supposed $\mathcal{B}$ be a nonempty subset of Banach spaces $\mathcal{A}$ and $\mathcal{J} : \mathcal{B} \to \mathcal{B}$ is a mapping that has fixed point p. Consider that $p_0 \in \mathcal{B}$ and $u_{n+1} = \mathcal{H}(T, u_n)$ is an algorithmic process related to $\mathcal{H}$. Consider $\{x_n\}$ be a sequence that estimated $\{u_n\}$. If $\mu_n = \|x_{n+1} - \mathcal{H}(T, x_n)\|$, then $u_{n+1} = \mathcal{H}(T, u_n)$ stable in respect to T (T-stable) if satisfied

$$\sum_{n=0}^{\infty} \mu_n < \infty \Longrightarrow \lim_{n\to\infty} x_n = p.$$

The result from Berinde gives a powerful tool to show the stability of the iteration process in the main theorems.

Lemma II.2. *[35] Suppose $\{\tau_n\}$ and $\{\sigma_n\}$ be nonegatives real sequences and $0 \leq c < 1$, such that for all nonnegatives integer number n satisfied $\tau_{n+1} \leq c\tau_n + \sigma_n$. If $\sum_{n=0}^{\infty} \sigma_n = 0$ then $\sum_{n=0}^{\infty} \tau_n = 0$.*

III. Convergence and Stability Results

We start our discussion by defined a new method of iterative scheme and show that it strongly converges to a unique fixed point of contraction mapping. We also prove that our iteration method has a better convergence rate than the previous iteration methods mentioned before by many authors.

First, the new iteration process to estimate the fixed point of weak contraction mapping is defined as follows and we called as A-plus iteration.

$$\begin{cases} u_1 \in \mathcal{B} \\ w_n = \mathcal{J}((1-\tau_n)u_n + \tau_n \mathcal{J}u_n) \\ v_n = \mathcal{J}((1-\sigma_n)\mathcal{J}u_n + \sigma_n \mathcal{J}w_n) \\ u_{n+1} = \mathcal{J}((1-\rho_n)\mathcal{J}w_n + \rho_n \mathcal{J}v_n) \end{cases} \tag{11}$$

Using the iteration algorithm (11), we prove that the A-plus iteration algorithm has a higher convergence rate than some other iteration methods. We also prove that the iteration A-plus iteration process has a stable property with respect to $\mathcal{J}$. Furthermore, we give some numerical examples and simulations to show that A-plus iteration is better than others in approximate fixed points of weak contraction mapping.

Theorem III.1. *Suppose that $\mathcal{B}$ be a nonempty closed and convex subset of Banach space $\mathcal{A}$ and $\mathcal{J}:\mathcal{B}\to\mathcal{B}$ be weak contraction mapping that satisfied* (2). *If $\{u_n\}$ be a sequnce defined by A-plus iterative scheme with $\{\tau_n\},\{\sigma_n\},\{\rho_n\}$ are real sequences in $(0,1)$, then $\{u_n\}$ converges strongly to unique fixed point of $\mathcal{J}$.*

Proof: We know that $\mathcal{J}$ is weak contraction mapping that satisfied (2), so $\mathcal{J}$ has one fixed point in $\mathcal{B}$. Let $t \in Fix(\mathcal{J})$. So, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|\mathcal{J}u_n - t\| &= \|\mathcal{J}u_n - \mathcal{J}t\| \\ &\le k\|u_n - t\| + K\|t - \mathcal{J}u_n\| \\ &= k\|u_n - t\| \end{aligned} \tag{12}$$

According to definition of A-plus iteration process (11), then

$$\begin{aligned} \|w_n - t\| &= \|\mathcal{J}((1-\tau_n)u_n + \tau_n\mathcal{J}u_n) - \mathcal{J}t\| \\ &\le k\|(1-\tau_n)u_n + \tau_n\mathcal{J}u_n - t\| \\ &= k\|(1-\tau_n)(u_n - t) + \tau_n(\mathcal{J}u_n - t)\| \\ &\le [(1-\tau_n)\|u_n - t\| + \tau_n\|\mathcal{J}u_n - t\|] \\ &\le [(1-\tau_n)\|u_n - t\| + \tau_n k\|u_n - t\|] \\ &= k(1-\tau_n(1-k))\|u_n - t\| \end{aligned} \tag{13}$$

Using the last inequality (13), we have

$$\begin{aligned} \|v_n - t\| &= \|\mathcal{J}((1-\sigma_n)\mathcal{J}u_n + \sigma_n\mathcal{J}w_n) - \mathcal{J}t\| \\ &\le k\left[(1-\sigma_n)\|\mathcal{J}u_n - t\| + \sigma_n\|\mathcal{J}w_n - t\|\right] \\ &\le k\left[(1-\sigma_n)k\|u_n - t\| + \sigma_n k\|w_n - t\|\right] \\ &\le k^2\left[(1-\sigma_n)\|u_n - t\| + \right. \\ &\quad \left. \sigma_n k(1-\tau_n(1-k))\|u_n - t\|\right] \\ &\le k^2\left[1-(\sigma_n + k\sigma\tau_n)(1-k)\right]\|u_n - t\| \end{aligned}$$

Also, it follows from A-plus iteration, we get

$$\begin{aligned} \|u_{n+1} - t\| =& \|\mathcal{J}((1-\rho_n)\mathcal{J}w_n + \rho_n\mathcal{J}v_n) - \mathcal{J}t\| \\ \le& k\|(1-\rho_n)\mathcal{J}w_n + \rho_n\mathcal{J}v_n - t\| \\ \le& k\left[(1-\rho_n)\|\mathcal{J}w_n - p\| + \rho_n\|\mathcal{J}v_n - t\|\right] \\ \le& k\left[(1-\rho_n)k\|w_n - t\| + \rho_n k\|v_n - t\|\right] \\ \le& k^2[(1-\rho_n)k(1-\tau_n(1-k))\|u_n - t\| + \\ & \rho_n k^2\left[1-(\sigma_n + k\sigma\tau_n)(1-k)\right]\|u_n - t\| \\ =& k^3[(1-\rho_n)(1-\tau_n(1-k))\|u_n - t\| + \\ & \rho_n k\left[1-(\sigma_n + k\sigma\tau_n)(1-k)\right]\|u_n - t\| \end{aligned}$$

Since $0 < k < 1$ and sequences $\{\tau_n\},\{\sigma_n\},\{\rho_n\} \in (0,1)$, then $(1-\tau_n(1-k)) < 1$ and also $[1-(\sigma_n + k\sigma\tau_n)(1-k)] < 1$. Therefore, we get

$$\begin{aligned} \|u_{n+1} - t\| &\le k^3[(1-\rho_n) + \rho_n k]\|u_n - t\| \\ &= k^3(1-\rho_n(1-k))\|u_n - t\| \end{aligned} \tag{14}$$

Now, by using the last inequalities (14), we have

$$\begin{aligned} \|u_n - t\| &\le k^3[1-\rho_{n-1}(1-k)]\|u_{n-1} - t\| \\ \|u_{n-1} - t\| &\le k^3[1-\rho_{n-2}(1-k)]\|u_{n-2} - t\| \\ &\vdots \\ \|u_2 - t\| &\le k^3[1-\rho_1(1-k)]\|u_1 - t\| \\ \|u_1 - t\| &\le k^3[1-\rho_0(1-k)]\|u_0 - t\| \end{aligned}$$

Hence, the inequality (14) becomes

$$\|u_{n+1} - t\| \le k^{3(n+1)}\|u_0 - t\|\prod_{i=0}^{n}[1-\rho_i(1-k)] \tag{15}$$

Since $\rho_i(1-k) < 1$ and using inequalities $1-\lambda \le e^{-\lambda}$ for all $\lambda \in [0,1]$, we get

$$\|u_{n+1} - t\| \le k^{3(n+1)}\|u_0 - t\|e^{-(1-k)\sum_{i=0}^{n}\rho_i} \tag{16}$$

So,

$$\lim_{n\to\infty}\|u_{n+1} - t\| \le \lim_{n\to\infty} k^{3(n+1)}\|u_0 - t\|e^{-(1-k)\sum_{i=0}^{n}\rho_i} \tag{17}$$

or $\lim_{n\to\infty}\|u_n - t\| = 0$. Thus, sequences $\{u_n\}$ converges strongly to a unique fixed point of $\mathcal{J}$. ■

The following result proves that the A-plus iteration process (11) has property almost stable with respect to a weak contraction mapping $\mathcal{J}$.

Theorem III.2. *A self-mapping $\mathcal{A}$ and $\mathcal{J}:\mathcal{B}\to\mathcal{B}$ be weak contraction mapping that satisfied* (2) *with $\mathcal{B}$ be a nonempty closed and convex subset of Banach space , then A-plus iterative scheme* (11) *is almost stable with respect to a weak contraction mapping $\mathcal{J}$.*

Proof: Consider a sequance $\{u_n\}$ that generated by A-plus iteration scheme (11) and $\{s_n\}$ be a sequence that approximate sequence $\{u_n\}$. Consider that

$$u_{n+1} = \mathcal{G}(\mathcal{J}, u_n)$$

and

$$\lambda_n = \|s_{n+1} - \mathcal{G}(\mathcal{J}, u_n)\|$$

for all $n \in \mathbb{N}$. In this context, we will show that

$$\sum_{n=0}^{\infty} \lambda_n < \infty \text{ implies } \lim_{n\to\infty} s_n = t$$

We have

$$\begin{aligned}\|s_{n+1} - t\| &\le \|s_{n+1} - \mathcal{G}(\mathcal{J}, s_n)\| + \|\mathcal{G}(\mathcal{J}, s_n) - t\| \\ &= \lambda_n + \|s_{n+1} - t\| \\ &\le \lambda_n + k^3(1 - \rho_n(1-k))\|s_n - t\|\end{aligned}$$

Since $(1 - \rho_n(1-k)) < 1$, then

$$\|s_{n+1} - t\| \le k^3\|s_n - t\| + \lambda_n$$

Since $\sum_{n=0}^{\infty} \lambda_n < \infty$, then $\sum_{n=0}^{\infty} \|s_n - t\| < \infty$. Hence,

$$\lim_{n\to\infty} \|s_n - t\| = 0 \text{ or } \lim_{n\to\infty} s_n = t$$

Therefore, A-plus iteration process (11) is almost stable with respect to weak contraction $\mathcal{J}$. ■

The following theorem investigates the rate of convergence of A-plus iteration algorithm compared with iteration process mentioned before for almost contraction mapping.

Theorem III.3. *Consider that $\mathcal{B} \subset \mathcal{A}$ with $\mathcal{A}$ is a closed and convex subset of Banach space and $\mathcal{J} : \mathcal{B} \to \mathcal{B}$ be weak contraction mapping that satisfied* (2). *Suppose that* $\{u_{(1);n}\}$, $\{u_{(2);n}\}$, $\{u_{(3);n}\}$, *and* $\{u_{(4);n}\}$ *be sequences defined by Thakur* (5), *Ali, et. al.* (6) *(F^* iteration), Jubair, et.al.* (7), *Piri, et.al.* (8) *respectively and* $\{u_n\}$ *be a sequence defined by A-plus algorithm* (11) *with* $\{\tau_n\}$, $\{\sigma_n\}$, $\{\rho_n\} \in [0, 1]$. *If t is a fixed point of $\mathcal{J}$, then A-plus iteration method converges to t faster than other iteration processes mentioned before.*

Proof: Consider the inequalities (15) in Theorem III.1, so

$$\|u_{n+1} - t\| \le k^{3(n+1)}\|u_0 - t\| \prod_{i=0}^{n}[1 - \rho_i(1-k)]$$

Since $1 - \rho_i(1-k) < 1$ for all $0 \le i \le n$, then

$$\|u_{n+1} - t\| \le k^{3(n+1)}\|u_0 - t\|$$

Let $\tau_n = k^{3(n+1)}\|u_0 - p\|$.
From iteration process defined by Thakur, et.al. (5), then

$$\begin{aligned}\|u_{(1);n+1} - t\| &\le k^2[1 - \tau_n\sigma_n(1-k)]\|u_{(1);n} - t\| \\ &\le k^{2(n+1)}\|u_{(1);0} - t\| \prod_{i=0}^{n}[1 - \tau_n\sigma_n(1-k)]\end{aligned}$$

Since $[1 - \tau_n\sigma_n(1-k)] < 1$ for all $0 \le i \le n$, then

$$\|u_{(1);n+1} - t\| \le k^{2(n+1)}\|u_{(1);0} - t\|$$

Let $\tau_{(1);n} = k^{2(n+1)}\|u_{(1);0} - t\|$. Therefore,

$$\lim_{n\to\infty} \frac{\tau_n}{\tau_{(1);n}} = \lim_{n\to\infty} \frac{k^{3(n+1)}\|u_0 - t\|}{k^{2(n+1)}\|u_{(1);0} - t\|} = 0$$

Therefore, according to Definition II.3, sequence $\{u_n\}$ has better convergence rate than sequence $\{u_{(1);n}\}$ to fixed point t.
Next, as showed by Ali, et.al. (6), gives

$$\|u_{(2);n+1} - t\| \le k^2(1 - (1-k)\tau_n)\|u_{(2);n} - t\|$$

Since $(1 - (1-k)\tau_n) < 1$, show

$$\|u_{(2);n+1} - t\| \le k^{2(n+1)}\|u_{(2);0} - t\|$$

Let $\tau_{(2);n} = k^{2(n+1)}\|u_{(2);0} - t\|$. We get

$$\lim_{n\to\infty} \frac{\tau_n}{\tau_{(2);n}} = \lim_{n\to\infty} \frac{k^{3(n+1)}\|u_0 - t\|}{k^{2(n+1)}\|u_{(2);0} - t\|} = 0$$

It shows that sequence $\{u_n\}$ converges faster than sequence $(u_{(2);n})_{n\ge1}$ to fixed point t. Also, from Jubair, et. al. (7), we get

$$\|u_{(3);n+1} - t\| \le k^2(1 - (1-k)\rho_n)(1 - (1-k)\tau_n\sigma_n)$$
$$\|u_{(3);n} - t\|$$

Since $(1 - (1-k)\rho_n) < 1$ and $(1 - (1-k)\tau_n\sigma_n) < 1$, then

$$\|u_{(3);n+1} - t\| \le k^{2(n+1)}\|u_{(3);0} - t\|$$

Let $\tau_{(3);n} = k^{2(n+1)}\|u_{(3);0} - t\|$. We get

$$\lim_{n\to\infty} \frac{\tau_n}{\tau_{(3);n}} = \lim_{n\to\infty} \frac{k^{3(n+1)}\|u_0 - t\|}{k^{2(n+1)}\|u_{(3);0} - t\|} = 0$$

It shows that iterative scheme defined by $\{u_n\}$ has better convergence rate than sequence $\{u_{(3);n}\}$ to fixed point t. Result from Piri, et. al. (8), gives

$$\|u_{(4);n+1} - t\| \le k^2(1 - (1-k)\tau_n)(1 - (1-k)\sigma_n)$$
$$\|u_{(4);n} - t\|$$

Since $(1 - (1-k)\tau_n) < 1$ and $(1 - (1-k)\sigma_n) < 1$, then

$$\|u_{(4);n+1} - t\| \le k^{2(n+1)}\|u_{(4);0} - t\|$$

Let $\tau_{(4);n} = k^{2(n+1)}\|u_{(4);0} - t\|$. We get

$$\lim_{n\to\infty} \frac{\tau_n}{\tau_{(4);n}} = \lim_{n\to\infty} \frac{k^{3(n+1)}\|u_0 - t\|}{k^{2(n+1)}\|u_{(4);0} - t\|} = 0$$

It means that iterative scheme defined by $\{u_n\}$ is better than the sequence $\{u_{(4);n}\}$ in relation to convergence rate to fixed point t. Therefore, A-plus iteration method converges to fixed point of $\mathcal{J}$ faster than algorithms defined by Thakur, et. al., Ali, et.al., Jubair, et.al., Piri, et.al. related to weak contraction mapping. ■

The following theorem aims to compare the rate of convergence between A-plus scheme in (11) and D iterative method defined by Hussain et. al.

Theorem III.4. *Suppose that $\mathcal{B}$ be a nonempty closed and convex subset of Banach space $\mathcal{A}$ and $\mathcal{J} : \mathcal{B} \to \mathcal{B}$ be weak contraction mapping that satisfied* (2). *Suppose that* $\{u_{(5);n}\}$ *is D iteration process proposed by Hussain et. al. [31] and* $\{u_n\}$ *be A-plus iteration scheme with* $\{\tau_n\}$,$\{\sigma_n\}$, $\{\rho_n\} \in (0, 1)$ *are control sequnces and satisfying $\rho_n > \tau_n\sigma_n$ for all natural numbers n. Then A-plus iteration method is faster than D iteration process.*

Proof: From inequalities 15 in Theorem III.1, we get

$$\|u_{n+1} - t\| \le k^{3(n+1)}\|u_0 - t\| \prod_{i=1}^{n}(1 - \rho_i(1-k))$$

For some $i \in \{0, 1, 2, ..., n\}$, then

$$\|u_{n+1} - t\| \le k^{3(n+1)}\|u_0 - t\|(1 - \rho_i(1-k))^{(n+1)}$$

Let $\tau_n = k^{3(n+1)}\|u_0 - t\|(1-\rho_i(1-k))^{(n+1)}$.
As shown by Hussain, et.al. (9), we have

$$\begin{aligned}\|u_{(5);n+1} - t\| &\le k^3[(1-\sigma_n) + \sigma_n k(1-\tau_n+\tau_n k)] \\ &\quad \|u_{(5);n} - t\| \\ &\le k^3[(1-\sigma_n) + \sigma_n(1-\tau_n+\tau_n k)] \\ &\quad \|u_{(5);n} - t\| \\ &= k^3[1-\sigma_n\tau_n + \tau_n\sigma_n k)]\|u_{(5);n} - t\| \\ &= k^3[1-\tau_n\sigma_n(1-k)]\|u_{(5);n} - t\|\end{aligned}$$

Hence,

$$\|u_{(5);n+1} - t\| \le k^{3(n+1)}\|u_{(5);0} - t\| \prod_{i=0}^{n}(1-\tau_i\sigma_i(1-k))$$

For some $i \in \{0,1,2,...,n\}$, we have

$$\|u_{(5);n} - t\| \le k^{3(n+1)}\|u_{(5);0} - t\|(1-\tau_i\sigma_i(1-k))^{n+1}$$

Let $\tau_{(5);n} = k^{3(n+1)}\|u_{(5);0} - t\|(1-\tau_i\sigma_i(1-k))^{n+1}$.
Therefore,

$$\frac{\tau_n}{\tau_{(5);n}} = \frac{\|u_0 - t\|(1-\rho_i(1-k))^{(n+1)}}{\|u_{(5);0} - t\|(1-\tau_i\sigma_i(1-k))^{n+1}}$$

Since $\rho_n > \tau_n\sigma_n$ for all $n \in \mathbb{N}$, then

$$(1-\rho_i(1-k)) < (1-\tau_i\sigma_i(1-k))$$

Taking limit for $n \to \infty$, we get $\lim_{n\to\infty}\frac{\tau_n}{\tau_{(5);n}} = 0$. So, for given conditions, A-plus iteration scheme converges faster than D iteration process to fixed point of T. ■

From the above theorem, A-plus iteration process and D iteration are three-step iteration algorithms. The selected sequences must satisfy the condition $\rho_n > \tau_n\sigma_n$ in order to reach the fixed point of mapping quickly.

The example below supports our previous result.

Example III.1. Let $\mathcal{B} = [0, 50]$ be subset of all real numbers with usual norm. The set $\mathcal{B}$ is a Banach space. If $\mathcal{J} : \mathcal{B} \to \mathcal{B}$ be a self-mapping such that for all $u \in \mathcal{B}$ satisfied $\mathcal{J}u = u + 1 - \ln u$. Then $\mathcal{J}$ is a weak contraction mapping that satisfied (2). The number e is a fixed point of $\mathcal{J}$ [36].

We will estimate the fixed point $e \approx 2.718282$ of weak contraction $\mathcal{J}$ and show that A-plus iteration process converges faster to the point 2.718282 than the others iteration scheme mentioned before. Choose initial value $u_0 = 40$ and sequences $(\tau_n)_{n\ge1} = 0.5$, $(\sigma_n)_{n\ge1} = 0.4$, $(\rho_n)_{n\ge1} = 0.6$ are sequences in $(0, 1)$. All conditions in Theorem III.3 and Theorem III.4 are satisfied. Therefore, we can approximate the fixed point of $\mathcal{J}$ using iterative methods in III.3 and III.4. The result from numerical simulation is shown in Table I and Figure 1.

From Table I, the A-plus iteration process demonstrates a superior convergence rate compared to other iterative algorithms. Specifically, the A-plus method requires only 15 iterations to converge to the fixed point of T, whereas the D iteration needs 17 iterations, and the method by Piri et. al. requires 20 iterations. The remaining algorithms (Thakur et al., F^* Iteration, Jubair et al.) exhibit slower convergence, exceeding 20 iterations. Also, Figure 1 supports these findings by illustrating the stability of all iterative methods under the weak contraction mapping $\mathcal{J}$. The graph confirms that the values stabilize without significant change as the number of iterations increases, ultimately converging to the fixed point.

IV. Applications to Integral Equation

The application of fixed point theory often encounters problems with integral equations. Integral equations can be defined as an operator that satisfies certain conditions that can be investigated to determine the existence of a fixed point which is the solution of integral problems. We will use results from the iteration process (11) to obtain the estimation solution of the following type integral equations

$$\mathcal{H}(t) = \mathcal{F}(t) + \int_x^y \mathcal{G}(t,p,\mathcal{H}(p))\,dp \tag{18}$$

for all $t, p \in [x, y]$, where $\mathcal{F} : [x, y] \to \mathbb{R}$ and $\mathcal{H} : [x, y]^2 \times \mathbb{R} \to \mathbb{R}$ are continuous functions.
Let $\mathcal{X}$ be the set of all functions that continuous on interval $[x, y]$ and written as $\mathcal{C}[x, y]$. Suppose the norm $\|.\|_\infty$ defined by $\|\mathcal{G} - \mathcal{J}\|_\infty = \sup_{t\in[a,b]}\{|\mathcal{G}(t) - \mathcal{J}(t)|\}$. Clearly, $(X, \|.\|_\infty)$ is Banach spaces. The following theorem shows that A-plus scheme converges to the solution mapping that satisfied conditions (1) and (2).

Theorem IV.1. *Let $\mathcal{X}$ be Banach space and $\mathcal{Y}$ be a nonempty closed and convex subset of $\mathcal{X}$ and $\{u_n\}$ be a sequence defined by A-plus iteration scheme* (11)*. Suppose that $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is an operator defined by*

$$\mathcal{T}(\mathcal{H}(t)) = \mathcal{F}(t) + \int_x^y \mathcal{G}(t,p,\mathcal{H}(p))\,dp \tag{19}$$

Consider that the conditions below hold:

1) *the function $\mathcal{F}(t)$ and $\mathcal{G} : [x, y]^2 \times \mathbb{R} \to \mathbb{R}$ are continuous functions;*
2) *there is constant number $\tau > 0$ such that*

$$|\mathcal{G}(t,p,\mathcal{H}(p)) - \mathcal{G}(t,p,\mathcal{J}(p))| \le \tau\,|\mathcal{H}(p) - \mathcal{J}(p)| \tag{20}$$

3) *for all $t, p \in [x, y]$, $\tau(y - x) < 1$*

Then the integral equation (18) *has a single solution $p^* \in \mathcal{C}[x, y]$. If $\mathcal{T}$ is a operator that hold conditions* (1) *and* (2)*, then $\{u_n\}$ converges to solution of integral equations* (18)*.*

Proof: Consider any $t, p \in [x, y]$ and mapping $\mathcal{H}, \mathcal{J} \in \mathcal{Y}$, then

$$\begin{aligned}&|\mathcal{T}(\mathcal{H}(t)) - \mathcal{T}(\mathcal{J}(t))| \\ &= \left|\mathcal{F}(t) + \int_x^y \mathcal{G}(t,p,\mathcal{H}(p))\,dp - (\mathcal{F}(t) + \int_x^y \mathcal{G}(t,p,\mathcal{J}(p))\,dp\right)\right| \\ &\le \int_x^y |\mathcal{G}(t,p,\mathcal{H}(p))\,dp - \mathcal{G}(t,p,\mathcal{J}(p))|\;dp \\ &\le \tau\int_x^y |\mathcal{H}(p) - \mathcal{J}(p)|\;dp \\ &= \tau\int_x^y \sup_{t\in[a,b]}|\mathcal{H}(t) - \mathcal{J}(t)|\;dp \\ &= \tau(y-x)\|\mathcal{H}(t) - \mathcal{J}(t)\|_\infty\end{aligned}$$

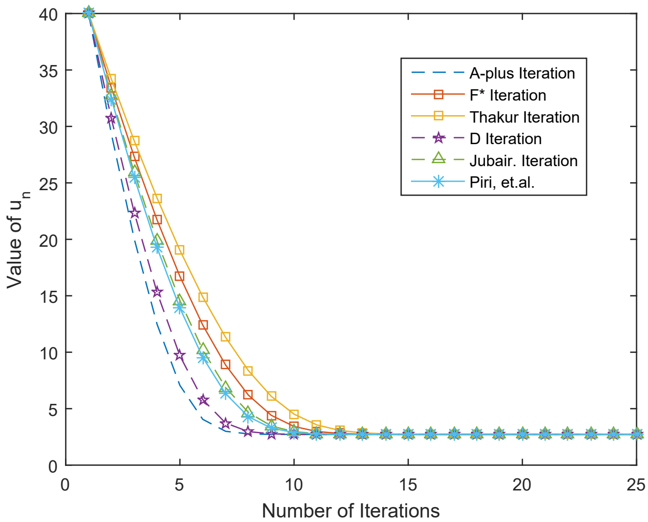

Fig. 1. Graphical representation of convergence rate behavior of some iteration methods for Example III.1

TABLE I
CONVERGENCE RATE OF SOME ITERATIVE ALGORITHM TO FIXED POINT OF WEAK CONTRACTION MAPPING IN EXAMPLE III.1

No.	Thakur, et. al.	F^* Iter	Jubair, et.al.	Piri, et.al.	D Iter	A-plus
1	40.000000	40.000000	40.000000	40.000000	40.000000	40.000000
2	34.181739	33.417327	32.662207	32.413697	30.675096	29.313403
3	28.719542	27.301102	25.918257	25.467707	22.371145	20.029232
4	23.652489	21.710676	19.854400	19.258920	15.288624	12.475957
5	19.026713	16.717534	14.576334	13.906610	9.679942	7.059590
6	14.896217	12.406176	10.208676	9.550581	5.806661	4.055887
7	11.322473	8.870418	6.879647	6.328003	3.729241	2.996176
8	8.370361	6.196074	4.664199	4.293443	2.967886	2.765450
9	6.095226	4.413779	3.466752	3.279066	2.771837	2.725895
10	4.513480	3.423229	2.964073	2.890725	2.729315	2.719500
11	3.559663	2.976471	2.792513	2.767830	2.720534	2.718476
12	3.072267	2.806398	2.740002	2.732193	2.718741	2.718313
13	2.857217	2.747478	2.724574	2.722161	2.718375	2.718287
14	2.770956	2.727853	2.720099	2.719361	2.718301	2.718283
15	2.737961	2.721408	2.718806	2.718582	2.718286	2.718282
16	2.725592	2.719302	2.718433	2.718365	2.718283	2.718282
17	2.720991	2.718615	2.718325	2.718305	2.718282	2.718282
18	2.719285	2.718390	2.718294	2.718288	2.718282	2.718282
19	2.718653	2.718317	2.718285	2.718284	2.718282	2.718282
20	2.718419	2.718293	2.718283	2.718282	2.718282	2.718282
21	2.718333	2.718286	2.718282	2.718282	2.718282	2.718282
22	2.718301	2.718283	2.718282	2.718282	2.718282	2.718282
23	2.718289	2.718282	2.718282	2.718282	2.718282	2.718282
24	2.718284	2.718282	2.718282	2.718282	2.718282	2.718282
25	2.718283	2.718282	2.718282	2.718282	2.718282	2.718282

Since $\tau(b-a) < 1$, then $\mathcal{T}$ is contraction mapping and has one fixed point. Hence, the integral equation (18) has a single solution in $\mathcal{C}[x, y]$. Next, we will prove that the sequence $\{u_n\}$ defined by (11) converges to a solution of the integral equation. Suppose that

$$
\begin{aligned}
&|\mathcal{T}(\mathcal{H}(t)) - \mathcal{T}(\mathcal{J}(t))| \\
&\leq |\mathcal{T}(\mathcal{H}(t)) - \mathcal{H}(t)| + |\mathcal{H}(t) - \mathcal{T}(\mathcal{J}(t))| \\
&= |\mathcal{T}(\mathcal{H}(t)) - \mathcal{H}(t)| + \Big|\mathcal{F}(t) + \int_x^y \mathcal{G}(t, p, \mathcal{H}(p))\, dp - \\
&\quad \mathcal{F}(t) + \int_x^y \mathcal{G}(t, p, \mathcal{H}(p))\, dp\Big| \\
&\leq \sup_{t\in[x,y]} |\mathcal{T}(\mathcal{H}(t)) - \mathcal{H}(t)| + \tau \int_x^y \sup_{p\in[x,y]} |\mathcal{H}(p) - \mathcal{J}(p)|
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&|\mathcal{T}(\mathcal{H}(t)) - \mathcal{T}(\mathcal{J}(t))| \\
&\leq \|\mathcal{T}(\mathcal{G}(t)) - \mathcal{J}(t)\|_\infty + \tau \int_a^b \sup_{t\in[a,b]} |\mathcal{G}(t) - \mathcal{J}(t)|\, dp \\
&= \|\mathcal{T}(\mathcal{H}(t)) - \mathcal{J}(t)\|_\infty + \tau \int_x^y \|\mathcal{H}(t) - \mathcal{J}(t)\|_\infty\, dp \\
&= \|\mathcal{T}(\mathcal{H}(t)) - \mathcal{J}(t)\|_\infty + \tau(y-x)\|\mathcal{H}(t) - \mathcal{J}(t)\|_\infty
\end{aligned}
$$

Clearly, that mapping $\mathcal{T}$ is weak contraction mapping and satisfied condition (2). According to Theorem III.1, the sequence $\{u_n\}$ defined by (11) converges to the solution of integral equation (18). ■

The following an example are given to show that the previous theorem is a very powerful method to gain the solution of integral problems.

Example IV.1. Suppose the following integral equation

$$\mathcal{H}(t) = (1-2t^2) + \int_0^1 \frac{1}{3} t^2 p \mathcal{H}(p)\, dp \tag{21}$$

We will approximate the solution of integral equation using iterative algorithm defined in (11).

This integral equation is an example of integral equation (18) with $\mathcal{F}(t) = 1-2t^2$ and $\mathcal{G}(t) = \frac{1}{3}t^2 p \mathcal{H}(p)$ for all $t, p \in [0,1]$. The exact solution of above integral (21) is $\mathcal{H}(t) = 1-2t^2$ for $t \in [0,1]$. Consider that $\mathcal{T}$ be a mapping with definition as follow

$$\mathcal{T}(\mathcal{H}(t)) = (1-2t^2) + \int_0^1 \frac{1}{3} t^2 p \mathcal{H}(p)\, dp \tag{22}$$

Now, for arbitrary continuous functions $\mathcal{H}, \mathcal{J}$ on $[0,1]$ and $t, p \in [0,1]$, we have

$$\begin{aligned} |\mathcal{G}(t,p,\mathcal{H}(p)) - \mathcal{G}(t,p,\mathcal{J}(p))| &= \left| \frac{1}{3}t^2 p\mathcal{H}(p) - \frac{1}{3}t^2 p\mathcal{J}(p) \right| \\ &\leq \frac{1}{3} |\mathcal{H}(p) - \mathcal{J}(p)| \end{aligned}$$

All assumptions in Theorems IV.1 are satisfied. So, the sequence $\{u_n\}$ defined by (11) converges to the solution of integral equation above, so the solution of integral equation (21) can be approximated by A-plus iteration process. Let approximate values of $\mathcal{H}(t)$ with initial guess $\mathcal{H}(t) = t(1-t)$ using sequences $\tau_n = 0.5, \sigma_n = 0.4, \rho_n = 0.6$. Table II and Figure IV show approximated value of $\mathcal{H}(t)$. In Table II shows the exact solution and approximation solution using some $t \in [0,1]$. The number $\mathcal{H}_1(t)$ describes the estimation solution of integral equation using A-plus iteration procedure (11) with just 1 iteration while $\mathcal{H}_3(t)$ using 3 iteration process. Absolute error is defined as the absolute value of difference between the exact solution and $\mathcal{H}_3(t)$. The absolute error from calculations gives information that the proposed method can be used to estimate the solution of integral equations. Also, from FigureIV, we can observe that the approximation solution is very close to the exact solution. It means that iterative methods defined in (11) is useful to find the solution of integral equation problems in the form of (18).

The following example gives an illustration for finding a solution of integral type (18) with difficulty in finding exact solution. The A-plus algorithm will be used to find its solution.

Example IV.2. Given integral equation as follow

$$\mathcal{P}(t) = 2\sin(\frac{t}{2}) + \int_0^{2\pi} \frac{1}{8} \sin(\frac{t}{2}) \sin(p) \mathcal{P}(p)\, dp \tag{23}$$

Consider that $\mathcal{T}$ be a mapping with definition as follow

$$\mathcal{T}(\mathcal{P}(t)) = 2\sin(\frac{t}{2}) + \int_0^{2\pi} \frac{1}{8} \sin(\frac{t}{2}) \sin(p) \mathcal{P}(p)\, dp \tag{24}$$

Now, for arbitrary continuous functions $\mathcal{P}, \mathcal{Q}$ and $t, p \in [0, 2\pi]$, then

$$\begin{aligned} &\left| \frac{1}{8}\sin(\frac{t}{2})\sin(p)\mathcal{P}(p) - \frac{1}{8}\sin(\frac{t}{2})\sin(p)\mathcal{Q}(p) \right| \\ &\leq \frac{1}{8} |\mathcal{P}(p) - \mathcal{Q}(p)| \end{aligned}$$

and we have $\frac{1}{8}2\pi < 1$. So, all assumptions in Theorems IV.1 are satisfied. The solutions of above integral (23) can obtained using the iterative method (11) with $\tau_n = 0.5, \sigma = 0.4, \rho = 0.6$. From Table III, the solutions of $\mathcal{P}(t)$ are given for some $t \in [0, 2\pi]$ by applying A-plus scheme for 1 iterations, 3 iterations, and 10 iterations.

The following result shows that the A-plus iteration process can be used to estimate the solution of the nonlinear fractional differential equation (FDE). Given the non-linear FDE

$$D^\alpha g(t) = \mathcal{F}(t, g(t)) \tag{25}$$

with $0 < t < 1$, $1 < \alpha < 2$, and integral boundary value condition

$$g(0) = 0 \text{ and } g(1) = \int_0^\eta g(s) ds \;\; (0 < \eta < 1)$$

where $\mathcal{F}$ is continuous function and D^α is Caputo fractional derivative. Let $(\mathcal{X}, \|.\|_\infty)$ be a Banach spaces where $\mathcal{X}$ is spaces of continuous function from $[0,1]$ to $\mathbb{R}$ and $\|g\|_\infty = \sup_{0 \leq t \leq 1} |g(t)|$.

The following results shows that solution of nonlinear FDE can be estimated by iterative methods defined by (11).

Theorem IV.2. *Let $\mathcal{H} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be a self-mapping defined by*

$$\begin{aligned} &\mathcal{H}(g(t)) = \\ &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}(s, g(s))\, ds - \\ &\frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \mathcal{F}(s, g(s))\, ds + \\ &\frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-k)^{\alpha-1} \mathcal{F}(k, g(k))\, dk \right) ds \end{aligned}$$

for $g(t) \in \mathcal{C}[0,1]$ and $t \in [0,1]$. If $|\mathcal{F}(t,x) - \mathcal{F}(t,y)| \leq \frac{\Gamma(\alpha+1)}{5}|x-y|$ holds for $t \in [0,1]$ and $x, y \in \mathbb{R}$, then the A-plus iteration process (11) converges to a solution of FDE (25).

Proof: From Baleanu, et.al. [37], $g \in \mathcal{C}[0,1]$ is solution of nonlinear FDE (25) if and only if it is solution of integral equation

$$\begin{aligned} &g(t) = \\ &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}(s, g(s))\, ds - \\ &\frac{2t}{(2-\mu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \mathcal{F}(s, g(s))\, ds + \\ &\frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-k)^{\alpha-1} \mathcal{F}(k, g(k))\, dk \right) ds \end{aligned}$$

Therefore, we just need to find $g^* \in \mathcal{C}[0,1]$ such that $\mathcal{H}(g^*) = g^*$.

TABLE II
SOLUTION OF INTEGRAL EQUATION (21) IN EXAMPLE IV.1 USING EXACT AND APPROXIMATION METHOD

t	Exact Solution	$\mathcal{H}_1$	$\mathcal{H}_3$	abs error
0	1	1	1	0
0.1	0.98	0.98025	0.980000000002621	$2.62145860574492.10^{-12}$
0.2	0.92	0.921	0.920000000010486	$1.04857234006772.10^{-11}$
0.3	0.82	0.82225	0.820000000023593	$2.35929054070994.10^{-11}$
0.4	0.68	0.684	0.680000000041943	$4.19428936027089.10^{-11}$
0.5	0.5	0.50625	0.500000000065536	$6.55357990098082.10^{-11}$
0.6	0.28	0.289	0.280000000094372	$9.43716216283974.10^{-11}$
0.7	0.02	0.03225	0.0200000001284504	$1.2845024349728.10^{-10}$
0.8	−0.28	−0.264	−0.279999999832229	$1.67771740944289.10^{-10}$
0.9	−0.62	−0.59975	−0.619999999787664	$2.12336148663894.10^{-10}$
1	−1	−0.975	−0.999999999737857	$2.62143307061535.10^{-10}$

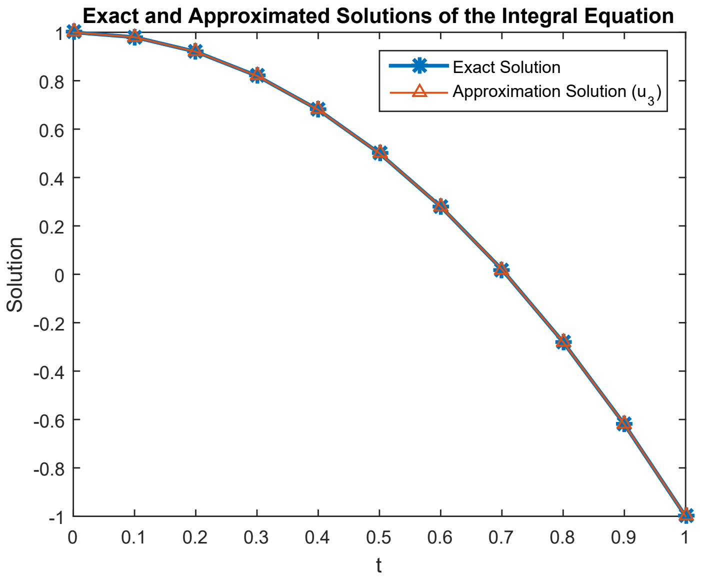

Fig. 2. Exact solution comparing with approximation solution in Example IV.1

Let $g, h \in \mathcal{C}[0,1]$, so we have

$$
\begin{aligned}
&|\mathcal{H}g(t) - \mathcal{H}h(t)| \\
&= \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}(s, g(s))\, ds - \\
&\quad \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} F(s, g(s))\, ds + \\
&\quad \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-k)^{\alpha-1} \mathcal{F}(k, g(k))\, dk \right) ds - \\
&\quad \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{F}(s, h(s))\, ds - \\
&\quad \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} F(s, h(s))\, ds + \\
&\quad \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-k)^{\alpha-1} \mathcal{F}(k, g(k))\, dk \right) ds \Bigg|
\end{aligned}
$$

Using the condition $|\mathcal{F}(t,x) - \mathcal{F}(t,y)| \le \frac{\Gamma(\alpha+1)}{5}|x-y|$, we have

$$
\begin{aligned}
&|\mathcal{H}g(t) - \mathcal{H}h(t)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} \frac{\Gamma(\alpha+1)}{5} |g(s) - h(s)| ds + \\
&\quad \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 |1-s|^{\beta-1} \frac{\Gamma(\alpha+1)}{5} |g(s) - h(s)|\, ds \\
&\quad + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \\
&\quad \int_0^\eta \left(\int_0^s |s-k|^{\alpha-1} \frac{\Gamma(\alpha+1)}{5} |g(k) - h(k)|\, dk \right) ds \\
&\le \frac{\Gamma(\alpha+1)}{5} \|g-h\|_\infty \sup_{0 \le t \le 1} \Bigg[\int_0^t |t-s|^{\alpha-1} ds + \\
&\quad \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 |1-s|^{\alpha-1} ds + \\
&\quad \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s |s-k|^{\alpha-1} dk \right) ds \Bigg]
\end{aligned}
$$

TABLE III
SOLUTION OF INTEGRAL EQUATION (23) IN EXAMPLE IV.2 USING A-PLUS ESTIMATION METHOD

t	$\mathcal{P}(t)$		
	1 iteration	3 iterations	10 iterations
0	0	0	0
$\frac{\pi}{3}$	1.19617062491707	0.999882669542861	0.999882669542861
$\frac{\pi}{2}$	1.69181820956047	1.41419599555358	1.41419599555358
$\frac{3\pi}{4}$	2.21040690258827	1.8476858639537	1.8476858639537
π	2.39253018912932	1.99992327401818	1.99992327401818
$\frac{7\pi}{6}$	2.31104232318298	1.93180731861806	1.93180731861806
$\frac{3\pi}{2}$	1.69181820956047	1.41419599555358	1.41419599555358
$\frac{7\pi}{4}$	0.915544710941122	0.765306613113932	0.765306613113932
2π	$3.33066907387547.10^{-16}$	$2.22044604925031.10^{-16}$	$2.22044604925031.10^{-16}$

So, we obtain

$$|\mathcal{H}g\,(t) - \mathcal{H}h\,(t)| \leq \frac{\Gamma(\alpha+1)}{5}|g(t)-h(t)| + 0.|h(t) - \mathcal{H}g(t)|$$

Since $\dfrac{\Gamma(\alpha+1)}{5} < 1$, then $\mathcal{H}$ is weak contraction mapping. Therefore,using Theorem III.1, the iterative method defined by (11) converges to solution of nonlinear FDE (25). ■

Example IV.3. Consider the nonlinear FDE

$$D^{1.2}g(t) - \sin(2t) = 0, \quad 0 < t < 1 \tag{26}$$

with boundary value integral equation

$$g(0) = 0 \quad g(1) = \int_0^{0.1} g(s)ds$$

Using Theorem III.1, the solution of nonlinear FDE will be estimated. Consider that $\mathcal{H}$ be a self-mapping

$$\mathcal{H}g(t) = \frac{1}{\Gamma(1.2)}\int_0^t (t-s)^{0.2}\sin(2s)\,ds - \frac{2t}{(2-0.1^2)\,\Gamma(1.2)}\int_0^1 (1-s)^{0.2}\sin(2s)\,ds + \frac{2t}{(2-0.1^2)\,\Gamma(1.2)}\int_0^{0.1}\left(\int_0^s (s-k)^{0.2}\sin(2k)\,dk\right)ds$$

All conditions in Theorem IV.2 is satisfied, so we can use A-plus iteration process to estimate the solution of (26). By choosing $g(t) = t - t^2$ and control sequence $\tau_n = 0.65, \sigma_n = 0.2, \rho_n = 0.4$ with 5 iteration process, then the solution of (26) for some value of $t \in [0,1]$ is shown in Table IV.

TABLE IV
APRROXIMATION SOLUTION OF FRACTIONAL INTEGRAL EQUATION (26) IN EXAMPLE IV.3

.

t	Approximation solution
0	0
0.2	-0.0985450826193333
0.4	-0.139585804664431
0.5	-0.138837400272051
0.7	-0.102848660268232
0.9	-0.038170983263466
1	-0.00289216510638879

V. CONCLUSION

In this study, we introduced the A-plus iteration, a novel algorithm for estimating fixed points of almost contraction mappings in closed convex Banach spaces. We proved that the A-plus iteration converges to a unique fixed point and demonstrated its almost $\mathcal{J}$-stable. Furthermore, comparative analysis revealed that the A-plus method achieves a faster convergence rate than existing iterative schemes. To illustrate its practical utility, we applied the algorithm to approximate solutions for integral equations. Numerical examples were provided to validate the theoretical results and underscore the efficiency of the proposed method.

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