Some Results on Fs-Binary Relations of Different Kinds

Gudivada Srinivasa Rao, Vaddiparthi Yogeswara and Biswajit Rath

***Abstract*— Y. Vaddiparthi et al. introduced Fs-binary relations on a pair of Fs-sets in a research article, "Some Modifications on Binary Relations on Fs-sets-Consequences," and proved the collection of all Fs-binary relations with suitable partial orders is an infinitely distributive lattice. In this paper, we introduce various kinds of Fs-binary relations, viz. increasing Fs-binary relation, decreasing Fs-binary relation and preserving Fs-binary relation and prove any non-empty collection of all increasing (decreasing) Fs-binary relations on a pair of Fs-sets is a $\vee$-complete lattice ($\wedge$-complete lattice) with the same partial order and also prove that the collection of all preserving Fs-binary relations with the same partial order is an infinitely distributive sublattice of the infinitely distributive lattice of all Fs-binary relations on the same pair of Fs-sets with the same partial order. Also, lastly, we define composition between suitable Fs-binary relations and introduce (1) a category of Fs-binary relations. (2) a category of increasing Fs-binary relations (3) a category of decreasing Fs-binary relations and (4) a category of preserving Fs-binary relations and prove some related results.**

***Index Terms*—Fs-set, Fs-subset, Fs-objects, Fs-binary relation,composition of Fs-binary relations, category of Fs-binary relations.**

I. Introduction

Fuzzy relations were introduced in the pioneering work on fuzzy sets by L. Zadeh [21]. Several mathematicians as well as computer scientists in later years extensively studied [0, 1] valued fuzzy relations. Vaddiparthi Yogeswara et al. [9] introduced a kind of fuzzy set which is called an Fs-set in the year 2013. An Fs-set is a four tuple in which the first two components are crisp sets such that the second component is a subset of the first component. The fourth component is a complete Boolean algebra, and the third component is a function with the domain as its second component and the co-domain as its fourth component. As well as the third component is a combination of two subcomponent functions which are written in brackets preceded by the third component. The first and the second components are respectively, domains of the first subcomponent function and the second subcomponent function, and the fourth component is a co-domain of both the subcomponent functions. Also, the first subcomponent function, whenever it is restricted to the second component, the resultant is more valued than the second subcomponent function. The third component, which is the combination of two subcomponents, is called the membership function of the given Fs-set. The way of writing the membership function as a combination of more sub-functions helps to study the introduced theory more accurately [1], [2]. Based on this definition of Fs-set, Vaddiparthi Yogeswara et al. [9] introduced the theory of Fs-sets and the theory of Fs-binary relations [20] on a pair of Fs-sets [10] and proved the collection of all Fs-binary relations with a suitable partial order is an infinitely distributive lattice. In this paper, we introduce various kinds of Fs-binary relations, viz. increasing Fs-binary relations, decreasing Fs-binary relations and preserving Fs-binary relations, and prove that the collection of all increasing (decreasing) Fs-binary relations on a pair of Fs-sets is a $\vee$-complete lattice ($\wedge$-complete lattice) with a suitable partial order and also prove that the collection of all preserving Fs-binary relations with the same partial order is an infinitely distributive sublattice of the infinitely distributive lattice of all Fs-binary relations [19] on the same pair of Fs-sets with the same partial order. The necessary examples wherever needed are also given. Lastly, we define the composition between suitable Fs-binary relations and introduce (1) a category of Fs-binary relations, (2) a category of increasing Fs-binary relations, (3) a category of decreasing Fs-binary relations and (4) a category of preserving Fs-binary relations [4] and prove some related results. The necessary information for smooth reading of this paper is given in Section III of this paper.

The necessary information for smooth reading of this paper is given in section-III of this paper.

Let $\mathscr{A} = (A_1, A, \overline{A}(\mu_{1A_1}, \mu_{2A}), L)$ be an universal Fs-set [9] where L is a complete Boolean algebra.

II. Fs-binary relation

Let L_{B_0} and L_{C_0} be complete sub algebras of L.

Definition II.1.
Let $\mathscr{B}_0 = (B_{10}, B_0, \overline{B}_0(\mu_{1B_{10}}, \mu_{2B_0}), L_{B_0}), \mathscr{C}_0 = (C_{10}, C_0, \overline{C}_0(\mu_{1C_{10}}, \mu_{2C_0}), L_{C_0})$ be a pair of Fs-subsets [9] of $\mathscr{A}$. A triplet $\overline{g} = (g_1, g, \varphi)$ with $g \subseteq g_1$ (Crisp sub set sense)is said to be an Fs-binary relation from $\mathscr{B}_0$ into $\mathscr{C}_0$,denoted by $\overline{g} \subseteq \mathscr{B}_0 \times \mathscr{C}_0$ if, and only if there exist two objects, call Fs-objects $\mathscr{B}_1 = (B_{11}, B_1, \overline{B}_1(\mu_{1B_{11}}, \mu_{2B_1}), L_{B_0})$ and $\mathscr{C}_1 = (C_{11}, C_1, \overline{C}_1(\mu_{1C_{11}}, \mu_{2C_1}), L_{C_0})$ such that

(a) $B_0 \subseteq B_1 \subseteq B_{11} \subseteq B_{10}$,$C_0 \subseteq C_1 \subseteq C_{11} \subseteq C_{10}$,$\mu_{1B_{11}} \subseteq B_{11} \times L_{B_0}$ $\{\mu_{1C_{11}} \subseteq C_{11} \times L_{C_0}\}$,$\mu_{2B_1} \subseteq B_1 \times L_{B_0}$ $\{\mu_{2C_1} \subseteq C_1 \times L_{C_0}\}$ and $\mu_{1B_{11}} b \{\mu_{1C_{11}} c\}$ exists means there exists $\alpha \in L_{B_0}$ $(\beta \in L_{C_0})$,$c_1 \in C_{11} (b_1 \in B_{11})$ such that $(b, \alpha) \in \mu_{1B_{11}}$ $\{(c, \beta) \in \mu_{1C_{11}}\}$ and $(b, c_1) \in g_1 \{(b_1, c) \in g_1\}$ (here $\subseteq$ stands in crisp set sense). Similarly, for $\mu_{2B_1} b$ $\{\mu_{2C_1} c\}$ exists means there exists $\alpha \in L_{B_0}$ $(\beta \in L_{C_0})$ such that $(b, \alpha) \in \mu_{2B_1}$ $\{(c, \beta) \in \mu_{2C_1}\}$ and $(b, c_1) \in g$ $\{(b_1, c)$

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$\in g\}$. Also $\mu_{2B_1}b \subseteq \mu_{1B_{11}}b$,where b is such that $(b,\alpha) \in \mu_{2B_1}$for some $\alpha \in L_{B_0}$ and $(b,c) \in g$ for some $c \in C_1$.
Similarly, $\mu_{2C_1}c \subseteq \mu_{1C_{11}}c$,where c is such that $(c,\beta) \in \mu_{2C_1}$for some $\beta \in L_{C_0}$ and $(b,c) \in g$ for some $b \in B_1$.
(b) $g_1 = \{(b,c)/b \in B_{11}, c \in C_{11}\} \subseteq B_{10} \times C_{10}$(Crisp subset of).

Here $B_{11} =$ domain of g_1 and $C_{11} = Co-domain\ of g_1$ and $B_{11} \times\ C_{11}$ is such that $g_1 \subseteq\ B_{11} \times\ C_{11}$.

Here on words, we write domain of g_1 and *Co-domain of* g_1 as $domg_1$ and $Codg_1$ respectively.

Note that if there exist $X \times Y$ such that $g_1 \subseteq X \times Y \subseteq B_{11} \times C_{11}$ then $X \times Y = B_{11} \times C_{11}$ Here, we can easily observe $B_{11} \times C_{11}$ is the least or smallest cartesian product containing g_1.
(c) $g = \{(b,c)/b \in B_1, c \in C_1\} \supseteq B_0 \times C_0$(Crisp subset of).

Here $B_1 =$ *domain of* g and $C_1 =$ *Co-domain of* g and $B_1 \times C_1$ is such that $g \subseteq\ B_1 \times C_1$.
Similarly, as in **(b)** we can easily observe that $B_1 \times C_1$ is the least cartesian product containing g.

Whenever $B_{11} \subsetneq B_1$ or $C_{11} \subsetneq C_1$ or $\mu_{1B_{11}}b < \mu_{2B_1}b$ for at least one $b \in B_{11}$ with $(b,c) \in g_1$ for some $c \in C_{11}$ or $\mu_{1C_{11}}c < \mu_{2C_1}c$ for at least one $c \in C_{11}$ with $(b,c) \in g_1$ for some $b \in B_{11}$, we assume $\overline{g}$ is Fs-empty binary relation, denoted by $\emptyset$, here $\subsetneq$ denote proper subset (crips set sense) symbol.

Also, for $(b,c) \in B_{11} \times C_{11}$we can observe $c \in \{g_1 b\}$ or $c \notin \{g_1 b\}$.
(d) $(\mu_{1B_{10}}/B_{11}) \geqq \vee\mu_{1B_{11}}$i.e., for each $b \in B_{11}$ such that $(b,c) \in g_1$ for some $c \in C_{11}$, $\mu_{1B_{10}}b \geqq \vee_{(b,c)\in g_1} \{\mu_{1B_{11}}b\}$. Here $c \in C_{11}$ varies . It can also be denoted by $\mu_{1B_{10}}b \geqq \vee \{\mu_{1B_{11}}b\}$.

And $(\mu_{1C_{10}}/C_{11}) \geqq \vee\mu_{1C_{11}}$i.e., for each $c \in C_{11}$ such that $(b,c) \in g_1$ for some $b \in B_{11}, \mu_{1C_{10}}c \geqq \vee_{(b,c)\in g_1} \{\mu_{1C_{11}}c\}$. Here $b \in B_{11}$ varies.It can also be denoted by $\mu_{1C_{10}}c \geqq \vee \{\mu_{1C_{11}}c\}$.
(e) $\mu_{2B_0} \leqq \vee(\mu_{2B_1}/B_0)$ i.e., for each $b \in B_0$ such that $(b,c) \in g$ for some $c \in C_0$, $\mu_{2B_0}b \leq \vee_{(b,c)\in B_0\times C_0 \subseteq g} \{\mu_{2B_1}b\}$. Here $c \in C_0$ may vary. It can also be denoted by $\mu_{2B_0}b \leqq \vee \{\mu_{2B_1}b\}$.

And $\mu_{2C_0} \leqq \vee(\mu_{2C_1}/C_0)$ i.e., for each $c \in C_0$ such that $(b,c) \in B_0 \times C_0$ for some $b \in B_0$, $\mu_{2C_0}c \leqq \vee_{(b,c)\in B_0\times C_0} \{\mu_{2C_1}c\}$.

Here $b \in B_0$ may vary. It can also be denoted by. $\mu_{2C_0}c \leqq \vee \{\mu_{2C_1}c\}$.
(f) $\varphi : L_{B_0} \to L_{C_0}$ is a complete Boolean algebra homomorphism.
(g) Whenever $\mathscr{B}_1 = \Phi_{\mathscr{A}}$ = The Fs-empty set or $\mathscr{C}_1 = \Phi_{\mathscr{A}}$ then the corresponding Fs-binary relation is called Fs-empty binary relation, denoted by $\emptyset$. This $\emptyset$ is considered same as given in **II.1(c)**.
(h) We can denote $\overline{g}$ with Fs-objects $\mathscr{B}_1$ and $\mathscr{C}_1$ by $(\overline{g}, \mathscr{B}_1, \mathscr{C}_1)$ also.

Remarks II.2. Observe $\vee\mu_{2B_1}b \leqq \vee\mu_{1B_{11}}b$ and $\vee\mu_{2C_1}b \leqq \vee\mu_{1C_{11}}b$.

Remarks II.3. $(\overline{g}_i, \overline{\mathscr{B}}_i, \overline{\mathscr{C}}_i)$ in general we mean, $\overline{g}_i = (g_{1i}, g_i, \varphi)$, $\mathscr{B}_i = (B_{1i}, B_i, \overline{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L)$ and $\mathscr{C}_i = (C_{1i}, C_i, \overline{C}_i(\mu_{1C_{1i}}, \mu_{2C_i}), M)$.

Remarks II.4. Whenever $B_{11}, C_{11}, B_1$and C_1 all are non-empty and $\mu_{1B_{11}}, \mu_{1C_{11}}, \mu_{2B_1}$ and μ_{2C_1} exist and are defined for every element and unique then Fs-objects $\mathscr{B}_1$ and $\mathscr{C}_1$ are usual Fs-subsets[9].

Remarks II.5. A part of the diagrammatic representation of Fs- binary relation is as in below

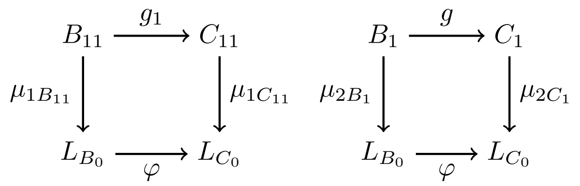

Fig. 1. Partial diagrammatic representation of Fs-binary relation

Example II.6. Example of an Fs-binary relation
Consider $\overline{g} = (g_1, g, \varphi)$, $G_{11} = \{a_1, a_2\}$, $G_1 = \{a_2\}$,$H_{11} = \{b_1, b_2\}$,
$H_1 = \{b_2\}$.
$g_1 = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}, g = \{(a_2, b_2)\}$

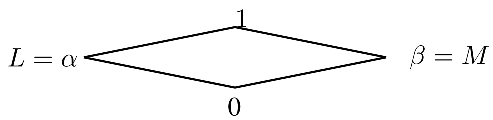

Fig. 2. Complete Boolean Algebra

$\mu_{1G_{11}}a_1 = \alpha$, $\mu_{1G_{11}}a_2 = \alpha$, $\mu_{2G_1}a_2 = 0$ i.e., $\mu_{1G_{11}}a_2 \geq \mu_{2G_1}a_2$
$\mu_{1H_{11}}b_1 = \beta$, $\mu_{1H_{11}}b_2 = \beta$, $\mu_{2H_1}b_2 = \beta$ i.e., $\mu_{1H_{11}}b_2 \geq \mu_{2H_1}b_2$
Define $\varphi : L \to M$ as $\varphi(\alpha) = \beta, \varphi(\beta) = \alpha, \varphi(0) = 0, \varphi(1) = 1$ then φ is a Boolean algebra homomorphism and $\overline{g}$ is a Fs-binary relation from $(G_{11}, G_1, \overline{G}_1(\mu_{1G_{11}}, \mu_{2G_1}), L)$ into $(H_{11}, H_1, \overline{H}_1(\mu_{1H_{11}}, \mu_{2H_1}), M)$.

Proposition II.1. *Suppose* $\mathscr{B}, \mathscr{C} \subseteq \mathscr{A}$ *and* $\varphi : L_{B_1} \to L_{C_1}$ *is a complete Boolean algebra homomorphism. Then the following are true*

(a) $\vee\mu_{1C_{11}}(g_1|_{B_1}) \geqq \vee\mu_{2C_1}(g)$
(b) $\vee(\varphi \circ \mu_{1B_{11}}|_{B_1}) \geqq \vee(\varphi \circ \mu_{2B_1})$
(c) $\varphi \circ (\vee\mu_{2B_1}b)^c = (\vee(\varphi \circ \mu_{2B_1}))^c$ *for any* $b \in B_1$.

Proof of (a): For $gb = \{c_i\}_{i\in I}, g_1 b = \{c_i\}_{i\in I} \cup \{c_j\}_{j\in J}$
We have

$$\begin{aligned}
&\vee\mu_{2C_1}c_i \leqq \vee\mu_{1C_{11}}c_i, \forall i \in I \\
\implies \vee_{i\in I}(\vee\mu_{2C_1}c_i) &\leqq \vee_{i\in I}(\vee\mu_{1C_{11}}c_i) \\
&\leqq (\vee_{i\in I}(\vee\mu_{1C_{11}}c_i)) \vee (\vee_{j\in J} \vee \mu_{1C_{11}}c_j) \\
&= \vee\mu_{1C_{11}}(g_1 b)
\end{aligned}$$

i.e. $\vee\mu_{2C_1}(gb) \leqq \vee\mu_{2C_1}(g_1 b)$. ■

Proof of (b): We have $B_1 \subseteq B_{11}$.
For any $b \in B_1$ assume $\mu_{2B_1}b = \{\alpha_i\}_{i\in I}$ and $\mu_{1B_{11}}b = \{\alpha_i\}_{i\in I} \vee \{\alpha_j\}_{j\in J}$
we have

$$\begin{aligned}
&\vee_{i\in I}\alpha_i \leqq (\vee_{i\in I}\alpha_i) \vee (\vee_{j\in J}\alpha_j), \text{ i.e.} \\
&\vee\mu_{2B_1}b \leqq \vee\mu_{1B_{11}}b \\
\implies &\varphi(\vee\mu_{2B_1}b) \leqq \varphi(\vee\mu_{1B_{11}}b) = \vee(\varphi \circ \mu_{1B_{11}}b) \\
\implies &\vee(\varphi \circ \mu_{2B_1}b) \leqq \vee(\varphi \circ \mu_{1B_{11}}b), \forall b \in B_1
\end{aligned}$$

i.e. $\vee(\varphi \circ \mu_{2B_1}) \leqq \vee(\varphi \circ \mu_{1B_{11}}|_{B_1})$. ■

Proof of (c) :

$$\begin{aligned}\textbf{LHS} &= \varphi \circ (\vee \mu_{2B_1} b)^c \\ &= \varphi \circ (\wedge (\mu_{2B_1} b)^c) \\ &= \wedge [\varphi \circ (\mu_{2B_1} b)^c] \\ &= \wedge [(\varphi \circ \mu_{2B_1} b)^c] \\ &= [\vee (\varphi \circ \mu_{2B_1} b)]^c = \textbf{RHS}\end{aligned}$$

i.e. $\varphi \circ (\vee \mu_{2B_1} b)^c = (\vee(\varphi \circ \mu_{2B_1}))^c$, for any $b \in B_1$. ■

Definition II.7.
Two non-empty Fs-binary relations $(\overline{g}_1, \mathscr{B}_1, \mathscr{C}_1)$ and $(\overline{g}_2, \mathscr{B}_2, \mathscr{C}_2)$ are said to be equal if, and only if

(1) (a) $g_{11} = g_{12}$
    (b) $g_1 = g_2$

(2) (a) For $\mu_{1B_{11}} \subseteq B_{11} \times L$, $\mu_{1B_{12}} \subseteq B_{12} \times L$, $\vee \mu_{1B_{11}} b = \vee \mu_{1B_{12}} b$ for each $b \in B_{11} = B_{12}$ whenever $\mu_{1B_{11}} b$ and $\mu_{1B_{12}} b$ exist.
    (b) For $\mu_{1C_{11}} \subseteq C_{11} \times M$, $\mu_{1C_{12}} \subseteq C_{12} \times M$, $\vee \mu_{1C_{11}} c = \vee \mu_{1C_{12}} c$ for each $c \in C_{11} = C_{12}$ whenever $\mu_{1C_{11}} c$ and $\mu_{1C_{12}} c$ exist.

(3) (a) For $\mu_{2B_1} \subseteq B_1 \times L, \mu_{2B_2} \subseteq B_2 \times L, \vee \mu_{2B_1} b = \vee \mu_{2B_2} b$ for each $b \in B_1 = B_2$ whenever $\mu_{2B_1} b$ and $\mu_{2B_2} b$ exist.
    (b) For $\mu_{2C_1} \subseteq C_1 \times M, \mu_{2C_2} \subseteq C_2 \times M, \vee \mu_{2C_1} c = \vee \mu_{2C_2} c$ for each $c \in C_1 = C_2$ whenever $\mu_{2C_1} c$ and $\mu_{2C_2} c$ exist .

Example II.8. The following example shows equality of two Fs-binary relations $(\overline{g}_1, \mathscr{B}_1, \mathscr{C}_1)$ and $(\overline{g}_2, \mathscr{B}_2, \mathscr{C}_2)$.
Here $\overline{g}_1 = (\overline{g}_1, \mathscr{B}_1, \mathscr{C}_1), B_{11} = \{b, c, d\}, B_1 = \{b, c\}$, L is the same coomplete Boolean Algebra as in Example II.6

$\mu_{1B_{11}} b = \{\alpha, \beta\}, \mu_{2B_1} b = \{\alpha, 0\}, \mu_{1B_{11}} c = \{\alpha, 1\}$,
$\mu_{2B_1} c = \{\alpha, 0\}$.
$\vee \mu_{1B_{11}} b = 1, \vee \mu_{2B_1} b = \alpha, \vee \mu_{1B_{11}} c = 1, \vee \mu_{2B_1} c = \alpha$
$C_{11} = \{b, c, d\}, C_1 = \{b, c\}$.
$\mu_{1C_{11}} b = \{\alpha, 1\}, \mu_{2C_1} b = 0$,
$\mu_{1C_{11}} c = \{\beta\}, \mu_{2C_1} c = \{0\}$
$\vee \mu_{1C_{11}} b = 1, \vee \mu_{2C_1} b = 0, \vee \mu_{1C_{11}} c = \beta, \vee \mu_{2C} c = 0$
$g_{11} = \{(b, c), (c, b)\}, g_1 = \{(b, c)\}$
$\overline{g}_2 = (\overline{g}_2, \mathscr{B}_2, \mathscr{C}_2), B_{12} = \{b, c, f\}, B_2 = \{b\}$
$\mu_{1B_{12}} b = \{\alpha, 1\}, \mu_{2B_2} b = \{\alpha, 0\}$
$\vee \mu_{1B_{12}} b = 1, \vee \mu_{2B_2} b = \alpha$
$C_{12} = \{b, c, f\}, C_2 = \{c\}$
$\mu_{1C_{12}} c = \{\beta, 0\}, \mu_{2C_2} c = \{0\}$
$\vee \mu_{1C_{12}} c = \beta, \vee \mu_{2C_2} c = 0$
$g_{12} = \{(b, c), (c, b)\}, g_2 = \{(b, c)\}$.

Remarks II.9. We can easily observe from **(1)(a)**, $domg_{11} = domg_{12}$ and $codg_{11} = codg_{12}$ (or) $B_{11} = B_{12}$, $C_{11} = C_{12}$. Similarly, that in **(2)(a)**.

Theorem II.10. *The Collection of all Fs-binary relations from an Fs-subset $\mathscr{B}_0$ into another Fs-subset $\mathscr{C}_0$, denoted by $R^2(\mathscr{B}_0, \mathscr{C}_0)$ is an infinitely distributive lattice [8] with partial order $\leqq$ defined for $(\overline{g}_1, \mathscr{B}_1, \mathscr{C}_1)$ and $(\overline{g}_2, \mathscr{B}_2, \mathscr{C}_2)$ in $R^2(\mathscr{B}_0, \mathscr{C}_0)$*

$$(\overline{g}_1, \mathscr{B}_1, \mathscr{C}_1) \leqq (\overline{g}_2, \mathscr{B}_2, \mathscr{C}_2) \textit{ if, and only if}$$

$$g_{11} \subseteq g_{12}, g_1 \supseteq g_2,$$

$$\vee \mu_{1B_{11}} \leqq \vee \mu_{1B_{12}}, \vee \mu_{1C_{11}} \leqq \vee \mu_{1C_{12}} \textit{ and}$$
$$\vee \mu_{2B_1} \geqq \vee \mu_{2B_2}, \vee \mu_{2C_1} \geqq \vee \mu_{2C_2}.$$

Remarks II.11. For supremum $(\vee)$ and infimum $(\wedge)$ of any family of Fs-binary relations, one can refer [19].

III. Fs-binary Relations of Different Kinds

Definition III.1. An Fs-binary relation $\bar{g} = (g_1, g, \varphi) \in R^2(\mathscr{B}_0, \mathscr{C}_0)$ is said to be an increasing Fs-binary relation, from $\mathscr{B}_0$ into $\mathscr{C}_0$, denoted by $(\bar{g} \uparrow, \mathscr{B}_1, \mathscr{C}_1) \subseteq \mathscr{B}_0 \times \mathscr{C}_0$ or simply $\bar{g} \uparrow$, if, and only if (from figure),

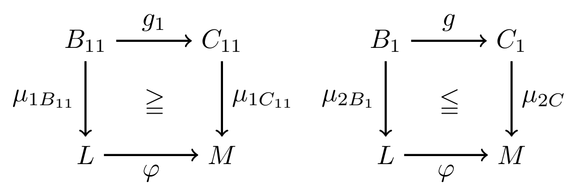

Fig. 3. Increasing Fs-binary Relation

1) If both $\vee(\mu_{1C_{11}} g_1 b)$ and $\varphi\ o(\vee \mu_{1B_{11}} b)$, exist for any $b \in B_{11}$ then $\vee(\mu_{1C_{11}} g_1 b) \geqq \varphi\ o(\vee \mu_{1B_{11}} b)$,
2) If both $\vee(\mu_{2C_1} g b)$ and $\varphi o(\vee \mu_{2B_1} b)$, exist for any $b \in B_1$ then $\vee(\mu_{2C_1} g b) \leqq \varphi o(\vee \mu_{2B_1} b)$,

For some b, if any of the above four expressions does not exist, then the corresponding inequality is not tenable.

For an example of an increasing Fs-binary relation one can see Example III.3.

Theorem III.2. *The Collection of all increasing Fs-binary relations, denoted by $R_i^2(\mathscr{B}_0, \mathscr{C}_0)$ with partial order $\leqq$ given in $R^2(\mathscr{B}_0, \mathscr{C}_0)$ [19] is a $\vee$-Complete lattice.*

Proof: Initially let us prove this theorem for two increasing Fs-binary relations and then we can generalize it to any arbitrary family of increasing Fs-binary relations.

In general,we will have $\bar{g}_i = (g_{1i}, g_i, \varphi)$ (RemarksII.3)with Fs-objects $\mathscr{B}_i$ and $\mathscr{C}_i$.

Given $\bar{g}_1 \uparrow$ and $\bar{g}_2 \uparrow$. Assuming all relevant expression are existing, we have for $\bar{g}_1 \uparrow$,

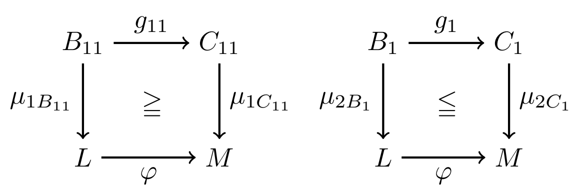

Fig. 4. Increasing Fs-binary Relation

1) $\vee(\mu_{1C_{11}} g_{11} b) \geqq \varphi\ o(\vee \mu_{1B_{11}} b)$, for $b \in B_{11}$
2) $\vee(\mu_{2C_1} g_1 b) \leqq \varphi\ o(\vee \mu_{2B_1} b)$, for $b \in B_1$,
$\bar{g}_1$ with Fs-objects $\mathscr{B}_1$ and $\mathscr{C}_1$
where $domg_{11} = B_{11}, codg_{11} = C_{11}, domg_1 = B_1$, $codg_1 = C_1, \mu_{1B_{11}} \subseteq B_{11} \times L, \mu_{1C_{11}} \subseteq C_{11} \times M$, $\mu_{2B_1} \subseteq B_1 \times L$, $\mu_{2C_1} \subseteq C_1 \times M$ and similarly, for $\bar{g}_2 \uparrow$,we have

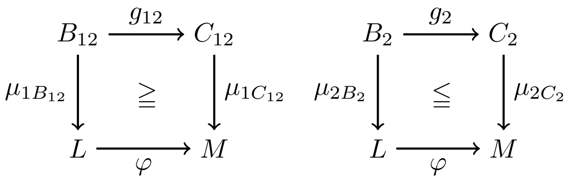

Fig. 5. Increasing Fs-binary Relaton

3) $\vee(\mu_{1C_{12}} g_{12} b) \geqq \varphi\ o(\vee\mu_{1B_{12}} b)$, for $b \in B_{12}$
4) $\vee(\mu_{2C_2} g_2 b) \leqq \varphi\ o(\vee\mu_{2B_2} b)$, for $b \in B_2$,
$\bar{g}_2$ with Fs-objects $\mathscr{B}_2$ and $\mathscr{C}_2$
where $domg_{12} = B_{12}, codg_{12} = C_{12}, domg_2 = B_2$, $codg_2 = C_2, \mu_{1B_{12}} \subseteq B_{12} \times L, \mu_{1C_{12}} \subseteq C_{12} \times M$, $\mu_{2B_2} \subseteq B_2 \times L, \mu_{2C_2} \subseteq C_2 \times M$.
Suppose $\bar{g}_1 \vee \bar{g}_2 = \bar{g}_3$ with Fs-objects $\mathscr{B}_3$ and $\mathscr{C}_3$
Need to show
$\bar{g}_3$ is an increasing Fs-binary relation.
Sufficient to show whenever all the relevant expressions exist, then we have to show with the help of the following diagram

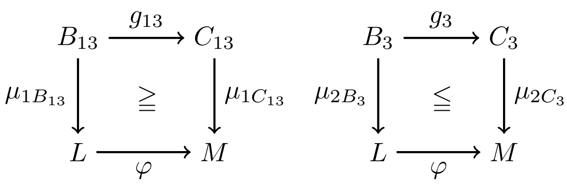

Fig. 6. Increasing Fs-binary Relaton

5) $\vee(\mu_{1C_{13}} g_{13} b) \geqq \varphi o(\vee\mu_{1B_{13}} b)$, for $b \in B_{13}$
6) $\vee(\mu_{2C_3} g_3 b) \leqq \varphi o(\vee\mu_{2B_3} b)$, for $b \in B_3$
where $g_{13} = g_{11} \cup g_{12}$

$$\begin{aligned} domg_{13} &= dom\,(g_{11} \cup g_{12}) = domg_{11} \cup domg_{12} \\ &= B_{11} \cup B_{12} \end{aligned}$$

$$\begin{aligned} codg_{13} &= cod\,(g_{11} \cup g_{12}) = codg_{11} \cup codg_{12} \\ &= C_{11} \cup C_{12} \end{aligned}$$

$$g_3 = g_1 \cap g_2$$

$$\begin{aligned} dom\ g_3 &= dom(g_1 \cap g_2) \subseteq domg_1 \cap domg_2 \\ &= B_1 \cap B_2 \end{aligned}$$

$$\begin{aligned} cod\ g_3 &= cod(g_1 \cap g_2) \subseteq codg_1 \cap codg_2 \\ &= C_1 \cap C_2 \end{aligned}$$

For $\mu_{1B_{13}} \subseteq B_{13} \times L, \vee\mu_{1B_{13}} b$ is defined by,

$$\vee\,\mu_{1B_{13}} b = \{(\vee\mu_{1B_{11}}) \vee (\vee\mu_{1B_{12}})\}\,(b)$$

For $\mu_{1C_{13}} \subseteq C_{13} \times M, \vee\mu_{1C_{13}} c$ is defined by,

$$\vee\,\mu_{1C_{13}} c = \{(\vee\mu_{1C_{11}}) \vee (\vee\mu_{1C_{12}})\}\,(c)$$

For $\mu_{2B_3} \subseteq B_3 \times L, \vee\mu_{2B_3} b$ is defined by,

$$\vee\,\mu_{2B_3} b = (\vee\mu_{2B_1} b) \wedge (\vee\mu_{2B_2} b)$$

For $\mu_{2C_3} \subseteq C_3 \times M, \vee\mu_{2C_3} c$ is defined by,

$$\vee\,\mu_{2C_3} c = (\vee\mu_{2C_1} c) \wedge (\vee\mu_{2C_2} c).$$

Proof(5):
Case 1.

$$\begin{aligned} \vee(\mu_{1C_{13}} g_{13} b) &= (\vee\mu_{1C_{11}} g_{13} b) \vee (\vee\mu_{1C_{12}} g_{13} b)\ [19] \\ &\quad \text{(assuming } g_{13}b \text{ is in both domains of } \mu_{1C_{11}} \text{ and } \mu_{1C_{12}}) \\ &= \left[\vee\mu_{1C_{11}}\,(g_{11}b \cup g_{12}b)\right] \vee \left[\vee\mu_{1C_{12}}\,(g_{11}b \cup g_{12}b)\right] \\ &\quad \text{(assuming } b \text{ is in both domains of } g_{11} \text{ and } g_{12}) \\ &= [\vee(\mu_{1C_{11}} g_{11} b \cup \mu_{1C_{11}} g_{12} b)] \vee [\vee(\mu_{1C_{12}} g_{11} b \cup \mu_{1C_{12}} g_{12} b)] \\ &\quad \text{(assuming both } g_{11}b \text{ and } g_{12}b \text{ are in all domains of } \mu_{1C_{11}} \text{ and } \mu_{1C_{12}}) \\ &\geqq (\vee\mu_{1C_{11}} g_{11} b) \vee (\vee\mu_{1C_{12}} g_{12} b) \\ &\geqq \{\varphi o(\vee\mu_{1B_{11}} b)\} \vee \{\varphi o(\vee\mu_{1B_{12}} b)\} \\ &\qquad \text{from (1) and (3)} \\ &= \varphi\ o\{(\vee\mu_{1B_{11}} b) \vee (\vee\mu_{1B_{12}} b)\} \\ &= \varphi o\,(\vee\mu_{1B_{13}} b) \end{aligned}$$

so that $\vee(\mu_{1C_{13}} g_{13} b) \geqq \varphi\ o(\vee\mu_{1B_{13}} b)$.
Case 2. Suppose $(b, *) \in g_{11}$ and $(b, *) \notin g_{12}$
Then, it follows $b \notin B_{12} = dom\ \mu_{1B_{12}}$.
Hence,

$$\begin{aligned} \vee(\mu_{1C_{13}} g_{13} b) &= \vee\{\mu_{1C_{13}}\,(g_{11} \cup g_{12})\,(b)\} \\ &= \vee(\mu_{1C_{13}} g_{11} b) \\ &= \{(\vee\mu_{1C_{11}}) \vee (\vee\mu_{1C_{12}})\}\,(g_{11} b) \\ &\geqq \vee(\mu_{1C_{11}} g_{11} b) \\ &\geqq \varphi\ o\,(\vee\mu_{1B_{11}} b) \\ &= \varphi\ o\,(\vee\mu_{1B_{13}} b) \\ (\because \vee\mu_{1B_{13}} b &= \{(\vee\mu_{1B_{11}}) \vee (\vee\mu_{1B_{12}})\}\,(b)) \end{aligned}$$

so that $\vee(\mu_{1C_{13}} g_{13} b) \geqq \varphi\ o(\vee\mu_{1B_{13}} b)$.
Case 3. Suppose $(b, *) \notin g_{11}$ and $(b, *) \in g_{12}$
Then, it follows $b \notin B_{11} = dom\ \mu_{1B_{11}}$.
Hence,

$$\begin{aligned} \vee(\mu_{1C_{13}} g_{13} b) &= \vee\{\mu_{1C_{13}}\,(g_{11} \cup g_{12})\,(b)\} \\ &= \vee(\mu_{1C_{13}} g_{12} b) \\ &= \{(\vee\mu_{1C_{11}}) \vee (\vee\mu_{1C_{12}})\}\,(g_{12} b) \\ &\geqq \vee(\mu_{1C_{12}} g_{12} b) \\ &\geqq \varphi\ o\,(\vee\mu_{1B_{12}} b) = \varphi\ o\,(\vee\mu_{1B_{13}} b) \\ (\because \vee\mu_{1B_{13}} b &= \{(\vee\mu_{1B_{11}}) \vee (\vee\mu_{1B_{12}})\}\,(b)) \end{aligned}$$

so that $\vee(\mu_{1C_{13}} g_{13} b) \geqq \varphi\ o(\vee\mu_{1B_{13}} b)$. ■

Proof(6):

$$\begin{aligned} \vee(\mu_{2C_3} g_3 b) &= (\vee\mu_{2C_1} g_3 b) \wedge (\vee\mu_{2C_2} g_3 b)\ [19] \\ &= \left[\vee\mu_{2C_1}\,(g_1 b \cap g_2 b)\right] \wedge \left[\vee\mu_{2C_2}\,(g_1 b \cap g_2 b)\right] \\ &\leqq (\vee\mu_{2C_1} g_1 b) \wedge (\vee\mu_{2C_2} g_2 b) \\ &\leqq (\varphi\ o(\vee\mu_{2B_1} b)) \wedge (\varphi\ o(\vee\mu_{2B_2} b)) \end{aligned}$$

from (2) and (4)

$$= \varphi\ o((\vee \mu_{2B_1} b) \wedge (\vee \mu_{2B_2} b))$$
$$= \varphi\ o(\vee \mu_{2B_3} b)$$

so that $\vee(\mu_{2C_3} g_3 b) \leqq\ \varphi\ o(\vee \mu_{2B_3} b)$. ■

Given a family $\{\bar{g}_i\}_{i\in I}$ with $\bar{g}_i \uparrow$,for each $i \in I$

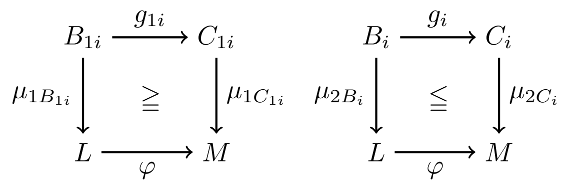

Fig. 7. Increasing Fs-binary Relaton

7) $\vee(\mu_{1C_{1i}} g_{1i} b) \geqq\ \varphi\ o(\vee \mu_{1B_{1i}} b)$, for $b \in\ B_{1i}$,for each $i \in I$
8) $\wedge(\mu_{2C_i} g_i b) \leqq\ \varphi\ o(\wedge \mu_{2B_i} b)$, for $b \in B_i$, for each $i \in I$
Suppose $\vee_{i\in I} \bar{g}_i = \bar{g}$.We use notations as in [19]
Here $\bar{g}_i \subseteq \mathscr{B}_i \times \mathscr{C}_i, \bar{g} \subseteq \vee_{i\in I}(\mathscr{B}_i \times \mathscr{C}_i)$.
Now $\bar{g}$ with Fs-objects $\mathscr{D}$ and $\mathscr{E}$
Here $\bar{g} = (g_1, g, \varphi)$,where
$g_1 = \cup_{i\in I} g_{1i} \subseteq \cup_{i\in I}(B_{1i} \times C_{1i})$,
$g = \cap_{i\in I} g_i \subseteq \cap_{i\in I}(B_i \times C_i)$
For $\mu_{1D_1} \subseteq\ D_1 \times\ L, \vee \mu_{1D_1} b$ is defined by

$$\vee \mu_{1D_1} b = \bigvee_{(b,*)\in \cup_{i\in I} g_{1i}} (\vee \mu_{1B_{1i}} b\ ) \tag{III.1}$$

For $\mu_{1E_1} \subseteq E_1 \times M, \vee \mu_{1E_1} c$ is defined by

$$\vee \mu_{1E_1} c = \bigvee_{(*,c)\in \cup_{i\in I} g_{1i}} (\vee \mu_{1C_{1i}} c) \tag{III.2}$$

For $\mu_{2D} \subseteq D \times L, \vee \mu_{2D} b$ is defined by

$$\wedge \mu_{2D} b = \bigwedge_{(b,*)\in \cap_{i\in I} g_i} (\vee \mu_{2B_i} b) \tag{III.3}$$

For $\mu_{2E} \subseteq E \times M, \vee \mu_{2E} c$ is defined by

$$\wedge \mu_{2E} c = \bigwedge_{(*,c)\in \cap_{i\in I} g_i} (\vee \mu_{2C_i} c) \tag{III.4}$$

Need to show

9) $\vee(\mu_{1E_1} g_1 b) \geqq\ \varphi\ o(\vee \mu_{1D_1} b)$, for $b \in\ D_1$
10) $\wedge(\mu_{2E} g b) \leqq\ \varphi\ o(\wedge \mu_{2D} b)$, for $b \in\ D$

Proof(9):

$$\begin{aligned}
\vee(\mu_{1E_1} g_1 b) &= \bigvee_{(*,g_1 b)\in \cup_{i\in I} g_{1i}} (\vee \mu_{1C_{1i}} g_1 b) \\
&= \bigvee_{i\in I} \bigvee_{(*,g_1 b)\in g_{1i}} \mu_{1C_{1i}} g_1 b [19] \\
&= \bigvee_{i\in I} \bigvee_{(*,g_{1j} b)\in g_{1j}} \mu_{1C_{1i}} \left(\bigcup_{j\in I} g_{1j}\ b\right) \\
&= \bigvee_{i\in I} \left[\bigcup_{j\in I} \mu_{1C_{1i}}\ g_{1j} b\right] \\
&\geqq \bigvee_{i\in I} [\vee(\mu_{1C_{1i}} g_{1i} b)] \\
&\geqq \bigvee_{i\in I} [\varphi o(\vee \mu_{1B_{1i}} b)] \\
&= \varphi \left[\bigvee_{i\in I} \vee \mu_{1B_{1i}} b\right] \\
&= \varphi \left[\bigvee_{i\in I} \bigvee_{(b,*)\in g_{1i}} \mu_{1B_{1i}} b\ \right]
\end{aligned}$$

$(\because$ here $\vee \mu_{1B_{1i}} b = \bigvee_{(b,*)\in g_{1i}} \mu_{1B_{1i}} b\ )$

$$\begin{aligned}
&= \varphi \left[\bigvee_{(b,*)\in \cup_{i\in I} g_{1i}} (\vee \mu_{1B_{1i}} b\ )\right] \\
&= \varphi o(\vee \mu_{1D_1} b
\end{aligned}$$

so that,$\vee(\mu_{1E_1} g_1 b) \geqq\ \varphi o(\vee \mu_{1D_1} b)$. ■

Proof(10):

$$\begin{aligned}
\wedge(\mu_{2E} g b) &= \bigwedge_{(*,gb)\in \cap_{i\in I} g_i} (\vee \mu_{2C_i} g b) \\
&= \bigwedge_{i\in I} \bigvee_{(*,gb)\in g_i} \mu_{2C_i} g b\ [19] \\
&= \bigwedge_{i\in I} \bigvee_{(*,g_j b)\in g_j} \mu_{2C_i} \left(\bigcap_{j\in I} g_j\ b\right) \\
&= \bigwedge_{i\in I} \left[\bigcap_{j\in I} \mu_{2C_i}\ g_j b\right] \\
&\leqq \bigwedge_{i\in I} [\vee(\mu_{2C_i} g_i b)] \\
&\leqq \bigwedge_{i\in I} [\varphi o(\vee \mu_{2B_i} b)] \text{ from(8)} \\
&= \varphi \left[\bigwedge_{i\in I} \vee \mu_{2B_i} b\right] \\
&= \varphi \left[\bigwedge_{i\in I} \bigvee_{(b,*)\in g_i} \mu_{2B_i} b\ \right]
\end{aligned}$$

$(\because$ here $\vee \mu_{2B_i} b = \bigvee_{(b,*)\in g_i} \mu_{2B_i} b)$

$$\begin{aligned}
&= \varphi \left[\bigwedge_{(b,*)\in \cap_{i\in I} g_i} (\vee \mu_{2B_i} b\ )\right] \\
&= \varphi o(\wedge \mu_{2D} b)
\end{aligned}$$

so that, $\wedge(\mu_{2E} g b) \leqq \varphi o(\wedge \mu_{2D} b)$. ■

Hence the theorem. ■

Example III.3. A collection of all increasing Fs-binary relations, denoted by $\mathrm{R}_{\mathrm{i}}^2(\mathscr{B}_0, \mathcal{C}_0)$ with partial order $\leqq$ given in $\mathrm{R}^2(\mathscr{B}_0, \mathscr{C}_0)$ is not a $\wedge$ - Complete lattice.
Here $\overline{g}_1$ and $\overline{g}_2$ are increasing Fs-binary relations but $\overline{g}_1 \wedge \overline{g}_2$ is not an increasing Fs-binary relation. $\overline{g}_1 = (\overline{g}_1, \overline{\mathscr{B}}_1, \overline{\mathscr{C}}_1)$, with $g_{11} = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}, g_1 = \{(a_2, b_2)\}$
$B_{11} = domg_{11} = \{a_1, a_2\}, B_1 = domg_1 = \{a_2\}$,
$C_{11} = codg_{11} = \{b_1, b_2\}, C_1 = codg_1 = \{b_2\}$, L is the same complete Boolean Algebra as in Example II.6 .
Define $\varphi : L \to M$ as $\varphi(\alpha) = \beta, \varphi(\beta) = \alpha, \varphi(0) = 0, \varphi(1) = 1$.
$\mu_{1B_{11}} a_1 = 1, \mu_{1B_{11}} a_2 = 1, \mu_{2B_1} a_2 = 1$ $i.e., \mu_{1B_{11}} a_2 \geqq \mu_{2B_1} a_2$
$\mu_{1C_{11}} b_1 = \alpha, \mu_{1C_{11}} b_2 = \beta, \mu_{2C_1} b_2 = \beta i.e., \mu_{1C_{11}} b_2 \geqq \mu_{2c_1} b_2$
$\overline{g}_2 = (\overline{g}_2, \overline{\mathscr{B}}_2, \overline{\mathscr{C}}_2)$, with $g_{12} = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}, g_2 = \{(a_2, b_2)\}$
$B_{12} = \{a_1, a_2\}, B_2 = \{a_2\}, C_{11} = \{b_1, b_2\}, C_1 = \{b_2\}$.
$\mu_{1B_{12}} a_1 = 1, \mu_{1B_{12}} a_2 = 1, \mu_{2B_2} a_2 = 0$ $i.e., \mu_{1B_{12}} a_2 \geqq \mu_{2B_2} a_2$
$\mu_{1C_{12}} b_1 = 0, \mu_{1C_{12}} b_2 = 1, \mu_{2C_2} b_2 = 0$ $i.e., \mu_{1C_{12}} b_2 \geqq \mu_{2C_2} b_2$.

Definition III.4. An Fs-binary relation $\overline{g} = (g_1, g, \varphi) \in R^2(\mathscr{B}_0, \mathscr{C}_0)$ is said to be a decreasing Fs-binary relation, from $\mathscr{B}_0$ into $\mathscr{C}_0$, denoted by $(\overline{g} \downarrow, \mathscr{B}_1, \mathscr{C}_1) \subseteq \mathscr{B}_0 \times \mathscr{C}_0$ or simply $\overline{g} \downarrow$,if, and only if (from Fig 8)

1) If both $\vee(\mu_{1C_{11}} g_1 b)$ and $\varphi\ o(\vee \mu_{1B_{11}} b)$, exist for any $b \in\ B_{11}$ then $\vee(\mu_{1C_{11}} g_1 b) \leqq \varphi\ o(\vee \mu_{1B_{11}} b)$,

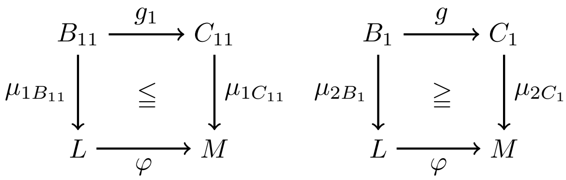

Fig. 8. Decreasing Fs-binary Relation

2) If both $\vee(\mu_{2C_1}gb)$ and $\varphi o(\vee\mu_{2B_1}b)$, exist for any $b \in B_1$ then $\vee(\mu_{2C_1}gb) \geqq \varphi o(\vee\mu_{2B_1}b)$,

For some b, if any of the above four expressions does not exist, then the corresponding inequality is not tenable.

For an example of an decreasing Fs-binary relation one can see Example III.6

Theorem III.5. *The Collection of all decreasing Fs-binary relations, denoted by $R_d^2(\mathscr{B}_0, \mathscr{C}_0)$ with partial order $\leqq$ given in $R^2(\mathscr{B}_0, \mathscr{C}_0)$ [19] is a $\wedge$-Complete lattice.*

Proof: Initially let us prove this theorem for two decreasing Fs-binary relations and then we can generalize it to any arbitrary family of decreasing Fs-binary relations.

In general, we have $\bar{g}_i$ with Fs-objects $\mathscr{B}_i$ and $\mathscr{C}_i$.

Given $\bar{g}_1 \downarrow$ and $\bar{g}_2 \downarrow$. Assuming all relevant expression are existing, we have for $\bar{g}_1 \downarrow$,

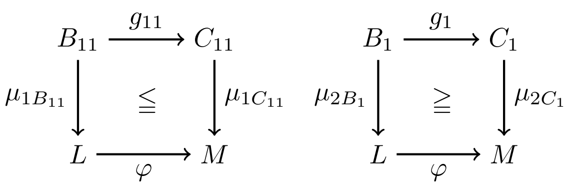

Fig. 9. Decreasing Fs-binary relation

1) $\vee(\mu_{1C_{11}}g_{11}b) \leqq \varphi\ o(\vee\mu_{1B_{11}}b)$, for $b \in B_{11}$
2) $\vee(\mu_{2C_1}g_1b) \geqq \varphi\ o(\vee\mu_{2B_1}b)$, for $b \in B_1$,
   $\bar{g}_1$ with Fs-objects $\mathscr{B}_1$ and $\mathscr{C}_1$
   where $domg_{11} = B_{11}, codg_{11} = C_{11}$,
   $domg_1 = B_1, codg_1 = C_1, \mu_{1B_{11}} \subseteq B_{11} \times L$,
   $\mu_{1C_{11}} \subseteq C_{11} \times M, \mu_{2B_1} \subseteq B_1 \times L$,
   $\mu_{2C_1} \subseteq C_1 \times M$ and
   Similarly, for $\bar{g}_2 \downarrow$,we have

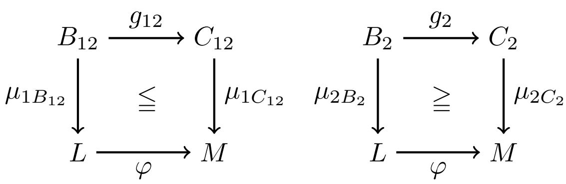

Fig. 10. Decreasing Fs-binary Relation

3) $\vee(\mu_{1C_{12}}g_{12}b) \leqq \varphi\ o(\vee\mu_{1B_{12}}b)$, for $b \in B_{12}$
4) $\vee(\mu_{2C_2}g_2b) \geqq \varphi\ o(\vee\mu_{2B_2}b)$, for $b \in B_2$,
   $\bar{g}_2$ with Fs-objects $\mathscr{B}_2$ and $\mathscr{C}_2$
   where $domg_{12} = B_{12}, codg_{12} = C_{12}, domg_2 = B_2$,
   $codg_2 = C_2, \mu_{1B_{12}} \subseteq B_{12} \times L, \mu_{1C_{12}} \subseteq C_{12} \times M$,
   $\mu_{2B_2} \subseteq B_2 \times L, \mu_{2C_2} \subseteq C_2 \times M$
   Suppose $\bar{g}_1 \wedge \bar{g}_2$ be with Fs-objects $\mathscr{B}_4$ and $\mathscr{C}_4$
   Need to show
   $\bar{g}_4$ is decreasing Fs-binary relation.
   Sufficient to show whenever all the relevant expressions exist, then we have to show with the help of the following diagram.

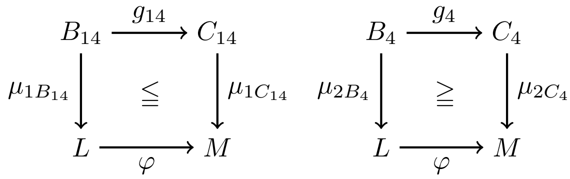

Fig. 11. Decreasing Fs-binary Relation

5) $\vee(\mu_{1C_{14}}g_{14}b) \leqq \varphi\ o(\vee\mu_{1B_{14}}b)$, for $b \in B_{14}$
6) $\vee(\mu_{2C_4}g_4b) \geqq \varphi\ o(\vee\mu_{2B_4}b)$, for $b \in B_4$
   where $g_{14} = g_{11} \cap g_{12}$

$$
\begin{aligned}
domg_{14} &= dom\,(g_{11} \cap g_{12}) \subseteq domg_{11} \cap\ domg_{12} \\
&= B_{11} \cap\ B_{12} \\
codg_{14} &= cod\,(g_{11} \cap g_{12}) \subseteq codg_{11} \cap codg_{12} = C_{11} \cap C_{12} \\
g_4 &= g_1 \cup g_2 \\
dom\ g_4 &= dom(g_1 \cup\ g_2) =\ domg_1 \cup\ domg_2 \\
&= B_1 \cup\ B_2 \\
cod\ g_4 &= cod(g_1 \cup\ g_2) =\ codg_1 \cup\ codg_2 \\
&= C_1 \cup\ C_2
\end{aligned}
$$

For $\mu_{1B_{14}} \subseteq B_{14} \times L, \vee\mu_{1B_{14}}b$ is defined by,

$$\vee\mu_{1B_{14}}b = \{(\vee\mu_{1B_{11}}) \wedge (\vee\mu_{1B_{12}})\}(b)$$

For $\mu_{1C_{14}} \subseteq C_{14} \times M, \vee\mu_{1C_{14}}c$ is defined by,

$$\vee\mu_{1C_{14}}c = \{(\vee\mu_{1C_{11}}) \wedge (\vee\mu_{1C_{12}})\}(c)$$

For $\mu_{2B_4} \subseteq B_4 \times L, \vee\mu_{2B_4}b$ is defined by,

$$\vee\mu_{2B_4}b = (\vee\mu_{2B_1}b) \vee (\vee\mu_{2B_2}b)$$

For $\mu_{2C_4} \subseteq C_4 \times M, \vee\mu_{2C_4}c$ is defined by,

$$\vee\mu_{2C_4}c = (\vee\mu_{2C_1}c) \vee (\vee\mu_{2C_2}c)$$

Proof(5):

$$
\begin{aligned}
\vee(\mu_{1C_{14}}g_{14}b) &= (\vee\mu_{1C_{11}}g_{14}b) \wedge (\vee\mu_{1C_{12}}g_{14}b)\ [19] \\
&= [\vee\mu_{1C_{11}}(g_{11}b \cap g_{12}b)] \\
&\qquad \wedge [\vee\mu_{1C_{12}}(g_{11}b \cap g_{12}b)] \\
&\leqq (\vee\mu_{1C_{11}}g_{11}b) \wedge (\vee\mu_{1C_{12}}g_{12}b) \\
&\leqq (\varphi o(\vee\mu_{1B_{11}}b)) \wedge (\varphi o(\vee\mu_{1B_{12}}b)) \\
&\qquad \text{from (1) and (3)} \\
&= \varphi o((\vee\mu_{1B_{11}}b) \wedge (\vee\mu_{1B_{12}}b)) \\
&= \varphi o(\vee\mu_{1B_{14}}b)
\end{aligned}
$$

so that, $\vee(\mu_{1C_{14}}g_{14}b) \leqq \varphi o(\vee\mu_{1B_{14}}b)$. ■

Proof(6): **Case 1**.

$$\vee(\mu_{2C_4}g_4b) = (\vee\mu_{2C_1}g_4b) \vee (\vee\mu_{2C_2}g_4b)\ [19]$$

(assuming g_4b is in both domains of μ_{2C_1} and μ_{2C_2})

$$
\begin{aligned}
&= [\vee\mu_{2C_1}(g_1b \cup g_2b)] \\
&\qquad \vee [\vee\mu_{2C_2}(g_1b \cup g_2b)]
\end{aligned}
$$

(assuming b is in both domains of g_1 and g_2)

$$= [\vee(\mu_{2C_1}g_1b \cup \mu_{2C_1}g_2b)]$$

$$\vee\left[\vee\left(\mu_{2C_2}g_1b\cup\mu_{2C_2}g_2b\right)\right]$$

(assuming both g_1b and g_2b are in all domains of μ_{2C_1} and μ_{2C_2})

$$\geqq(\vee\mu_{2C_1}g_1b)\vee(\vee\mu_{2C_2}g_2b)$$
$$\geqq(\varphi o(\vee\mu_{2B_1}b))\vee(\varphi o(\vee\mu_{2B_2}b))$$
from (2) and (4)
$$=\varphi o((\vee\mu_{2B_1}b)\vee(\vee\mu_{2B_2}b))$$
$$=\varphi o\left(\vee\mu_{2B_4}b\right)$$

so that, $\vee\left(\mu_{2C_4}g_4b\right)\geqq\ \varphi o(\vee\mu_{2B_4}b)$.

Case 2. Suppose $(b,*)\in g_1$ and $(b,*)\notin g_2$

Then, it follows $b\notin B_2=dom\ \mu_{2B_2}$

$$\text{Hence}\vee\left(\mu_{2C_4}g_4b\right)=\vee\left\{\mu_{2C_4}\left(g_1\cup g_2\right)(b)\right\}$$
$$=\vee\left(\mu_{2C_4}g_1b\right)$$
$$=\left\{(\vee\mu_{2C_1})\vee(\vee\mu_{2C_2})\right\}(g_1b)$$
$$\geqq\ \vee\left(\mu_{2C_1}g_1b\right)$$
$$\geqq\varphi o\left(\vee\mu_{2B_1}b\right)$$
$$=\varphi o\left(\vee\mu_{2B_4}b\right)(\because\vee\mu_{2B_4}b$$
$$=\left\{(\vee\mu_{2B_1})\vee(\vee\mu_{2B_2})\right\}(b))$$

so that $\vee\left(\mu_{2C_4}g_4b\right)\geqq\ \varphi o(\vee\mu_{2B_4}b)$.

Case 3. Suppose $(b,*)\notin g_1$ and $(b,*)\in g_2$

Then, it follows $b\notin B_1=dom\ \mu_{2B_1}$

$$\text{Hence}\vee\left(\mu_{2C_4}g_4b\right)=\vee\left\{\mu_{2C_4}\left(g_1\cup g_2\right)(b)\right\}$$
$$=\vee\left(\mu_{2C_4}g_2b\right)$$
$$=\left\{(\vee\mu_{2C_1})\vee(\vee\mu_{2C_2})\right\}(g_2b)$$
$$\geqq\ \vee\left(\mu_{2C_2}g_2b\right)$$
$$\geqq\varphi o\left(\vee\mu_{2B_2}b\right)$$
$$=\varphi o\left(\vee\mu_{2B_4}b\right)(\because\vee\mu_{2B_4}b$$
$$=\left\{(\vee\mu_{2B_1})\vee(\vee\mu_{2B_2})\right\}(b))$$

so that,$\vee\left(\mu_{2C_4}g_4b\right)\geqq\ \varphi o(\vee\mu_{2B_4}b)$. ■

Given a family $\{\overline{g}_i\}_{i\in I}$ with $\overline{g}_i\downarrow$,for each $i\in I$

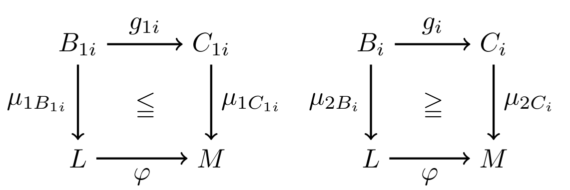

Fig. 12. Decreasing Fs-binary Relation

7) $\wedge\left(\mu_{1C_{1i}}g_{1i}b\right)\leqq\ \varphi\ o(\wedge\mu_{1B_{1i}}b)$, for $b\in\ B_{1i}$,for each $i\in I$
8) $\vee(\mu_{2C_i}g_ib)\geqq\ \varphi\ o(\vee\mu_{2B_i}b)$, for $b\in B_i$, for each $i\in I$

Suppose $\wedge_{i\in I}\overline{g}_i=\overline{h}$.We use notations as in [19]

Here $\overline{g}_i\subseteq\mathscr{B}_i\times\mathscr{C}_i$,$\overline{h}\subseteq\wedge_{i\in I}\left(\mathscr{B}_i\times\mathscr{C}_i\right)$.

Now $\overline{h}$ is with Fs-objects $\mathscr{G}$ and $\mathscr{H}$

Here $\overline{h}=(h_1,h,\varphi)$,where $h_1=\cap_{i\in I}g_{1i}$ $\subseteq\cap_{i\in I}\left(B_{1i}\times C_{1i}\right)$,$h=\cup_{i\in I}g_i\ \subseteq\cup_{i\in I}\left(B_i\times C_i\right)$

For $\mu_{1G_1}\subseteq G_1\times L$,$\vee\mu_{1G_1}b$ is defined by

$$\vee\mu_{1G_1}b=\bigwedge_{(b,*)\in\cap_{i\in I}g_{1i}}\left(\vee\mu_{1B_{1i}}b\ \right)\qquad\text{(III.5)}$$

For $\mu_{1H_1}\subseteq H_1\times M$,$\vee\mu_{1H_1}c$ is defined by

$$\vee\mu_{1H_1}c=\bigwedge_{(*,c)\in\cap_{i\in I}g_{1i}}\left(\vee\mu_{1C_{1i}}c\right)\qquad\text{(III.6)}$$

For $\mu_{2G}\subseteq G\times L$,$\vee\mu_{2G}b$ is defined by

$$\vee\mu_{2G}b=\bigvee_{(b,*)\in\cup_{i\in I}g_i}\left(\vee\mu_{2B_i}b\right)\qquad\text{(III.7)}$$

For $\mu_{2H}\subseteq H\times M$,$\vee\mu_{2H}c$ is defined by

$$\vee\mu_{2H}c=\bigvee_{(*,c)\in\cup_{i\in I}g_i}\left(\vee\mu_{2C_i}c\right)\qquad\text{(III.8)}$$

Need to show

9) $\wedge\left(\mu_{1H_1}h_1b\right)\leqq\varphi o(\wedge\mu_{1G_1}b)$, for $b\in G_1$
10) $\vee(\mu_{2H}hb)\geqq\ \varphi o(\vee\mu_{2G}b)$, for $b\in G$.

Proof(9):

$$\wedge\left(\mu_{1H_1}h_1b\right)=\bigwedge_{(*,h_1b)\in\cap_{i\in I}g_{1i}}\left(\vee\mu_{1C_{1i}}h_1b\right)$$
$$=\bigwedge_{i\in I}\bigvee_{(*,h_1b)\in g_{1i}}\mu_{1C_{1i}}h_1b$$
$$=\bigwedge_{i\in I}\bigvee_{(*,g_{1j}b)\in g_{1j}}\mu_{1C_{1i}}\left(\bigcap_{j\in I}g_{1j}\ b\right)$$
$$=\bigwedge_{i\in I}\left[\bigcap_{j\in I}\mu_{1C_{1i}}\ g_{1j}b\right]$$
$$\leqq\bigwedge_{i\in I}\left[\vee\left(\mu_{1C_{1i}}g_{1i}b\right)\right]$$
$$\leqq\bigwedge_{i\in I}\left[\varphi o(\vee\mu_{1B_{1i}}b)\right]$$
$$=\varphi\left[\bigwedge_{i\in I}\vee\mu_{1B_{1i}}b\right]$$
$$=\varphi\left[\bigwedge_{i\in I}\bigvee_{(b,*)\in g_{1i}}\mu_{1B_{1i}}b\ \right]$$
$$(\because\ \text{here}\ \vee\mu_{1B_{1i}}b=\bigvee_{(b,*)\in g_{1i}}\mu_{1B_{1i}}b)$$
$$=\varphi\left[\bigwedge_{(b,*)\in\cap_{i\in I}g_{1i}}\left(\vee\mu_{1B_{1i}}b\ \right)\right]=\varphi o(\wedge\mu_{1G_1}b)$$

so that $\wedge\left(\mu_{1H_1}h_1b\right)\leqq\varphi o(\wedge\mu_{1G_1}b)$. ■

Proof(10):

$$\vee\left(\mu_{2H}hb\right)=\bigvee_{(*,hb)\in\cup_{i\in I}g_i}\left(\vee\mu_{2C_i}hb\right)$$
$$=\bigvee_{i\in I}\bigvee_{(*,hb)\in g_i}\mu_{2C_i}hb$$
$$=\bigvee_{i\in I}\bigvee_{(*,g_jb)\in g_j}\mu_{2C_i}\left(\bigcup_{j\in I}g_j\ b\right)$$
$$=\bigvee_{i\in I}\left[\bigcup_{j\in I}\mu_{2C_i}\ g_jb\right]$$
$$\geqq\bigvee_{i\in I}\left[\vee\left(\mu_{2C_i}g_ib\right)\right]$$
$$\geqq\bigvee_{i\in I}\left[\varphi o(\vee\mu_{2B_i}b)\right]$$
$$=\varphi\left[\bigvee_{i\in I}\vee\mu_{2B_i}b\right]$$
$$=\varphi\left[\bigvee_{i\in I}\bigvee_{(b,*)\in g_i}\mu_{2B_i}b\ \right]$$
$$(\because\ \text{here}\vee\mu_{2B_i}b=\bigvee_{(b,*)\in g_i}\mu_{2B_i}b\ )$$
$$=\varphi\left[\bigvee_{(b,*)\in\cup_{i\in I}g_i}\left(\vee\mu_{2B_i}b\ \right)\right]=\varphi o(\vee\mu_{2G}b)$$

so that,$\vee\left(\mu_{2H}hb\right)\geqq\ \varphi o(\vee\mu_{2G}b)$. ■

Hence the theorem. ■

Example III.6. A collection of all decreasing Fs-binary relations, denoted by $\mathrm{R}^2_{\mathrm{d}}\left(\mathscr{B}_0,\mathcal{C}_0\right)$ with partial order $\leqq$ given as in $\mathrm{R}^2\left(\mathscr{B}_0,\mathscr{C}_0\right)$ is not a $\vee$ - Complete lattice.

Here $\overline{g}_1$ and $\overline{g}_2$ are decreasing Fs-binary relations but $\overline{g}_1\vee\overline{g}_2$

is not an decreasing Fs-binary relation.
$\overline{g}_1 = (\overline{g}_1, \overline{\mathscr{B}}_1, \overline{\mathscr{C}}_1)$, with $g_{11} = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}, g_1 = \{(a_2, b_2)\}$
$B_{11} = \{a_1, a_2\}, B_1 = \{a_2\}, C_{11} = \{b_1, b_2\}, C_1 = \{b_2\}$, L is the same coomplete Boolean Algebra as in Example II.6
Define $\varphi : L \to M$ as $\varphi(\alpha) = \beta, \varphi(\beta) = \alpha, \varphi(0) = 0, \varphi(1) = 1$.
$\mu_{1B_{11}} a_1 = \alpha, \mu_{1B_{11}} a_2 = \alpha, \mu_{2B_1} a_2 = 0 \; i.e., \mu_{1B_{11}} a_2 \geqq \mu_{2B_1} a_2$
$\mu_{1C_{11}} b_1 = \beta, \mu_{1C_{11}} b_2 = \beta, \mu_{2C_1} b_2 = \beta \; i.e., \mu_{1C_{11}} b_2 \geqq \mu_{2c_1} b_2$
$\overline{g}_2 = (\overline{g}_2, \overline{\mathscr{B}}_2, \overline{\mathscr{C}}_2)$, with $g_{12} = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}, g_2 = \{(a_2, b_2)\}$
$B_{12} = \{a_1, a_2\}, B_2 = \{a_2\}, C_{11} = \{b_1, b_2\}, C_1 = \{b_2\}$.
$\mu_{1B_{12}} a_1 = \alpha, \mu_{1B_{12}} a_2 = \alpha, \mu_{2B_2} a_2 = 0 \; i.e., \mu_{1B_{12}} a_2 \geqq \mu_{2B_2} a_2$
$\mu_{1C_{12}} b_1 = 1, \mu_{1C_{12}} b_2 = \beta, \mu_{2C_2} b_2 = 0 \; i.e., \mu_{1C_{12}} b_2 \geqq \mu_{2C_2} b_2$.

Definition III.7. An Fs-binary relation $\bar{g} = (g_1, g, \varphi) \in R^2(\mathscr{B}_0, \mathscr{C}_0)$ is said to be an preserving Fs-binary relation, from $\mathscr{B}_0$ into $\mathscr{C}_0$, denoted by $\bar{g} \subseteq \mathscr{B}_0 \times \mathscr{C}_0$ or $(\bar{g}, \mathscr{B}_1, \mathscr{C}_1)$ or simply $\bar{g}$,if, and only if (from figure),

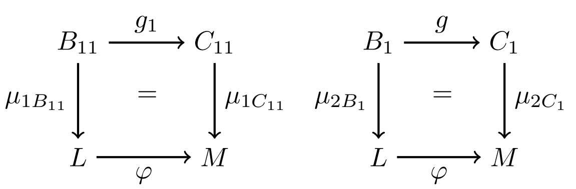

Fig. 13. Preserving Fs-binary Realtin

1) If both $\vee(\mu_{1C_{11}} g_1 b)$ and $\varphi \; o(\vee \mu_{1B_{11}} b)$, exist for any $b \in \; B_{11}$ then $\vee(\mu_{1C_{11}} g_1 b) = \varphi \; o(\vee \mu_{1B_{11}} b)$,
2) If both $\vee(\mu_{2C_1} g b)$ and $\varphi o(\vee \mu_{2B_1} b)$, exist for any $b \in B_1$ then $\vee(\mu_{2C_1} g b) = \varphi o(\vee \mu_{2B_1} b)$.

For some b, if any of the above four expressions does not exist, then the corresponding equality is not tenable.

Theorem III.8. *The Collection of all preserving Fs-binary relations, denoted by* $\mathrm{R}^2_{\mathrm{p}}(\mathscr{B}_0, \mathscr{C}_0)$ *is an infinitely distributive sub lattice of* $\left(\mathrm{R}^2(\mathscr{B}_0, \mathscr{C}_0), \leqq\right)$*[19].*

Proof follows from (III.2) and (III.5).

Remarks III.9. we can observe that $\mathrm{R}^2_p(\mathscr{B}_0, \mathscr{C}_0)$ is a sub poset of both $\mathrm{R}^2_i(\mathscr{B}_0, \mathscr{C}_0)$ and $\mathrm{R}^2_d(\mathscr{B}_0, \mathscr{C}_0)$.

Remarks III.10. $\mathrm{R}^2_i(\mathscr{B}_0, \mathscr{C}_0)$ and $\mathrm{R}^2_d(\mathscr{B}_0, \mathscr{C}_0)$ both are sub posets of $\left(\mathrm{R}^2(\mathscr{B}_0, \mathscr{C}_0), \leqq\right)$.

IV. Composition between suitable Fs-binary relations

Definition IV.1. Given any two Fs-binary relations $\bar{f} = (\bar{f}, \mathscr{B}, \mathscr{C})$ and $\bar{g} = (\bar{g}, \mathscr{C}, \mathscr{D})$, with $\bar{f} = (f_1, f, \varphi_1)$ and $\bar{g} = (g_1, g, \varphi_2)$, we define Composition between them, denoted by $\bar{g} \circ \bar{f}$ is given by $\bar{g} \circ \bar{f} = \left(\bar{g} \circ \bar{f}, \mathscr{B}, \mathscr{D}\right) = ((g_1 \circ f_1, g \circ f, \varphi_2 \circ \varphi_1), \mathscr{B}, \mathscr{D})$.

We can easily verify the following two theorems.

Theorem IV.2.

(a) *composition of increasing Fs-binary relations is an increasing Fs-binary relation.*
(b) *composition of decreasing Fs-binary relations is a decreasing Fs-binary relation.*
(c) *composition of preserving Fs-binary relations is a preserving Fs-binary relation.*

Theorem IV.3.

(a) *The class of all Fs-sets together with sets of morphisms: given by Fs-binary relations on Fs-sets, is a category, denoted by* $\mathbb{FSBR}$.
(b) *The class of all Fs-sets together with sets of morphisms: given by increasing Fs-binary relations on Fs-sets, is a category, denoted by* $\mathbb{FSBR}_i$.
(c) *The class of all Fs-sets together with sets of morphisms: given by decreasing Fs-binary relations on Fs-sets, is a category, denoted by* $\mathbb{FSBR}_d$.
(d) *The class of all Fs-sets together with sets of morphisms: given by preserving Fs-binary relations on Fs-sets, is a category, denoted by* $\mathbb{FSBR}_p$.

V. Conclusions and Future Work

We defined in this research paper an increasing Fs-binary relation, a decreasing Fs-binary relation and a preserving Fs-binary relation between any two Fs-subsets of an Fs-set. Also, we introduced the category of Fs-sets (increasing, decreasing and preserving) and proved some results.

Conflict of Interest: All authors have no conflict of interests.

Data Availability: The manuscript has no associate data Subsubsection text here.

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