Fuzzy Delta Graphs With Extorial Values

Iruthayaraj S, John Borg S and Britto Antony Xavier G

Abstract—**The main aim of this research is to develop fuzzy delta graphs and its product graphs. The delta graph is a graph whose vertices are weighted by ratio of generalized extorial function and the edges are identified by the differentiation and integration of generalized extorial function. Also we discuss all types of regularity and irregularity properties of fuzzy product graphs with examples.**

Index Terms—**Delta graph, Extorial function, Fuzzy graph, Fuzzy degrees of vertices and edges, Highly irregular fuzzy graph, Irregular fuzzy graph, Regular fuzzy graph, Totally regular fuzzy graph.**

I. Introduction

The authors, Santhimaheswari N.R and Sekar C (2016), focus on the behaviours of irregular fuzzy graphs and explore the concept of the d_m degree of vertices [16]. Moreover, the introduction of (m,k) regular fuzzy graphs and further investigations into totally (m,k) regular fuzzy graphs have opened avenues for comparing different classes of fuzzy graphs. By defining key concepts like the d_m degree and exploring regularity conditions, the field continues to evolve, offering insights into the complexities of uncertain information in graph theoretical frameworks. In[7], author clarifies how two different types of irregular fuzzy graphs neighbourly and highly irregular can be considered equivalent and establishes foundational results for the study of neighbourly irregular fuzzy graphs. To better understand, how these different types of fuzzy graphs interact and differ in their support neighbourly properties. The basic concepts of fuzzy graph is discussed in [10]. The transmission problem on graphs and digraphs are studied in[9]. In[3], the authors discussed how to investigate the solution of the difference equation. Also, the authors explore the properties of a newly defined extorial function and use it to solve a higher order difference equation with constant coefficients within the framework of discrete calculus. The paper includes examples to validate the results. Furthermore, the authors introduce a novel function, called the extorial function, which is defined by polynomials with polynomial factorials. Here we introduce generalized extorial function by adding shift values. We derive identities involving difference operators and generalized extorial function. Some interesting results on the relationship among generalized extorial function, as well as its sums are obtained. This study also describes how to use the generalized extorial function to arrive at fuzzy delta graphs and their product graphs. By choosing the right value on x, such as for electron charge in molecular structures, we can obtain applications in the field of physical science because the generated graph's structure is similar to chemical bonds.

Manuscript received January 10, 2025; revised June 30, 2025. This work was supported in part by the Carreno Grant Fellowship (SHC/Fr.Carreno Research Grant/2023/03), Sacred Heart Fellowship (SHC/SHFellowship/2024/19), Don Bosco Research Grant (SHC/DB Grant/2024/03) and DST for the FIST Fund (SR/FST/College-2017/130(c)).

Iruthayaraj S is a PhD candidate at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: iruthayarajsh@gmail.com)

John Borg S is an Assistant Professor at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: sjborg@gmail.com)

Britto Antony Xavier G is an Associate Professor at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail : brittoshc@gmail.com)

II. Basic definitions and theorems

In this section, we present the necessary basic definition and related theorems which will be used in the subsequent sections.

Definition II.1. [1] Let V and E be the set of vertices and edges of a graph G respectively. Let α and β be two functions from the vertex set V into [0,1] and edge set E into [0,1] satisfying the condition $\beta(v,w) \leq \alpha(v) \wedge \alpha(w)$. Then the pair $\widetilde{G} = (\alpha, \beta)$ is called as a fuzzy graph on G. $\widetilde{G} = (\alpha, \beta)$ may be denoted as $\widetilde{G} = (V, E, \alpha, \beta)$ for our convenient. The fuzzy vertex degree v is defined by $d_{\widetilde{G}}(v) = \sum_{(v,w)} \beta(v,w)$, for $(v,w) \in E$ and $\beta(v,w) = 0$ for (v,w) not in E, where the summation runs over the non loop edges (v,w). For a vertex $v \in V$, the total fuzzy vertex degree is defined as $td_{\widetilde{G}}(v) = d_{\widetilde{G}}(v) + \alpha(v)$.

Definition II.2. [7] Consider the fuzzy graph $\widetilde{G} = (V, E, \alpha, \beta)$ defined in II.1. If atleast one pair of adjacent vertices have distinct fuzzy vertex degree then the fuzzy graph is called irregular. If all the pairs of vertices have distinct fuzzy vertex degree then the fuzzy graph is known as strongly irregular. If every vertex adjacent to the vertices have distinct fuzzy vertex degrees then fuzzy graph is considered as highly irregular. If every pair of adjacent vertices is have distinct fuzzy vertex degree then fuzzy graph is said to be a neighbourly irregular.

Definition II.3. [16] Consider the fuzzy graph $\widetilde{G} = (V, E, \alpha, \beta)$ defined in II.1. For (v,w), the fuzzy edge degree is $d_{\widetilde{G}}(v,w) = d_{\widetilde{G}}(v) + d_{\widetilde{G}}(w) - 2\beta(v,w)$.
The minimum fuzzy edge degree is
$\delta_E(\widetilde{G}) = min\left\{d_{\widetilde{G}}(v,w) : (v,w) \in E\right\}$.
The maximum fuzzy edge degree is
$\Delta_E(\widetilde{G}) = max\left\{d_{\widetilde{G}}(v,w) : (v,w) \in E\right\}$. The total fuzzy edge degree is $td_{\widetilde{G}}(v,w) = d_{\widetilde{G}}(v) + d_{\widetilde{G}}(w) - \beta(v,w)$. The minimum total fuzzy edge degree is
$\delta_{tE}(\widetilde{G}) = min\left\{td_{\widetilde{G}}(v,w) : (v,w) \in E\right\}$.
The maximum total fuzzy edge degree is
$\Delta_{tE}(\widetilde{G}) = max\left\{td_{\widetilde{G}}(v,w) : (v,w) \in E\right\}$.

Definition II.4. [1] In fuzzy graph $\widetilde{G} = (V, E, \alpha, \beta)$, a sequence of distinct vertices $v_0, v_1 \cdots, v_m$, with $\beta(v_{j-1}, v_j) > 0$, $j = 1, 2, \cdots m$, is called a path P of length m and the degree of membership of a weakest arc is defined as its strength. If $v_0 = v_m$ and m is greater than 3 then P is called a fuzzy cycle. The connectedness strength between two vertex v and w is the strength of maximum of all paths from v to w denoted as $CONN_{\widetilde{G}}(v,w)$. If for all $v, w \in V$, $\beta_{\widetilde{G}}(v,w) = CONN_{\widetilde{G}}(v,w) > 0$ then the fuzzy graph $\widetilde{G}$ is called fuzzy connected.

Definition II.5. [13] Consider a fuzzy connected graph $\widetilde{G} = (V, E, \alpha, \beta)$. If fuzzy edge degrees of all pair of adjacent edges are distinct then the graph is called neighbourly edge irregular. If total fuzzy edge degree of all pairs of adjacent edges are distinct then the graph is called as totally irregular. If $\beta(v,w) = \alpha(v) \wedge \alpha(w) = min\{\alpha(v), \alpha(w)\} \forall\, v, w \in V$ then it is called complete. The values $p = \sum_{v\in V} \alpha(v)$ and $q = \sum_{(v,w)\in V\times V} \beta(v,w)$ are called order and size of the fuzzy complete graph. The scalar cardinality of S is defined as $\sum_{v\in S} \alpha(v)$, denoted as $|S|$.

Definition II.6. [16] Consider a connected fuzzy graph $\widetilde{G} = (V, E, \alpha, \beta)$. If fuzzy edge degrees are distinct for all pairs then it is called strongly edge irregular. If total edge degrees are distinct for all pairs, then it is called strongly edge totally irregular. If $d_{\widetilde{G}}(v) = constant$, $\forall\, v \in V$, then $\widetilde{G}$ is called regular fuzzy graph of degree $constant$. If each vertex has the $constant$ total vertex degree, then $\widetilde{G}$ is said to be the totally regular fuzzy graph of total degree $constant$.

Remark II.7. [16] Consider a connected fuzzy graph $\widetilde{G} = (V, E, \alpha, \beta)$. If $\widetilde{G}$ is strongly edge irregular fuzzy graph and strongly edge totally irregular fuzzy graph then it is need not be a constant function. A complete fuzzy graph need not be neighbourly irregular.

Definition II.8. [17] Consider a fuzzy graph $\widetilde{G} = (V, E, \alpha, \beta)$. The d_m degree of a vertex v in $\widetilde{G}$ is defined by $d_m(v) = \sum \beta^m(v,w)$, where the value of $\beta^m(v,w)$ is taken by $\beta^m(v,w) = sup\{\beta(v,v_1) \wedge \beta(v_1,v_2) \cdots \wedge \beta(v_{m-1},w)\}$ where, $(v, v_1, v_2 \cdots v_{m-1}, w)$ is the shortest path connecting v and w of length m. The minimum fuzzy d_m vertex degree of $\widetilde{G}$ is $\delta_m(\widetilde{G}) = min\{d_m(v) : v \in V\}$.
The maximum fuzzy d_m vertex degree of $\widetilde{G}$ is $\Delta_m(\widetilde{G}) = max\{d_m(v) : v \in V\}$.
If $d_m(v) = constant, \forall v \in V$, then $\widetilde{G}$ is said to be (m,c) regular fuzzy graph. The total fuzzy d_m vertex degree of a vertex v in V is $td_m(v) = \sum \beta^m(v,w) + \alpha(v) = d_m(v) + \alpha(v)$.
The minimum fuzzy td_m total vertex degree of $\widetilde{G}$ is $t\delta_m(\widetilde{G}) = min\{td_m(v) : v \in V\}$. The maximum fuzzy td_m total vertex degree of $\widetilde{G}$ is $t\Delta_m(\widetilde{G}) = max\{td_m(v) : v \in V\}$.

Definition II.9. [17] (Totally (m,c)-Regular Fuzzy Graph) If each vertex of $\widetilde{G}$ has the same total $d_m$vertex degree c, then $\widetilde{G}$ is said to be totally (m,c)-regular fuzzy graph. The following example illustrate $(2,c)$ regular fuzzy graph.

Theorem II.10. *[1] Consider a fuzzy graph $\widetilde{G} = (V, E, \alpha, \beta)$. Then α is constant function iff the following conditions are equivalent.*
(i) $\widetilde{G}$ is $(2,c)$ regular fuzzy graph.
(ii) $\widetilde{G}$ is totally $(2,c)$ regular fuzzy graph.

III. DELTA GRAPHS

In this section by employing fuzzy vertex values and difference of generalized extorial function, we developed new type of fuzzy delta graph and present basic concepts of the construction of delta graphs. Here we use the notation $\mathbb{N}(a) = \{a, a+1, a+2\cdots\}$

Lemma III.1. *For the positive integer's' we have*

$$\Delta^s x^{(m)} = m^{(s)} x^{(m-s)}, \tag{1}$$

where $m^{(s)} = m(m-1)(m-2)\ldots(m-(s-1))$.

Definition III.2. For $x \in \mathbb{R}$ and $m, \kappa \in \mathbb{N}(0)$, the generalized extorial function, denoted as $\mathcal{E}(\kappa, x^{(m)})$, is defined as

$$\sum_{r=0}^{\infty} \frac{x^{(m+r\kappa)}}{(m+r\kappa)!} = \mathcal{E}(\kappa, x^{(m)}) \tag{2}$$

Here, m and κ denote the initial power and the shift value of generalized extorial function(2)

Lemma III.3. *For $m \in \mathbb{N}(1)$, we have $\frac{x^{(m-1)}}{(m-1)!} = \Delta \frac{x^{(m)}}{m!}$.*

Proof: *The proof follows from* (1) *and* $\frac{m}{m!} = \frac{1}{(m-1)!}$ ∎

Lemma III.4. *For $m = 0, 1, 2\cdots\kappa$, $\kappa \in \mathbb{N}(2)$, and $0 \le r \le m$, $r \in \mathbb{N}(0)$, we have*

$$\Delta^r \mathcal{E}(\kappa, x^{(m)}) = \begin{cases} \mathcal{E}(\kappa, x^{((m-r) mod \kappa)}), & r > m \\ \mathcal{E}(\kappa, x^{(m-r)}), & r \le m \end{cases} \tag{3}$$

Proof: *Since $\mathcal{E}(\kappa, x^{(m)})$ is convergent for each $x \in \mathbb{R}$ and $(-\kappa)! = \infty$ for $\mathbb{N}(1)$, operating Δ to each term of $\mathcal{E}(\kappa, x^{(m)})$ in* (2)*, we get*

$$\Delta\mathcal{E}(\kappa, x^{(m)}) = \left(\frac{x^{(m-1)}}{(m-1)!} + \frac{x^{(m+\kappa-1)}}{(m+\kappa-1)!} + \cdots\right) = \mathcal{E}(\kappa, x^{(m-1)}).$$

Applying Δ^2 on $\mathcal{E}(\kappa, x^{(m)})$, we get

$$\Delta^2\mathcal{E}(\kappa, x^{(m)}) = \left(\frac{x^{(m-2)}}{(m-2)!} + \frac{x^{(m+\kappa-2)}}{(m+\kappa-2)!} + \cdots\right) = \mathcal{E}(\kappa, x^{(m-2)})$$

$$\vdots$$

In general applying Δ^r on $\mathcal{E}(\kappa, x^{(m)})$, we derive(3) ∎

Theorem III.5. *Let $\sum_{r=0}^{\kappa-1} \Delta^r y = e^x$ has a solution as $y = \mathcal{E}(\kappa, x^{(0)})$*

Proof: *Consider $y = \mathcal{E}(\kappa, x^{(0)})$, defined by* (2)*,*
$\Delta^0\mathcal{E}(\kappa, x^{(0)}) = \mathcal{E}(\kappa, x^{(0)})$, $\Delta^1\mathcal{E}(\kappa, x^{(0)}) = \mathcal{E}(\kappa, x^{(4)})$, $\Delta^2\mathcal{E}(\kappa, x^{(0)}) = \mathcal{E}(\kappa, x^{(3)})$, $\Delta^3\mathcal{E}(\kappa, x^{(0)}) = \mathcal{E}(\kappa, x^{(2)}) \cdots$, $\Delta^{\kappa-1}\mathcal{E}(\kappa, x^{(0)}) = \mathcal{E}(\kappa, x^{(1)})$.
By adding all the above expressions, we get
$\Delta^0\mathcal{E}(\kappa, x^{(0)}) + \Delta^1\mathcal{E}(\kappa, x^{(0)}) \cdots + \Delta^{\kappa-1}\mathcal{E}(\kappa, x^{(0)})$
$= (\Delta^0 + \Delta^1 + \Delta^2 \cdots + \Delta^{\kappa-1})\mathcal{E}(\kappa, x^{(0)}) = e^x$
The proof is complete. ∎

From the generalized extorial function defined by (2), we introduce a delta graph for any $\kappa \ge 2$ and $m = 0, 1\cdots\kappa-1$.

Definition III.6. The r^{th} order delta graph arrived from the generalized extorial function is a pair. $G_\kappa^r = (V_\kappa, E_\kappa^{+r})$, where $V_\kappa = \{\mathcal{E}(\kappa, x^{(m)}) | m = 0, 1\cdots\kappa-1\}$ is the vertex set.

$E_\kappa^{+r} = \left\{(\mathcal{E}(\kappa, x^{(m)}) \rightarrow \mathcal{E}(\kappa, x^{((m-r) mod \kappa)}) | m = 0, 1, \cdots \kappa - 1\right\}$ is the edge set. Here each extorial function $\mathcal{E}(\kappa, x^{(m)})$ is vertex, and $\mathcal{E}(\kappa, x^{(m)}) \rightarrow \mathcal{E}(\kappa, x^{(m-r)})$ is the label of directed edge in G_κ^r. Note that V_κ having κ vertices. The integral graph is obtained from difference graph by reversing the direction. The directed edge of κ^{th} order integral graph is given below,

$E_\kappa^{-r} = \left\{(\mathcal{E}(\kappa, x^{((m-r) mod \kappa)}) \rightarrow \mathcal{E}(\kappa, x^{(m)})) | m = 0, 1, \cdots \kappa - 1\right\}$

Definition III.7. The r^{th} order delta graph with κ vertices is obtained by replacing each directed edges into usual edge in the delta graph G_κ^r. This graph is denoted as $G_\kappa^r = (V_\kappa, E_\kappa^{+r})$. Let $G_p^r = (V_p, E_p^r)$, $G_q^s = (V_q, E_q^s)$ be the two delta graphs.

Consider $V_p = \left\{\mathcal{E}[p, x^{(0)}], \quad \mathcal{E}[p, x^{(1)}], \cdots \mathcal{E}[p, x^{(p-1)}]\right\}$
$V_q = \left\{\mathcal{E}[q, x^{(0)}], \quad \mathcal{E}[q, x^{(1)}], \cdots \mathcal{E}[q, x^{(q-1)}]\right\}$
$V_{pq} = \left\{(\mathcal{E}[p, x^{(0)}], \mathcal{E}[q, x^{(0)}]), (\mathcal{E}[p, x^{(0)}], \mathcal{E}[q, x^{(1)}]) \cdots, (\mathcal{E}[p, x^{(0)}], \mathcal{E}[q, x^{(q-1)}]), \quad (\mathcal{E}[p, x^{(1)}], \mathcal{E}[q, x^{(0)}]) \cdots, (\mathcal{E}[p, x^{(1)}], \mathcal{E}[q, x^{(q-1)}]) \quad (\mathcal{E}[p, x^{(p-1)}], \mathcal{E}[q, x^{(0)}]) \cdots, (\mathcal{E}[p, x^{(p-1)}], \mathcal{E}[q, x^{(q-1)}])\right\}$
$E_{pq}^{rs} = \left\{(\mathcal{E}[p, x^{(a)}], \mathcal{E}[q, x^{(b)}]) \rightarrow (\mathcal{E}[p, x^{((a-r) mod p)}], \mathcal{E}[q, x^{((b-s) mod q)}]\right\}$
as edge set. The product of G_p^r and G_q^s is a graph defined as $G_{p,q}^{r,s} = (V_{pq}, E_{pq}^{rs})$.

IV. Fuzzy delta graphs

In this section, we construct fuzzy delta graph. The fuzzy values of vertex, vertex degrees, total vertex degrees, edge values, edge degrees and total edge degrees are given in the Table I and II.

Definition IV.1. Let $G_1 = (V_1, E_1)$ be a graph with vertex set $V_1 = \{v_0, v_1, v_2, \cdots v_{n-1}\}$ and $G_2 = (V_2, E_2)$ be another graph of with vertex set $V_2 = \{u_0, u_1, u_2, \cdots u_{m-1}\}$.
The m-multi-copies of G_1 with G_2 is a graph $\overline{G} = (\overline{V}, \overline{E})$ where $\overline{V} = V_2 \cup m \times V_1$ and edge set
$\overline{E} = E_2 \cup m \times E_1 \cup \{u_0v_0, u_1v_0, u_2v_0 \cdots u_{m-1}v_0\}$.

Definition IV.2. Let $\widetilde{G_1} = (V_1, E_1, \sigma_1, \mu_1)$ be the fuzzy graph and $\widetilde{G_2} = (V_2, E_2, \sigma_2, \mu_2)$ be the fuzzy graph.
Let $\overline{G} = (\overline{V}, \overline{E})$ be multi-copies graph of G_1 with repect to G_2. Let $\overline{\sigma} = \sigma_1 \cup \sigma_2$ and $\overline{\mu} = \mu_1 \cup \mu_2$ then $\widetilde{G} = (\overline{V}, \overline{E}, \overline{\sigma}, \overline{\mu})$ is called multi-copies fuzzy graph.

Definition IV.3. Consider the delta graph of extorial function is $G_k^r = (V_k, E_\kappa^r)$, where $V_\kappa = \left\{\mathcal{E}(\kappa, \kappa^{(m)}) | m = 0, 1, 2 \cdots \kappa - 1\right\}$ is the vertex set and the edge set is
$E_\kappa^r = \left\{(\mathcal{E}(\kappa, \kappa^{(m)}) - \mathcal{E}(\kappa, \kappa^{((m-r) mod \kappa)})) | m = 0, 1, 2 \cdots \kappa - 1\right\}$.
Define $\alpha : V_k \rightarrow [0, 1]$ by $\alpha(\mathcal{E}(\kappa, \kappa^{(m)})) = \dfrac{\mathcal{E}(\kappa, \kappa^{(m)})}{\mathcal{E}(1, \kappa^{(0)})}$,
$\beta(\mathcal{E}(\kappa, \kappa^{(m)}), (\mathcal{E}(\kappa, \kappa^{(n)})) = \alpha(\mathcal{E}(\kappa, \kappa^{(m)})) \wedge \alpha(\mathcal{E}(\kappa, \kappa^{(n)}))$
$= min\left\{\alpha(\mathcal{E}(\kappa, \kappa^{(m)})), \alpha(\mathcal{E}(\kappa, \kappa^{(n)}))\right\}$. Then
$\widetilde{G} = (V_\kappa, E_\kappa^r, \alpha, \beta)$ is fuzzy delta graph of extorial function.

The illustration for the definitionIV.3 is given below. The 5-cycle $u_0u_1u_2u_3u_4u_0$ with the fuzzy vertex values, calculated by $\alpha(u_i) = \dfrac{\mathcal{E}(5, 5^{(i)})}{\mathcal{E}(1, 5^{(0)})}$ for $i = 0, 1, 2, 3, 4$
is shown in Figure 1. The calculations of fuzzy values for 5-cycles are illustrated as
$\beta(u_i, u_{i+1}) = \alpha(u_i) \wedge \alpha(u_{i+1}) = min\left\{\dfrac{\mathcal{E}(5, 5^{(i)})}{\mathcal{E}(1, 5^{(0)})}, \dfrac{\mathcal{E}(5, 5^{(i+1)})}{\mathcal{E}(1, 5^{(0)})}\right\}$
$i = 0, 1, 2, 3,\ u_{-1} = u_4, u_5 = u_0, \beta(u_4, u_0) = \alpha(u_4) \wedge \alpha(u_0)$
$= min\left\{\dfrac{\mathcal{E}(5, 5^{(4)})}{\mathcal{E}(1, 5^{(0)})}, \dfrac{\mathcal{E}(5, 5^{(0)})}{\mathcal{E}(1, 5^{(0)})}\right\}$.
The fuzzy vertex degree of our 5-cycle is calculated by $d_{\widetilde{G}}(u_i) = \beta(u_{i-1}, u_i) + \beta(u_i, u_{i+1})$, for $i = 1, 2, 3$ we have, $d_{\widetilde{G}}(u_0) = \beta(u_4, u_0) + \beta(u_0, u_1)$, $d_{\widetilde{G}}(u_4) = \beta(u_3, u_4) + \beta(u_4, u_0)$.
For illustration
$d_{\widetilde{G}}(u_2) = \beta(u_1, u_2) + \beta(u_2, u_3) = 0.15 + 0.31 = 0.46$.
The fuzzy total vertex degree of our 5-cycle is calculated by $td_{\widetilde{G}}(u_i) = d_{\widetilde{G}}(u_i) + \alpha(u_i)$.
For illustration
$td_{\widetilde{G}}(u_2) = d_{\widetilde{G}}(u_2) + \alpha(u_2) = 0.46 + 0.31 = 0.77$.
The fuzzy degree of an edge (u_i, u_{i+1}) is defined as $d_{\widetilde{G}}(u_i, u_{i+1}) = d_{\widetilde{G}}(u_i) + d_{\widetilde{G}}(u_{i+1}) - 2\beta(u_i, u_{i+1})$.
For illustration
$d_{\widetilde{G}}(u_1, u_2) = d_{\widetilde{G}}(u_1) + d_{\widetilde{G}}(u_2) - 2\beta(u_1, u_2)$
$= 0.21 + 0.46 - 2(0.15) = 0.37$.
The fuzzy total edge degree is defined as
$td_{\widetilde{G}}(u_i, u_{i+1}) = d_{\widetilde{G}}(u_i) + d_{\widetilde{G}}(u_{i+1}) - \beta(u_i, u_{i+1})$.
For illustration
$td_{\widetilde{G}}(u_1, u_2) = d_{\widetilde{G}}(u_1) + d_{\widetilde{G}}(u_2) - \beta(u_1, u_2)$
$= 0.21 + 0.46 - (0.15) = 0.53$
All the fuzzy values are given in Table I and II

TABLE I
VERTEX VALUES, VERTEX DEGREES AND TOTAL VERTEX DEGREES OF FUZZY 5-CYCLE DELTA GRAPH

V/D	u_1	u_2	u_3	u_4	u_0
$\alpha(u_i)$	0.15	0.31	0.31	0.15	0.06
$d_{\widetilde{G}}(u_i)$	0.21	0.46	0.46	0.21	0.12
$td_{\widetilde{G}}(u_i)$	0.36	0.77	0.77	0.36	0.18

TABLE II
EDGE VALUES, EDGE DEGREES AND TOTAL EDGE DEGREES OF FUZZY 5-CYCLE DELTA GRAPH

V/D	(u_0, u_1)	(u_1, u_2)	(u_2, u_3)	(u_3, u_4)	(u_4, u_0)
$\beta(u_i, u_{i+1})$	0.06	0.15	0.31	0.15	0.06
$d_{\widetilde{G}}(u_i, u_{i+1})$	0.21	0.37	0.31	0.37	0.21
$td_{\widetilde{G}}(u_i, u_{i+1})$	0.28	0.53	0.62	0.53	0.28

Example IV.4. *The diagram representation of all fuzzy values is shown in the Figure 1. From the Figure 1 there exists vertex $u_4 \in \widetilde{G}$ such that u_0, u_3 are adjacent vertices but $d_{\widetilde{G}}(u_0) \neq d_{\widetilde{G}}(u_3) \neq d_{\widetilde{G}}(u_4)$ the given graph is irregular fuzzy. Also $td_{\widetilde{G}}(u_0) \neq td_{\widetilde{G}}(u_3) \neq td_{\widetilde{G}}(u_4)$ the given graph is totally irregular fuzzy. Since, $d_{\widetilde{G}}(u_i) = d_{\widetilde{G}}(u_j)$ for some i, j. So, given graph is not strongly irregular fuzzy. Here vertex u_3 adjacent to the vertex u_2 have same vertex degrees. So, given graph is not highly irregular fuzzy. Clearly, every vertex adjacent to the vertices having distnct degrees. So, given graph is highly irregular fuzzy graph.*
Since, $d_{\widetilde{G}}(u_3) = d_{\widetilde{G}}(u_2)$, given fuzzy graph is not neighbourly irregular. Here, $\delta_E(\widetilde{G}) = 0.21$, $\Delta_E(\widetilde{G}) = 0.37$, $\delta_{tE}(\widetilde{G}) = 0.28$, $\Delta_{tE}(\widetilde{G}) = 0.62$.

Here the pair of adjacent edges (u_0, u_1) and (u_0, u_4) having same total degrees. $\widetilde{G}$ is not neighbourly edge totally irregular fuzzy graph. Furthermore, the given fuzzy graph $\widetilde{G}$ is not complete as $(u_0, u_3) \notin V_5$.
The order of the fuzzy graph is
$p = \alpha(u_0) + \alpha(u_1) + \alpha(u_2) + \alpha(u_3) + \alpha(u_4) = 0.98$.
Size of the graph is

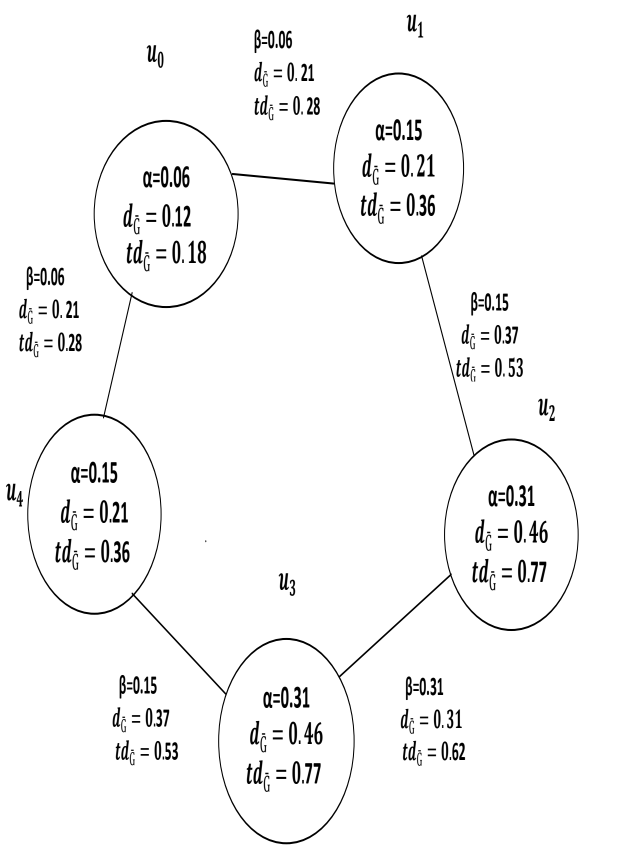

Fig. 1. Fuzzy Delta Graph of 5-Cycle

$q = \beta(u_0, u_1) + \beta(u_1, u_2) + \beta(u_2, u_3) + \beta(u_3, u_4) + \beta(u_4, u_0).$
$= 0.74$

Here, $td_{\widetilde{G}}(u_0, u_4) = td_{\widetilde{G}}(u_0, u_1)$ the graph $\widetilde{G}$ is not strongly edge totally irregular fuzzy graph.
$d_{\widetilde{G}}(u_0, u_4) = d_{\widetilde{G}}(u_0, u_1)$ the graph $\widetilde{G}$ is not strongly edge irregular fuzzy. Here, each vertex has not same degree.
So, $\widetilde{G}$ is not c-regular fuzzy graph. Also each vertex has not same total degree. So, $\widetilde{G}$ is not c-totally regular fuzzy graph.
The minimum total td_2-vertex degree of $\widetilde{G}$ is $t\delta_2(\widetilde{G}) = 0.03$
The maximum total td_2-vertex degree of $\widetilde{G}$ is $t\Delta_2(\widetilde{G}) = 0.74$
Since, $d_2(u_i) \neq c, \forall u_i \in V$ then $\widetilde{G}$ is not $(2, c)$-regular fuzzy graph. The minimum d_2-vertex degree of $\widetilde{G}$ is $\delta_2(\widetilde{G}) = 0.02$.
The maximum d_2-vertex degree of $\widetilde{G}$ is $\Delta_2(\widetilde{G}) = 0.54$. Here each vertex of $\widetilde{G}$ have not same total d_2-vertex degree c.
Hence, $\widetilde{G}$ is not totally $(2, c)$-regular fuzzy graph.

V. Product of Fuzzy delta graphs

Definition V.1. Let $\widetilde{G}_p^r = (V_p, E_p^r, \alpha_1, \beta_1)$ and $\widetilde{G}_q^s = (V_q, E_q^s, \alpha_2, \beta_2)$ be two fuzzy delta graphs. The product of fuzzy delta graphs are $\widetilde{G}_{p,q}^{r,s} = (V_{pq}, E_{pq}^{rs}, \alpha_{12}, \beta_{12})$ where
$\alpha_{12}(\mathcal{E}(p, x^{(m)}), \mathcal{E}(q, x^{(n)})) = \alpha_1(\mathcal{E}(p, x^{(m)}).\alpha_2(\mathcal{E}(q, x^{(n)}),$
$\beta_{12}((\mathcal{E}(p, x^{(m)}), (\mathcal{E}(q, x^{(n)}))=\alpha_{12}(\mathcal{E}(p, x^{(m)})) \wedge \alpha_{12}(\mathcal{E}(q, x^{(n)}))$
$= min\left\{\alpha_{12}(\mathcal{E}(p, x^{(m)}), \alpha_{12}(\mathcal{E}(q, x^{(n)}))\right\}.$

Consider the definition III.7 of product of delta and integral graph. When $r = 1$ and $s = 1$ taking 5-cycle $(u_0, u_1, u_2, u_3, u_4, u_0)$ and 3-cycle (v_0, v_1, v_2, v_0). The product of 5-cycle and 3-cycle of order $(1, 1)$ and graph having 15 vertices and 25 edges in 2. Consider two fuzzy graphs $\widetilde{G_1^1} = (V_1, E_1, \alpha_1, \beta_1)$ and $\widetilde{G_2^1} = (V_2, E_2, \alpha_2, \beta_2)$. Then the product of fuzzy graph $\widetilde{G_{1,2}^{1,1}} = G_1^1 \times G_2^1 = (V_{12}, E_{12}, \alpha_{12}, \beta_{12})$. The fuzzy vertex values, vertex degrees, total vertex degrees, edge values, edge degrees and total edge degrees are given in the Figure 2. In Figure 2 the fuzzy vertex values calculated as
$\alpha_{12}(u_i v_j) = \alpha_1(u_i)\alpha_2(v_j) = \dfrac{\mathcal{E}(5, 5^{(i)})}{\mathcal{E}(1, 5^{(0)})} \times \dfrac{\mathcal{E}(3, 3^{(j)})}{\mathcal{E}(1, 3^{(0)})},$
$i = 0, 1, 2, 3, 4$ and $j = 0, 1, 2$.

The fuzzy edge values calculated as follows
$\beta_{12}(u_i v_j, u_s v_r) = \alpha_{12}(u_i v_j) \wedge \alpha_{12}(u_s v_r)$
$= min\left\{\dfrac{\mathcal{E}(5, 5^{(i)})}{\mathcal{E}(1, 5^{(0)})} \times \dfrac{\mathcal{E}(3, 3^{(j)})}{\mathcal{E}(1, 3^{(0)})}, \dfrac{\mathcal{E}(5, 5^{(s)})}{\mathcal{E}(1, 5^{(0)})} \times \dfrac{\mathcal{E}(3, 3^{(r)})}{\mathcal{E}(1, 3^{(0)})}\right\}$
$i, s = 0, 1, 2, 3, 4$ and $j, r = 0, 1, 2$.

The fuzzy vertex degree calculated as follows
$d_{\widetilde{G_{1,2}^{1,1}}}(u_i v_j) = \sum_{(u_i v_j, u_r v_s) \in E_{12}} \beta_{12}(u_i v_j, u_r v_s),$ for $(u_i v_j, u_r v_s) \in E_{12}$ and $\beta_{12}(u_i v_j, u_r v_s) = 0$ for $(u_i v_j, u_r v_s) \notin E_{12}$ $j, s = 0, 1, 2$ and $i, r = 0, 1, 2, 3, 4$.

The total fuzzy vertex degree calculated as follows
$d_{\widetilde{G_{1,2}^{1,1}}}(u_i v_j) = \sum_{(u_i v_j, u_r v_s) \in E_{12}} \beta_{12}(u_i v_j, u_r v_s) + \alpha_{12}(u_i v_j),$
for $(u_i v_j, u_r v_s) \in E_{12}$ and $\beta_{12}(u_i v_j, u_r v_s) = 0$ for $(u_i v_j, u_r v_s) \notin E_{12}$ $j, s = 0, 1, 2$ and $i, r = 0, 1, 2, 3, 4$.

The fuzzy edge degree calculated as follows
$d_{\widetilde{G_{1,2}^{1,1}}}(u_i v_j, u_r v_s) = d_{\widetilde{G_{1,2}^{1,1}}}(u_i v_j) + d_{\widetilde{G_{1,2}^{1,1}}}(u_r v_s)$
$- 2\beta_{12}(u_i v_j, u_r v_s)$, and $\beta_{12}(u_i v_j, u_r v_s) = 0$
for $(u_i v_j, u_r v_s) \notin E_{12}$. where $i, s = 0, 1, 2, 3, 4$.
and $j, r = 0, 1, 2$.
The fuzzy total edge degree calculated as
$d_{\widetilde{G_{1,2}^{1,1}}}(u_i v_j, u_r v_s) = d_{\widetilde{G_{1,2}^{1,1}}}(u_i v_j) + d_{\widetilde{G_{1,2}^{1,1}}}(u_r v_s)$
$- \beta_{12}(u_i v_j, u_r v_s)$ where $\beta_{12}(u_i v_j, u_r v_s) = 0$
for $(u_i v_j, u_r v_s) \notin E_{12}$. $i, s = 0, 1, 2, 3, 4$ and $j, r = 0, 1, 2$.

Example V.2. *The diagram representing all these values of product graph 3-cycle and 5-cycle is given in the example. From Figure 2 there exists vertex $u_0 v_2 \in \widetilde{G_{1,2}^{1,1}}$ such that $u_4 v_2, u_0 v_1, u_0 v_0$ are adjacent vertices.*
But $d_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_0) \neq d_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_1) \neq d_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_2) \neq d_{\widetilde{G_{1,2}^{1,1}}}(u_4 v_2)$ the given graph is irregular fuzzy graph and $td_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_1) \neq td_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_2) \neq td_{\widetilde{G_{1,2}^{1,1}}}(u_4 v_2) \neq td_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_0)$ the given graph is totally irregular fuzzy graph.

A vertex $u_4 v_2$ adjacent to the vertex $u_4 v_1$ such that $d_{\widetilde{G_{1,2}^{1,1}}}(u_4 v_2) = d_{\widetilde{G_{1,2}^{1,1}}}(u_4 v_1)$. So, given graph is not strongly irregular fuzzy. Also the graph is not neighbourly irregular fuzzy. The minimum degree of an edge is $\delta_{E_{12}}(\widetilde{G}) = 0.01$.
The maximum degreee of an edge is $\Delta_{E_{12}}(\widetilde{G}) = 0.42$.
The minimum total degree of an edge is $\delta_{tE_{12}}(\widetilde{G}) = 0.09$.
The maximum total degreee of an edge is $\Delta_{tE_{12}}(\widetilde{G}) = 0.5$.

The adjacent edges $(u_0 v_2, u_4 v_2)$ and $(u_4 v_2, u_4 v_1)$ such that $d_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_2, u_4 v_2) = d_{\widetilde{G_{1,2}^{1,1}}}(u_4 v_2, u_4 v_1)$. So the graph is not neighbourly edge irregular fuzzy and not strongly edge irregular. Also $td_{\widetilde{G_{1,2}^{1,1}}}(u_0 v_2, u_4 v_2) = td_{\widetilde{G_{1,2}^{1,1}}}(u_4 v_2, u_4 v_1)$.
So the graph is not neighbourly edge totally irregular and not strongly edge totally irregular. Clearly, from the Figure 2 the fuzzy graph is not complete but connected. The order

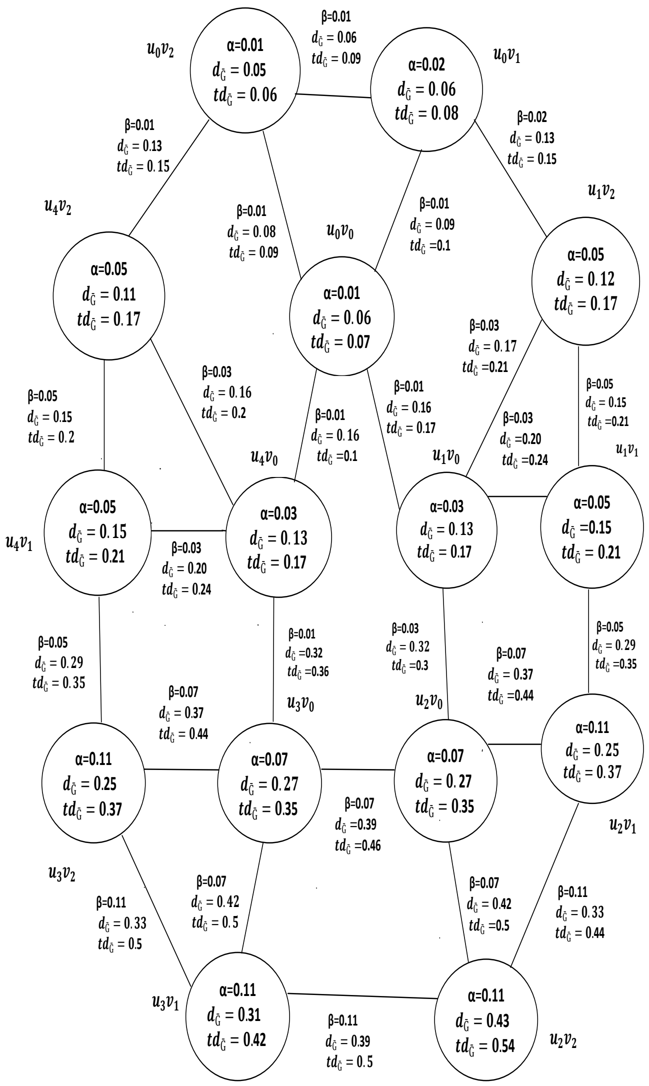

Fig. 2. Product of 5-Cycle and 3-Cycle of Fuzzy Delta Graph

of the fuzzy graph is $p = 0.88$ *and size* $q = 1.15$. *Here each vertex have not the same degree. So,* $\widetilde{G_{1,2}^{1,1}}$ *is not regular fuzzy graph and each vertex has not the same total vertex degree* c. *So, the fuzzy graph is not totally regular of total degree* c.

VI. Applications

Here is an application of the fuzzy weighted graph model in transportation network optimization, expressed in mathematical form.

Application: Urban transportation route optimization with fuzzy travel times. Let $G = (V, E, \widetilde{w})$ be a fuzzy weighted graph. Where, V is the set of nodes, $E \subseteq V \times V$ is the set of edges, $\widetilde{w} : E \to \widetilde{\mathbb{R}}$ is a fuzzy weight function assigning fuzzy numbers $\widetilde{w}_{ij}$ to each edge $(i, j) \in E$, representing uncertain travel time or cost. We define the fuzzy shortest path between source node $s \in V$ and destination node $t \in V$ as the path P is $P = \{s, v_1, v_2, \ldots, t\} \subseteq V$ minimizing the total fuzzy travel time:

$$\widetilde{W}(P) = \sum_{(i,j)\in P} \widetilde{w}_{ij}$$

The objective is:

$$\min_{P} \widetilde{W}(P)$$

Subject to:

P is a valid path from s to t in G, The aggregation of fuzzy weights is done using fuzzy addition (e.g., using α-cuts or defuzzification if needed for comparison).

Interpretation: This formulation allows urban planners or navigation systems to identify optimal routes under travel time uncertainty, offering more resilient and flexible routing decisions compared to crisp models. An important application of the fuzzy weighted graph model is in urban transportation planning, where uncertainty plays a significant role due to varying traffic conditions, weather disruptions, and road maintenance activities. In such scenarios, traditional crisp models often fail to account for the imprecise nature of travel costs, delays, or route reliability. The fuzzy weighted graph model allows planners to represent a transportation network as a graph where each edge is assigned a fuzzy weight that captures uncertain metrics like travel time, cost, or fuel consumption in the form of fuzzy numbers or membership functions. For example, the travel time between two locations may be represented as a triangular fuzzy number to reflect best-case, typical, and worst-case scenarios. Using this model, decision-makers can compute the most reliable or cost-effective routes under uncertain conditions, enabling more resilient and adaptive route planning. Moreover, by applying fuzzy shortest path algorithms on this graph, transportation systems can optimize traffic flow, improve public transit schedules, and support real-time route recommendation systems for drivers and logistics companies. Here is an application of the delta graph operator for modeling dynamic changes in networks, expressed in mathematical format.

Application: Dynamic network topology adjustment in communication networks. Let the initial state of a graph be:

$$G_t = (V_t, E_t)$$

Where, V_t is the set of vertices at time t,
$E_t \subseteq V_t \times V_t$ is the set of edges at time t.
We define a delta graph operator ΔG that captures the change in the graph over a time step Δt as:

$$\Delta G = G_{t+\Delta t} - G_t = (V_{t+\Delta t} \setminus V_t, E_{t+\Delta t} \setminus E_t)$$

The updated graph becomes:

$$G_{t+\Delta t} = G_t \oplus \Delta G$$

Here, $\oplus$ denotes a graph union with updates, including:
Vertex addition/removal : $V_{t+\Delta t} = V_t \cup V^+ \setminus V^-$
Edge addition/removal : $E_{t+\Delta t} = E_t \cup E^+ \setminus E^-$.
Where, V^+, E^+ are new vertices and edges added,
V^-, E^- are vertices and edges removed.
Use Case:

In a mobile ad hoc network (MANET), nodes join and leave dynamically due to movement. The delta graph operator ΔG models such topology changes. Routing algorithms can then adapt paths by recalculating on the updated graph $G_{t+\Delta t}$, ensuring efficient communication despite network volatility. This model can be extended by including fuzzy weights on edges or applying this operator iteratively to model time-evolving networks.

An important application of the delta graph operator lies in modeling dynamic changes in communication networks, such as the internet or wireless sensor networks. These networks often experience frequent structural updatesnodes may be added or removed, and connections may change due to failures, upgrades, or mobility. The delta graph operator is used to represent and manage such dynamic transitions by capturing the change between two graph states over time. For example, if G_t is the network graph at time t and G_{t+1} is the updated graph at time $t+1$, then the delta graph operator $\Delta(G_t, G_{t+1})$ encodes only the changes (added or deleted nodes/edges), making it efficient to track and update the network status. This is especially useful in adaptive routing protocols, where decisions are made based on real-time topological variations, ensuring efficient data transmission even under network instability.

VII. Conclusion

By defining the generalized extorial function and a new type delta graph have been introduced. After applying fuzzy concept on these delta graphs, product of fuzzy graph is established with fuzzy vertex and edge degrees. The regularity and irregularity behaviour like neighbourly edge, neighbourly irregular, strongly irregular, highly irregular, totally irregular, strongly edge irregular, strongly edge totally irregular are discussed. The fuzzy values assigned to with example each vertex may be replaced by change of electrons in the chemical graphs for getting applications in physical science.

References

[1] Al Mutab, Huda Mutab. "Fuzzy Graphs." Journal of Advances in Mathematics 17 (2019): 232-247.

[2] Alspach, B., and Parsons, T. D. (1979). Isomorphism of circulant graphs and digraphs. Discrete Mathematics, 25(2), 97-108.

[3] BORG, S. J., SATHINATHAN, T., and XAVIER, G. B. A. extorial Function And Its Properties In Discrete Calculus.

[4] Britto Antony Xavier.G, Gerly.T.G and Nasira Begum.H, Finite Series of Polynomials and Polynomial Factorials arising from Generalized q-Difference operator, Far East Journal of Mathematical Sciences,94(1)(2014), 47-63.

[5] Britto Antony Xavier.G, John Borg.S, M.Meganathan, Discrete heat equation model with shift values, Applied Mathematics, 2017, 8, 1343-1350.

[6] Britto Antony Xavier.G, Meganathan.M, Fractional order alpha Laplace and extorial transform by inverse difference operator, AIP conference proceedings 2095, 030016(2019) https://doi.org/10.1063/1.5097527.

[7] Gani, A. Nagoor, and S. R. Latha. "On irregular fuzzy graphs." Applied Mathematical Sciences 6.11 (2012): 517-523.

[8] G.Britto Antony Xavier, S.John Borg, S.Jaraldpushparaj, Extorial solutions for fractional and partial difference equations with application,AIP conference proceedings 2095, 030004(2019) https://doi.org/10.1063/1.5097515.

[9] Harary, F., and Schwenk, A. J. (1974). The communication problem on graphs and digraphs.

[10] Krishnaveni, J., & Maheswari, N. S. (2020). Support neighbourly irregular fuzzy graphs. International Journal of advanced research in engineering and technology, 11(1), 147-154.

[11] Maria Susai Manuel.M, Chandrasekar.V and Britto Antony Xavier.G, Solutions and Applications of Certain Class of Difference Equations, International Journal of Applied Mathematics, 24(6) (2011), 943-954.

[12] M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, Theory of Generalised Difference Operator and Its Applications, Far East Journal of Mathematical Sciences, Vol.20,No.2(2006), pp 163 - 171.

[13] N.R.Santhimaheswari and C. Sekar, On neighbourly edge irregular fuzzy graphs, International Journal of Mathematical Archive, 6(10) (2015) 224-231.

[14] R.P Agarwal Difference Equations and Inequalities, Marceldekker, New York, 2000.

[15] Ronald E.Mickens, Difference Equations, Van Nostrand Reinhold Company, New York, 1990.

[16] Santhimaheswari, N. R., and C. Sekar. "On strongly edge irregular fuzzy graphs." Kragujevac Journal of Mathematics 40.1 (2016): 125-135.

[17] Santhimaheswari, N. R., and C. Sekar. "On (r, m, k)-Regular Fuzzy Graphs." Mathematical Combinatorics 1 (2016): 18-26.

[18] Sathinathan, T., Sherine, V.R. And Xavier, G.B.A., Fractional Order of Alpha-Delta and its Sums on Extorial Functions. Advances in Mathematics: Scientific Journal 9 (2020), no.8, 60976106.

[19] Maria Susai Manuel, M., et al. "Theory and application of the generalized difference operator of the n th kind (Part I)." Demonstratio Mathematica 45.1 (2012): 95-106.

[20] Xavier, G., S. John Borg, and S. Jaraldpushparaj. "Extorial solutions for fractional and partial difference equations with applications." AIP conference proceedings. Vol. 2095. No. 1. AIP Publishing, 2019.