An Approximately Exact Smooth Penalty Function for Nonlinear Constrained Optimization Problem

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***Abstract*—Penalty functions have important applications in solving constrained optimization problems. In this paper, a new strategy is introduced for smooth approximation of l_1 penalty function. We further obtain error estimates between the optimal objective function values of constrained problem and the ones of new smooth penalty problem. As an example, we provide a smooth penalty function for mathematical programming with complementary constraints (MPCC)**

***Index Terms*—Constrained problem, Penalty function, Smooth penalty problem, Complementary constraints.**

I. Introduction

In this paper we consider the nonlinear optimization problem (NP):

$$\begin{aligned} &\min_{z\in\Re^n} \bar{f}(z) \\ &s.t.\ g_t(z)\le 0, \quad t=1,...,m, \\ &\qquad h_t(z)=0, \quad t=1,...,l, \end{aligned} \tag{1.1}$$

where the functions in this model are all continuously differentiable in $\Re^n$. Suppose in this paper the feasible set is bounded and closed.

The above model has been widely used in industry, engineering, management and other fields(see [1], [2], [3], [4]). In many optimization methods to solve this problem, exact penalty function has always taken an important role([5], [6], [7]). In [5] the l_1 penalty function is given as

$$P_1(z;\sigma)=\bar{f}(z)+\sigma(\|h(z)\|_1+\|g^+(z)\|_1), \tag{1.2}$$

with the penalty parameter $\sigma > 0$, and $g^+(z)$ with the components $\max\{0, g_i(z)\}$. Here the norm $\|\cdot\|_1$ is the l_1 norm. The corresponding penalty problem is

$$\min_{z\in\Re^n} P_1(z;\sigma). \tag{1.3}$$

In [8], the authors use a l_p exact penalty function constructed by the norm $\|\cdot\|_p$ to establish a global optimization algorithm for the optimization problem with general inequality constraints and simple convex inequality constraints.

For traditional exact penalty functions, when their penalty parameters are large enough, the global optimal solutions of unconstrained penalty problem are also the ones of the constrained problem. This is the benefit of its exactness. However, the non smoothness of traditional exact penalty functions also makes it difficult for many efficient fast algorithms based on function gradients to be effectively applied.

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In order to reduce the adverse effects of the non smoothness of penalty functions and achieve a certain degree of exactness, many scholars have studied smoothing techniques for traditional exact penalty functions([9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]).

In most literatures, the smoothing of non-smooth penalty function is mainly aimed at the optimization problem with only inequality constraints. In this paper, we give an approximately exact penalty function for optimization problems with both equality and inequality constraints. Our function has good smoothness and is a good approximation of the l_1 penalty function.

We give the arrangement of this article. A smoothing approach is proposed for (1.2) and the corresponding smoothed penalty problem is given in Section 2. The degree of approximation between the smooth penalty problem and the exact penalty problem is discussed, and error estimates between the corresponding optimal objective function values are obtained. An approximate penalty algorithm is also given. In Section 3, as an instance of our approximating technique, a smooth penalty function is constructed for the mathematical program with complementarity constraints. In Section 4 some conclusions are given.

II. Smooth penalty function and error estimates

We use in this section a smooth approximation to the absolute value function $|y|$ as

$$|y|\approx\varphi_1(y;\varepsilon):=\varepsilon(\ln 2+\ln(1+\cosh(y/\varepsilon))), \tag{2.1}$$

where

$$\cosh(z)=\frac{e^z+e^{-z}}{2},$$

and use another smooth function to approximate the function $y^+=\max\{0,y\}$ by

$$y^+\approx\varphi_2(y;\varepsilon):=\varepsilon\ln(1+\exp(y/\varepsilon)), \tag{2.2}$$

where $\varepsilon > 0$ is a parameter used to control the degree of approximation.

For $\varepsilon > 0$, the maximum difference between $|y|$ and $\varphi_1(y;\varepsilon)$ lies at the point $y=0$, which is $2\varepsilon\ln 2$. Furthermore, we can get the following error estimates

$$||y|-\varphi_1(y;\varepsilon)|\le\frac{8}{3}\varepsilon\exp(-|y|\varepsilon). \tag{2.3}$$

The maximum difference between y^+ and $\varphi_2(y;\varepsilon)$ lies at the point $y=0$, which is $\varepsilon\ln 2$, and we can get that

$$|y^+-\varphi_2(y;\varepsilon)|\le\frac{5}{3}\varepsilon\exp(-y^+\varepsilon). \tag{2.4}$$

Besides, the first derivative function of $\varphi_1(y;\varepsilon)$ is

$$\varphi_1'(y;\varepsilon)=\frac{\sinh(y/\varepsilon)}{\cosh(y/\varepsilon)+1}\in(-1,1), \tag{2.5}$$

where

$$\sinh(z)=\frac{e^z-e^{-z}}{2},$$

and its second derivative function is

$$\varphi_1''(y;\varepsilon)=\frac{1}{\varepsilon(\cosh(y/\varepsilon)+1)}\in(0,\frac{1}{2\varepsilon}). \quad (2.6)$$

The first derivative function of $\varphi_2(y;\varepsilon)$ is

$$\varphi_2'(y;\varepsilon)=\frac{\exp(y/\varepsilon)}{1+\exp(y/\varepsilon)}\in(0,1), \quad (2.7)$$

and its second derivative function is

$$\varphi_2''(y;\varepsilon)=\frac{1}{2(\cosh(y/\varepsilon)+1)\varepsilon}\in(0,\frac{1}{4\varepsilon}). \quad (2.8)$$

By the above smoothing functions, we now give a smooth approximation to the l_1 exact penalty function (1.2). The new smooth and approximately exact penalty function for nonlinear constrained optimization problem (1.1) is given as follows,

$$\begin{aligned}\psi(z;\sigma,\varepsilon)= \quad & \bar{f}(z)+\sigma\varepsilon(\sum_{t=1}^{m}\ln(1+\exp(g_t(z)/\varepsilon))\\ &+\sum_{t=1}^{l}(\ln 2+\ln(1+\cosh(h_t(z)/\varepsilon)))),\end{aligned} \quad (2.9)$$

and the corresponding penalty problem

$$\min_{z\in\Re^n}\psi(z;\sigma,\varepsilon). \quad (2.10)$$

From the above discussion, we can get the following conclusion

Theorem 2.1: For any $\sigma>0$, $\varepsilon>0$, and $z\in\Re^n$, we have

$$-\gamma_1\sigma\varepsilon\le\psi(z;\sigma,\varepsilon)-P_1(z;\sigma)\le\gamma_1\sigma\varepsilon,$$

where

$$\gamma_1=\frac{8}{3}l+\frac{5}{3}m.$$

Theorem 2.2: If for any $\sigma>0$ and $\varepsilon>0$, $\hat{z}$ and z^* are the optimal solution of the problem (2.10) and (1.1) respectively, then

$$-\gamma_1\sigma\varepsilon\le\psi(\hat{z};\sigma,\varepsilon)-P_1(z^*;\sigma)\le\gamma_1\sigma\varepsilon.$$

Proof. By the condition and Theorem 2.1, we know that

$$\begin{aligned}-\gamma_1\sigma\varepsilon &\le\psi(\hat{z};\sigma,\varepsilon)-P_1(\hat{z};\sigma)\\ &\le\psi(\hat{z};\sigma,\varepsilon)-P_1(z^*;\sigma)\\ &\le\psi(z^*;\sigma,\varepsilon)-P_1(z^*;\sigma)\le\gamma_1\sigma\varepsilon.\end{aligned}$$

□

Definition 2.3: z_ε is an ε-feasible solution of (1.1), if

$$g_t(z_\varepsilon)\le\varepsilon,\quad i=1,...,m,$$

$$|h_t(z_\varepsilon)|\le\varepsilon,\quad j=1,...,l.$$

Based on this definition,the conclusion below is given.

Theorem 2.4: If for any $\sigma>0$ and $\varepsilon>0$, $\hat{z}$ and $\tilde{z}$ are the optimal solution of the problem (2.10) and (1.3) respectively, $\tilde{z}$ is a feasible solution of (1.1), and $\hat{z}$ is an ε-feasible solution of (1.1), then there exists a constant $\gamma_2>0$, such that

$$-\gamma_2\sigma\varepsilon\le\bar{f}(\hat{z})-\bar{f}(\tilde{z})\le\gamma_1\sigma\varepsilon.$$

Proof. Since that $\hat{z}$ is an ε-feasible solution of (1.1), then

$$g_t(\hat{z})\le\varepsilon,\quad t=1,...,m,$$

$$|h_t(\hat{z})|\le\varepsilon,\quad t=1,...,l,$$

and

$$\begin{aligned}&\sum_{t=1}^{m}\ln(1+\exp(g_t(\hat{z})/\varepsilon))\\ &+\sum_{t=1}^{l}(\ln 2+\ln(1+\cosh(h_t(\hat{z})/\varepsilon)))\\ &\le l(\ln 2+\ln(1+\tfrac{e+e^{-1}}{2}))+m(\ln(1+e))\\ &=:\gamma_3.\end{aligned}$$

From Theorem 2.2 we know that

$$-\gamma_1\sigma\varepsilon\le\psi(\hat{z};\sigma,\varepsilon)-P_1(\tilde{z};\sigma)\le\gamma_1\sigma\varepsilon,$$

and

$$\begin{aligned}-\gamma_2\sigma\varepsilon\le\quad & \bar{f}(\hat{z})+\sigma\varepsilon(\sum_{t=1}^{m}\ln(1+\exp(g_t(\hat{z})/\varepsilon))\\ &+\sum_{t=1}^{l}(\ln 2+\ln(1+\cosh(h_t(\hat{z})/\varepsilon))))\\ &-(\bar{f}(\tilde{z})+\sigma(\sum_{t=1}^{m}g_t^+(z)+\sum_{t=1}^{l}|h_t(z)|)\le\gamma_1\sigma\varepsilon,\end{aligned}$$

where $\gamma_2=\gamma_1+\gamma_3$. Thus,

$$-\gamma_2\sigma\varepsilon\le\bar{f}(\hat{z})-\bar{f}(\tilde{z})\le\gamma_1\sigma\varepsilon.$$

□

From Theorem 2.4 we know that when $\varepsilon>0$ is sufficiently small, if the optimal solution of (2.10) is an ε-feasible solution of (1.1), then it approximately solves (1.1).

Definition 2.5: ([6]) We call the problem (1.1) is a convex constrained optimization problem if the functions $\bar{f}$ and g_i are all convex functions, and the functions h_t are all affine functions.

Definition 2.6: ([6]) The KKT conditions hold at z^*, if

$$\nabla\bar{f}(z^*)+\sum_{t=1}^{m}\mu_t^*\nabla g_t(z^*)+\sum_{t=1}^{l}\nu_t^*\nabla h_t(z^*)=0,$$

$$\mu_t^*g_t(z^*)=0,\quad \mu_t^*\ge 0,\quad g_t(z^*)\le 0,\quad t=1,...,m,$$

$$h_t(z^*)=0,\quad t=1,...,l,$$

where μ_t^*, ν_t^* are the corresponding Lagrangian multipliers.

From the above definitions, we have the following conclusion.

Theorem 2.7: Let the problem (1.1) be convex, and the KKT conditions hold at z^* with the corresponding Lagrangian multiplier (μ^*,ν^*). If

$$\sigma\ge\max\{\max_{t=1}^{m}\mu_t^*,\max_{t=1}^{l}|\nu_t^*|\},$$

then

$$\psi(z^*;\sigma,\varepsilon)\le\psi(z;\sigma,\varepsilon)+2\gamma_1\sigma\varepsilon,$$

for any $z\in\Re^n$.

Proof. Since the problem (1.1) is convex, and the KKT conditions hold at z^*, we have that z^* is an optimal solution of (1.1), and for any $z \in \Re^n$

$$\begin{array}{l} P_1(z;\sigma) \\ \geq \bar{f}(z^*) + \nabla \bar{f}(z^*)^T(z-z^*) + \sigma(\|g^+(z)\|_1 + \|h(z)\|_1) \\ = \bar{f}(z^*) - \sum_{t=1}^{m} \mu_i^* \nabla g_t(z^*)^T(z-z^*) \\ - \sum_{t=1}^{l} \nu^* \nabla h_t(z^*)^T(z-z^*) + \sigma(\sum_{t=1}^{m} g_t^+(z) + \sum_{t=1}^{l} |h_t(z)|) \\ \geq \bar{f}(z^*) - \sum_{t=1}^{m} \mu_i^*(g_t(z) - g_t(z^*)) \\ - \sum_{t=1}^{l} \nu_t^*(h_t(z) - h_t(z^*)) + \sigma(\sum_{t=1}^{m} g_t^+(z) + \sum_{t=1}^{l} |h_t(z)|) \\ = \bar{f}(z^*) - \sum_{t=1}^{m} \mu_t^* g_t(z) \\ - \sum_{t=1}^{l} \nu_t^* h_t(z) + \sigma(\sum_{t=1}^{m} g_t^+(z) + \sum_{t=1}^{l} |h_t(z)|). \end{array}$$

Then,

$$P_1(z;\sigma) \geq \bar{f}(z^*) + \sum_{t=1}^{m}(\sigma - \mu_t^*) g_t^+(z) + \sum_{t=1}^{l}(\sigma - |\nu_t^*|)|h_t(z)|.$$

So for

$$\sigma \geq \max\{\max_{t=1}^{m} \mu_t^*, \max_{t=1}^{l} |\nu_t^*|\},$$

we have that for any for any $z \in \Re^n$,

$$P_1(z;\sigma) \geq \bar{f}(z^*). \tag{2.11}$$

From Theorem 2.1 and (2.11), we know that for any $z \in \Re^n$, and $\varepsilon > 0$,

$$-\gamma_1\sigma\varepsilon \leq \psi(z;\sigma,\varepsilon) - P_1(z;\sigma) \leq \gamma_1\sigma\varepsilon. \tag{2.12}$$

Then

$$\begin{array}{l} \psi(z^*;\sigma,\varepsilon) - \psi(z;\sigma,\varepsilon) \\ \leq \psi(z^*;\sigma,\varepsilon) - P_1(z;\sigma) + \gamma_1\sigma\varepsilon \\ \leq \psi(z^*;\sigma,\varepsilon) - \bar{f}(z^*) + \gamma_1\sigma\varepsilon \\ = \psi(z^*;\sigma,\varepsilon) - P_1(z^*;\sigma) + \gamma_1\sigma\varepsilon \\ \leq 2\gamma_1\sigma\varepsilon. \end{array} \tag{2.13}$$

□

Theorem 2.7 shows that when the penalty parameter is greater than a threshold value related to the Lagrangian multiplier of the primal optimal solution z^*, the suboptimal property of any global optimal solution of the convex programming problem (1.1) can be defined by a function composed of penalty parameter and smooth parameter.

Theorem 2.8: Suppose in the problem (1.1), g_t are all convex, and h_t are all affine. If z^* is a local optimal solution of (2.10), then

$$e(z^*) := \|g^+(z^*)\|_1 + \|h(z^*)\|_1 \leq \kappa\varepsilon$$

for $\sigma = O(\frac{1}{\varepsilon})$, where the constant $\kappa > 0$.

Proof. Let $z^{(0)}$ be feasible for (1.1), then $e(z^{(0)}) = 0$. Set $d := z^{(0)} - z^*$ and $L_1 := \|d\|_1$. We consider $z^* + \eta d$, where $\eta \in [0,1]$. Since z^* is a local solution of (2.10) , it follows that there is a $\eta_1 > 0$,

$$\psi(z^* + \eta d;\sigma,\varepsilon) \geq \psi(z^*;\sigma,\varepsilon), \tag{2.14}$$

for any $\eta \in (0,\eta_1]$. Set $\overline{\eta} = \min\{1,\eta_1\}$, and

$$L_2 := \max\{\|\nabla f(\xi)\| \mid \xi \in N(z^*;\overline{\eta})\},$$

where

$$N(z;\overline{\eta}) = \{z \in \Re^n \mid \|z - z^*\| \leq \overline{\eta}\}.$$

By Theorem 2.1, we have that

$$\psi(z^* + \eta d;\sigma,\varepsilon) \leq P_1(z^* + \eta d;\sigma) + \gamma_1\sigma\varepsilon,$$
$$P_1(z^*;\sigma) \leq \psi(z^*;\sigma,\varepsilon) + \gamma_1\sigma\varepsilon.$$

On the other side, since $z^{(0)}$ is feasible for (1.1), we know

$$\begin{array}{rl} 0 & \leq g^+(z^* + \eta d) \\ & = \max\{0, g(z^* + \eta d)\} \\ & \leq \max\{0, (1-\eta)g(z^*) + \tau g(z^{(0)})\} \\ & \leq \max\{0, (1-\eta)g(z^*)\} + \max\{0, \eta g(z^{(0)})\} \\ & = (1-\eta)g^+(z^*), \end{array}$$

and

$$\begin{array}{rl} h(z^* + \eta d) & = h[(1-\eta)z^* + \eta z^{(0)}] \\ & = (1-\eta)h(z^*) + \eta h(z^{(0)}) \\ & = (1-\eta)h(z^*). \end{array}$$

So we have that

$$\begin{array}{l} P_1(z^* + \eta d;\sigma) \\ = \bar{f}(z^* + \eta d) + \sigma(\|g^+(z^* + \eta d)\|_1 + \|h(z^* + \eta d)\|_1) \\ \leq \bar{f}(z^*) + \eta L_1 L_2 + \sigma(1-\eta)(\|g^+(z^*)\|_1 + \|h(z^*)\|_1) \\ = P_1(z^*;\sigma) - \sigma\eta e(z^*) + \eta L_1 L_2, \end{array}$$

and

$$\psi(z^* + \eta d;\sigma,\varepsilon) \leq \psi(z^*;\sigma,\varepsilon) + 2\gamma_1\sigma\varepsilon - \sigma\eta e(z^*) + \eta L_1 L_2.$$

From (2.14), we know that $\forall \eta \in (0,\overline{\eta}]$,

$$e(z^*) \leq \frac{L_1 L_2}{\sigma} + \frac{2\gamma_1}{\eta}\varepsilon.$$

So for $\sigma = O(\frac{1}{\varepsilon})$, we get that

$$e(z^*) \leq \kappa\varepsilon,$$

where

$$\kappa \geq L_1 L_2 + \frac{2\gamma_1}{\overline{\eta}}.$$

□

Theorem 2.8 shows that when the constraint functions satisfies certain convexity and σ is sufficiently large while ε is sufficiently small, the local optimal solution of (2.10) is an approximate feasible solution of(1.1). It is worth noting that we do not require the objective function $\bar{f}$ to be convex here.

The above properties are shown that $\psi(z;\sigma,\varepsilon)$ is a good approximation of $P_1(z;\sigma)$.

We now give an approximate algorithm for (1.1).

Algorithm 2.1

Step 1. Set $\delta > 0$, $\alpha > 0$, $0 < \varrho < 1 < \tau$, and give $\varepsilon_1 > 0$, and $\sigma_1 > 0$. Set $k := 1$.

Step 2. Solve

$$\min_{z \in \Re^n} \psi(z;\sigma_k,\varepsilon_k),$$

and get the optimization solution z^k.

Step 3. If

$$e(z^k) = \|g^+(z^k)\|_1 + \|h(z^k)\|_1 \leq \alpha\varepsilon_k,$$

and $\varepsilon_k \leq \delta$, stop. Otherwise, adjust σ_k and ε_k as follows:

If $e(z^k) \leq \alpha\varepsilon_k$ but $\varepsilon_k > \delta$, set $\sigma_{k+1} := \sigma_k$ and $\varepsilon_{k+1} := \varrho\varepsilon_k$;

If $e(z^k) > \alpha\varepsilon_k$, *set* $\sigma_{k+1} := \tau\sigma_k$ *and*

$$\varepsilon_{k+1} := \max\{\max_{1\le t\le m} g_t(z^k), \max_{1\le t\le l} |h_t(z^k)|\}.$$

Set $k := k+1$, *and turn to Step 2.*

When we use Algorithm 2.1 to solve the convex programming problem (1.1), we can obtain the following conclusion by the error estimation analysis.

Theorem 2.9: We consider the convex problem (1.1). If for any $\sigma \in [\sigma_1, +\infty)$, $\varepsilon \in (0, \varepsilon_1]$, the solution set of (2.10) is not empty, then Algorithm 2.1 can obtains a δ-feasible solution after finite iterations.

III. Smooth penalty functions for MPCC

We now consider the MPCC model with the following form,

$$\begin{aligned} &\min_{z\in\Re^n} \bar{f}(z) \\ &s.t.\ g_t(z) \le 0, \quad t = 1, ..., m, \\ &\qquad h_t(z) = 0, \quad t = 1, ..., l, \\ &\qquad u_t(z) \ge 0, \quad t = 1, ..., p, \\ &\qquad v_t(z) \ge 0, \quad t = 1, ..., p, \\ &\qquad u_t(z)v_t(z) = 0, \quad t = 1, ..., p \end{aligned} \tag{3.1}$$

The constraints of a MPCC problem not only consists of standard inequality and equality constraints but also some additional complementarity-type constraints.

Although the special structure of the problem constraints has caused difficulties in solving, many different types of algorithms have been proposed in recent decades such as smoothing methods, the relaxation methods, and the penalty methods ([21], [22], [23], [24]).

With the similar ideas in Section 2, our new smooth penalty function for MPCC (3.1) is given as

$$\begin{aligned} \phi(z;\sigma,\varepsilon) \quad &= \bar{f}(z) + \sigma\varepsilon(\sum_{t=1}^{m} \ln(1+\exp(g_t(z)/\varepsilon)) \\ &+ \sum_{t=1}^{l} (\ln 2 + \ln(1+\cosh(h_t(z)/\varepsilon))) \\ &+ \sum_{t=1}^{p} \ln(1+\exp(-u_t(z)/\varepsilon)) \\ &+ \sum_{t=1}^{p} \ln(1+\exp(-v_t(z)/\varepsilon)) \\ &+ \sum_{t=1}^{p} (\ln 2 + \ln(1+\cosh(u_t(z)v_t(z)/\varepsilon)))), \end{aligned} \tag{3.2}$$

and the corresponding penalty problem is

$$\min_{z\in\Re^n} \phi(z;\sigma,\varepsilon). \tag{3.3}$$

Unlike the previous references([23], [24]), here we establish an unconstrained penalty problem, while the penalty problems in [23] and [24] are still constrained.

IV. Conclusion

We give an approximately exact and smooth penalty function for the nonlinear programming problems with equality and inequality constraints. This function has good smoothness and is a good approximation of the l_1 penalty function.

We also use this idea to construct a smooth penalty function for mathematical program with complementarity constraints.

Our future work will focus on the establishment of penalty algorithms for MPCC and the discussion of convergence. In addition, we will also explore the uses of our penalty algorithms in other special optimization models.

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