Fixed Theorems in New Extended B Metric Spaces and Applications to System of Linear Equations

B Padmavathi, M V R Kameswari, M Madhuri and A Bharathi

Abstract—**This study aims to introduce generalized $\delta - \vartheta$ extended Z-contractions within new extended b metric spaces and investigates the existence of fixed points for such contractions. To reinforce the developed theory, we present several illustrative examples. Additionally, we apply the theoretical results to solve a system of linear equations. As a practical demonstration, a numerical example is included to compute the current in an electrical circuit.**

Index Terms—**Fixed points, generalized $\delta - \vartheta$ extended Z-contractions, New extended b metric spaces, Triangular δ-orbital admissible map with respect to η, Simulation functions, System of linear equations.**

I. Introduction

Metric spaces play a crucial role in mathematics and its applications, inspiring several efforts to generalize their structure. These include the introduction of b-metric spaces[3], extended b metric spaces[6], new extended b metric spaces[1] and various other novel generalizations.

We begin with the following definitions.

Definition I.1: ([3]) Consider a mapping $d_{bm} : \mho \times \mho \to \mathcal{R}$, where $\mho$ is a nonempty. The mapping d_{bm} is termed as *b metric*, when a constant $\mathcal{S} > 1$, is such that d_{bm} fulfills the subsequent axioms: $\forall$ $\oe, \jmath$ and ⅄ $\in \mho$

1) $d_{bm}(\jmath, \oe) = 0 \iff \jmath = \oe$,
2) $d_{bm}(\jmath, \oe) = d_{bm}(\oe, \jmath)$,
3) $d_{bm}(\jmath, \oe) \leq \mathcal{S}[d_{bm}(\jmath, ⅄) + d_{bm}(⅄, \oe)]$.

Then $(\mho, d_{bm})$ is designated as *b metric space* shortly BMS.

An extended version of the generalized b-metric space, known as the extended b-metric space, was introduced by Kamran *et al.* [6].

Definition I.2: ([6]) Consider $\vartheta : \mho \times \mho \to [1, \infty)$ and $d_{ebm} : \mho \times \mho \to \mathcal{R}^+$ be two mappings, where $\mho$ is non empty set. d_{ebm} is *extended b metric* , if d_{ebm} satisfies the following: for all $\jmath, \oe$ and ⅄ $\in \mho$

i $d_{ebm}(\jmath, \oe) = 0$ if and only if $\jmath = \oe$,
ii $d_{ebm}(\jmath, \oe) = d_{ebm}(\oe, \jmath)$
iii $d_{ebm}(\jmath, \oe) \leq \vartheta(\jmath, \oe)[d_{ebm}(\jmath, ⅄) + d_{ebm}(⅄, \oe)]$.

Then $(\mho, d_{ebm})$ is designated as *extended b metric space*.

Recently, in 2019, new type of generalized b metric space specifically new extended b metric space added by Aydi *et al.* [1].

Definition I.3: ([1]) A mapping $\vartheta : \mho \times \mho \times \mho \to [1, \infty)$ and $d_{ebm} : \mho \times \mho \to \mathcal{R}^+$ be two mappings, where $\mho$ is non void. d_{ebm} is called *new extended b metric*, if d_{nebm} fulfills the subsequent axioms:$\forall$ $\jmath, \oe$ and ⅄ $\in \mho$

1) $d_{nebm}(\jmath, \oe) = 0$ if and only if $\jmath = \oe$,
2) $d_{nebm}(\jmath, \oe) = d_{nebm}(\oe, \jmath)$,
3) $d_{nebm}(\jmath, \oe) \leq \vartheta(\jmath, \oe, \Upsilon)[d_{nebm}(\jmath, ⅄) + d_{nebm}(⅄, \oe)]$.

Then $(\mho, d_{nebm})$ is termed as new *extended b metric space.*

If $\vartheta(\jmath, \oe, ⅄) = \vartheta(\jmath, \oe)$, the above definition coincide with Definition 1.2 and if $\vartheta(\jmath, \oe, ⅄) = s$, for $s \geq 1$, we get b metric space.

Definition I.4: ([1]) Consider a new extended b metric space $(\mho, d_{nebm})$ with ϑ.

1) A sequence $\{\Theta_n\}$ in $(\mho, d_{nebm})$ is d_{nebm} *convergent to* $\Theta^* \in \mho$ if for $\epsilon > 0$, there is $N_\epsilon \in N$ such that $d_{nebm}(\Theta_n, \Theta^*) \leq \epsilon$, for all $n \geq N_\epsilon$. i.e., $lim_{n\to\infty}\Theta_n = \Theta^*$.
2) A sequence $\{\Theta_n\}$ in $(\mho, d_{nebm})$ is d_{nebm} $Cauchy$ sequence if $\lim_{m,n\to+\infty} d_{nebm}(\Theta_n, \Theta_m) = 0$.
3) A new extended b metric space $(\mho, d_{nebm})$ is *complete* if every Cauchy sequence in $\mho$ is convergent to some point in $\mho$.

The concept of comparison functions was introduced by Rus [9] and has since been widely explored by various researchers to develop broader classes of contractive mappings.

Definition I.5: ([9]) A mapping $\complement : [0, \infty) \to [0, \infty)$ is a b-comparison function if it meets the following criteria:

1) $\complement$ is nondecreasing
2) $\exists\ p_o \in N$, $c \in [0, 1)$, $\mathcal{S} \geq 1$ and nonnegative series $\sum_{p=1}^{\infty} \vartheta_p$ which is convergent such that $\mathcal{S}^{p+1}\complement^{p+1}(t) \leq c\mathcal{S}^p\complement^p(t) + \vartheta_p$, for $p \geq p_0$ and any $t \geq 0$.

Lemma I.6: ([9]) If $\complement : [0, \infty) \to [0, \infty)$ is a b-comparison function then:

1) for any $\ell \in [0, \infty)$; the series $\sum_{i=1}^{\infty} s^i\complement^i(\ell)$ converges
2) $b_{\mathcal{S}} : [0, \infty) \to [0, \infty)$ defined as $b_{\mathcal{S}} = \sum_{i=1}^{\infty} \mathcal{S}^i\complement^i(\ell)$ is nondecreasing and is noncontinuous at $\ell = 0$.

A function $\complement$ is referred to as a b-comparison function if it satisfies the following properties:

1) $\complement(t) < t$

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2) $\lim_{n\to\infty} \complement^n(t) = 0$ for each $t > 0$.

From this point forward, we denote Φ, the class of all b- comparison functions and it will be assumed that all newly introduced extended b-metric spaces possess the property of continuity, ensuring that the associated distance function behaves continuously with respect to its arguments.

Lemma I.7 Consider a new extended b metric space on a non empty set $\mho$, $\mathcal{T}: \mho \to \mho$. If $\exists$ a sequence $\{\mathfrak{r}_n\}_{n\in N} \ni \mathfrak{r}_n > 1$ and $\vartheta(\Theta_n, \Theta_{n+1}, \Theta_{n+2}) < \mathfrak{r}_n$ for all $m > n$ and $n \in \mathcal{N}$. Moreover,

$$0 < d_{nebm}(\Theta_n, \Theta_{n+1}) \leq \complement(d_{nebm}(\Theta_n, \Theta_{n-1})) \quad (1)$$

for all $n \in \mathcal{N}$, $\Theta \in \Phi$ then the sequence $\{\Theta_n\}$ defined by $\Theta_n = \mathcal{T}\Theta_{n-1}$ for all $n \in \mathcal{N}$ is a Cauchy sequence in $\mho$.

Proof: We define a sequence $\{\Theta_n\}$ in $\mho$ by $\Theta_n = \mathcal{T}\Theta_{n-1}$. Through successive application of inequality (1), it follows that

$$d_{nebm}(\Theta_n, \Theta_{n+1}) \leq \complement^n(d_{nebm}(\Theta_0, \Theta_1)).$$

In view of property $\complement$, we get

$$\lim_{n\to\infty} d_{nebm}(\Theta_n, \Theta_{n+1}) = 0.$$

In view of Definition I.3, we attain

$$\begin{aligned}
& d_{nebm}(\Theta_n, \Theta_m) \\
& \leq \vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) \\
& \quad [d_{nebm}(\Theta_n, \Theta_{n+1}) + d_{nebm}(\Theta_{n+1}, \Theta_m)] \\
& \leq \vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) d_{nebm}(\Theta_n, \Theta_{n+1}) + \\
& \quad \vartheta(\Theta_n, \Theta_{n+1}, \Theta_m)\vartheta(\Theta_{n+1}, \Theta_{n+2}, \Theta_m) \\
& \quad [d_{nebm}(\Theta_{n+1}, \Theta_{n+2}) + d_{nebm}(\Theta_{n+2}, \Theta_m)] \\
& \leq \vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) d_{nebm}(\Theta_n, \Theta_{n+1}) + \vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) \\
& \quad \vartheta(\Theta_{n+1}, \Theta_{n+2}, \Theta_m) d_{nebm}(\Theta_{n+1}, \Theta_{n+2}) \ldots .. + \\
& \quad \vartheta(\Theta_n, \Theta_{n+1}, \Theta_m)\vartheta(\Theta_{n+1}, \Theta_{n+2}, \Theta_m)\vartheta(\Theta_{n+2}, \Theta_{n+3}, \Theta_m) \\
& \quad \ldots\ldots..\vartheta(\Theta_{m-2}, \Theta_{m-1}, \Theta_m) d_{nebm}(\Theta_{m-1}, \Theta_m) \\
& \leq \vartheta(\Theta_1, \Theta_2, \Theta_m)\vartheta(\Theta_2, \Theta_3, \Theta_m)\vartheta(\Theta_3, \Theta_4, \Theta_m) \ldots\ldots \\
& \quad \vartheta(\Theta_n, \Theta_{n+1}, \Theta_m)\complement^n(d_{nebm}(\Theta_0, \Theta_1) \\
& \quad + \vartheta(\Theta_1, \Theta_2, \Theta_m)\vartheta(\Theta_2, \Theta_3, \Theta_m)\vartheta(\Theta_3, \Theta_4, \Theta_m) \ldots\ldots \\
& \quad \vartheta(\Theta_{n+1}, \Theta_{n+2}, \Theta_m)\complement^{n+1}(d_{nebm}(\Theta_0, \Theta_1)) \\
& \quad + \ldots . + \vartheta(\Theta_1, \Theta_2, \Theta_m)\vartheta(\Theta_2, \Theta_3, \Theta_m)\vartheta(\Theta_3, \Theta_4, \Theta_m) \ldots . \\
& \quad \vartheta(\Theta_{m-2}, \Theta_{m-1}, \Theta_m)\ \complement^{m-1}(d_{nebm}(\Theta_0, \Theta_1)).
\end{aligned}$$

Let $\Omega_n = \sum_{j=1}^{n} \complement^j(d_{nebm}(\Theta_0, \Theta_1)) \prod_{i=1}^{j} \vartheta(\Theta_i, \Theta_{i+1}, \Theta_m)$, for all $n \in \mathcal{N}$.

We deduce that $\vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) \leq \Omega_{m-1} - \Omega_{n-1}$ for all $m > n$.

Let $\sum_{n=1}^{\infty} \complement^n(d_{nebm}(\Theta_0, \Theta_1)) \prod_{i=1}^{n} \vartheta(\Theta_i, \Theta_{i+1}, \Theta_m)$.

Let $p = max\{p_1, p_2, ...p_n\}$. We have

$$a_n = \complement^n(d_{nebm}(\Theta_0, \Theta_1)) \prod_{i=1}^{j} \vartheta(\Theta_i, \Theta_{i+1}, \Theta_m)$$

$$\leq \complement^n(d_{nebm}(\Theta_0, \Theta_1))p^n.$$

In light of Lemma I.6, we have that the series $\complement^n(d_{nebm}(\Theta_0, \Theta_1))p^j$ converges.

Applying comparison tests for series convergence, we derive $\sum_{n=1}^{\infty} \complement^n(d_{nebm}(\Theta_0, \Theta_1)) \prod_{i=1}^{j} \vartheta(\Theta_i, \Theta_{i+1}, \Theta_m)$ converges, and hence

$$\lim_{n,m\to\infty} d_{nebm}(\Theta_n, \Theta_m) = 0,$$

thus $\{\Theta_n\}$ is a Cauchy sequence.

Definition I.8: ([8]) Consider $\mho$ a nonempty set and $\delta: \mho \times \mho \to [0, \infty)$. A mapping $\mathcal{T}: \mho \to \mho$ is δ- *orbital admissible mapping* if $œ \in \mho$ whenever $\delta(œ, \mathcal{T}œ) \geq 1$ it follows that $\delta(\mathcal{T}œ, \mathcal{T}^2œ) \geq 1$.

Definition I.9: ([8]) Consider $\mho$ a nonviod set and $\delta: \mho \times \mho \to [0, \infty)$. $\mathcal{T}: \mho \to \mho$ is *triangular δ- orbital admissible mapping* if $\jmath, œ \in \mho$ whenever $\delta(œ, \jmath) \geq 1$ and $\delta(\jmath, \mathcal{T}\jmath) \geq 1$ it follows that $\delta(œ, \mathcal{T}\jmath) \geq 1$.

Definition I.10: ([4]) Consider a selfmap $\mathcal{T}: \mho \to \mho$ and $\delta, \vartheta: \mho \times \mho \to [0, \infty)$ be two mappings, $\mathcal{T}$ is *δ-orbital admissible mapping with respect to ϑ* if for every $œ \in \mho$ the condition
$\delta(œ, \mathcal{T}œ) \geq \vartheta(œ, \mathcal{T}œ)$ ensures that
$\delta(\mathcal{T}œ, \mathcal{T}^2œ) \geq \vartheta(\mathcal{T}œ, \mathcal{T}^2œ)$.

Definition I.11: ([4])Let $\mathcal{T}: \mho \to \mho$ and $\delta, \vartheta: \mho \times \mho \to [0, \infty)$, $\mathcal{T}$ is *triangular δ-orbital admissible mapping with respect to ϑ* : if for all $œ, \jmath \in \mho$

1) δ-orbital admissible mapping with respect to ϑ.
2) $\delta(œ, \jmath) \geq \vartheta(œ, \jmath)$ *and* $\delta(\jmath, \mathcal{T}\jmath) \geq \vartheta(\jmath, \mathcal{T}\jmath) \Rightarrow \delta(œ, \mathcal{T}\jmath) \geq \vartheta(œ, \mathcal{T}\jmath)$.

Lemma I.12: ([4]) Let $\mathcal{T}$ be a triangular δ-orbital admissible mapping with respect to ϑ. Suppose that there exists an element $\exists\ u_0 \in \mho$ such that $\delta(u_0, \mathcal{T}u_0) \geq \vartheta(u_0, \mathcal{T}u_0)$, we define a sequence $\{u_n\}$ by setting $u_{n+1} = \mathcal{T}u_n$. Then for $m, n \in N$ with $m < n$, it follows that $\delta(u_m, u_n) \geq \vartheta(u_m, u_n)$.

Definition I.13: ([5]) Consider $\delta, \vartheta: \mho \times \mho \to [0.\infty)$, where $\mho$ is a nonvoid set. A mapping $\mathcal{T}: \mho \to \mho$ is $\delta - \vartheta$ continuous if for each sequence $\{u_n\}$ in $\mho$ with $\delta(u_n, u_{n+1}) \geq \vartheta(u_n, u_{n+1})$ for all $n \in N$ and $u_n \to u$ as $n \to \infty$ implies $\mathcal{T}u_n \to \mathcal{T}u$ as $n \to \infty$.

Alternatively, Khojasteh *et al.* [7] introduced a novel class of mappings known as simulation functions and demonstrated that numerous existing results in the literature follow as direct consequences of their established findings.

Definition I.14: ([7]) A mapping $\zeta_{sm}: [0, \infty) \times [0, \infty) \to (-\infty, \infty)$ defines a *simulation function* if ζ_{sm} meets the subsequent conditions:

1) $\zeta_{sm}(0, 0) = 0$
2) $\zeta_{sm}(\mathfrak{u}, \mathfrak{v}) < \mathfrak{v} - \mathfrak{u}$, for all $\mathfrak{u}, \mathfrak{v} > 0$
3) if $\{\mathfrak{u}_n\}$ and $\{\mathfrak{v}_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} \mathfrak{u}_n = \lim_{n\to\infty} \mathfrak{v}_n = l \in (0, \infty)$ then $\limsup_{n\to\infty} \zeta_{sm}(\mathfrak{v}_n, \mathfrak{u}_n) < 0$.

Several examples of simulation functions can be found in the related literature, such as [4,7].

In a recent study, Chifu and Karpinar [4] proposed the notion of admissible extended Z-contraction mappings within the framework of extended b-metric spaces and derived fixed point results for these mappings.

Definition I.15: ([4]) Let $(\mho, d_{ebm})$ be a extended b metric space equipped with the function $\vartheta : \mho \times \mho \to [1, \infty)$. A mapping $\mathcal{T} : \mho \to \mho$ is referred as *admissible extended Z contraction* if there exists a simulation function $\zeta_{sm} \in Z$ such that

$$\zeta_{sm}(\delta(œ, ȷ) d_{ebm}(\mathcal{T}œ, \mathcal{T}ȷ), \complement(M_{\vartheta}(œ, ȷ)) \geq 0, \quad (2)$$

where $\complement \in \Psi$ and for all $ȷ, œ \in \mho$,

$$M_{\vartheta}(œ, ȷ) = max\{d_{ebm}(œ, ȷ), d_{ebm}(œ, \mathcal{T}œ), d_{ebm}(ȷ, \mathcal{T}ȷ)\}.$$

Definition I.16: ([3]) Let $(\mho, d_{ebm})$ be an extended b metric space endowed with a function $\vartheta : \mho \times \mho \to [1, \infty)$. Assume that there exists a sequence $\{\Upsilon_n\}_{n \in \mathcal{N}}$ such that $\vartheta(\Theta_n, \Theta_m) < \Upsilon_n$ for all $m > n$. Furthermore, if $\mathcal{T} : \mho \to \mho$ is admissible extended Z- contraction satisfying:

1) $\mathcal{T}$ is triangular δ-orbital admissible mapping;
2) there exists $\Theta_0 \in \mho$ such that $\delta(\Theta_0, \mathcal{T}\Theta_0) \geq 1$;
3) $\mathcal{T}$ is continuous;

or

4) if $\{u_n\}$ is a sequence in $\mho$ such that $\delta(u_n, u_{n+1}) \geq 1$ for all n and $u_n \to u \in \mho$ as $n \to \infty$, then there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\delta(u_{n(k)}, u) \geq 1$ for all k.

Under these conditions, it follows that $\mathcal{T}$ admits a fixed point $\Theta^* \in \mho$. Further, $\{\mathcal{T}^n\Theta_0\}$ is converges to Θ^*.

Definition I.17:([2]) Let $(\mho, d_{nebm})$ be a new extended b metric space associated with the function $\vartheta : \mho \times \mho \times \mho \to [1, \infty)$. A function $\mathcal{T} : \mho \to \mho$ is termed a *nonlinear contraction* if there exist a continuous, increasing function φ with $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ such that the following conditions hold:

$$\zeta_{sm}(\mathcal{T}œ, \mathcal{T}ȷ) \leq \varphi(M(œ, ȷ)) \quad (3)$$

for all $œ, ȷ \in \mho$ and

$$M(œ, ȷ) = max\{d_{nebm}(œ, ȷ), d_{nebm}(œ, \mathcal{T}œ), d_{nebm}(ȷ, \mathcal{T}ȷ)\}.$$

Defintion I.18: ([2]) Let $(\mho, d_{nebm})$ be a new extended b metric space associated with the function $\vartheta : \mho \times \mho \times \mho \to [1, \infty)$. The mapping $\mathcal{T} : \mho \to \mho$ satisfying condition (3) in addition to the following conditions:

1) $\sup_{m \geq 1} lim_{n \to \infty} \frac{\varphi^{n+1}(t)}{\varphi^n(t)} \vartheta(\Theta_{n+1}, \Theta_{n+2}, \Theta_m) < 1$, where $\Theta_n = \mathcal{T}^n\Theta_0$, $n \in \mathcal{N}$.
2) For $ȷ \in \mho$, $lim_{n \to \infty} \vartheta(ȷ, \Theta_n, \Theta_{n+1})$ and $lim_{n \to \infty} \vartheta(ȷ, \Theta_n, \mathcal{T}ȷ)$ exists and are finite.

Then $\mathcal{T}$ has a fixed point in $\mho$.

Motivated by the works of Chifu and Karpinar[4], Karan et. al [6]., and Aydi et. al., [2], we discuss generalized $\delta - \vartheta$ extended Z-contractions (Definition II.1) within the frame work of new extended b metric spaces. We obtained fixed points for these contractions (Theorems II.2, II.3 and II.4). We provide illustrative examples (Examples III.3. III.4 and III.5) to support the developed theory. As a practical application, we obtain solution to a system of linear equations (Theorem IV.1). Finally, we provide a numerical example demonstrating the computation of electric current flowing through a circuit.

II. Main Results

Defintion II.1: Let $(\mho, d_{nebm})$ be a new extended b metric space associated with the function $\vartheta : \mho \times \mho \times \mho \to [1, \infty)$. A mapping $\mathcal{T} : \mho \to \mho$ is an *generalized $\delta - \vartheta$ extended Z contraction* for all $ȷ, œ \in \mho$ if there exists a simulation function $\zeta_{sm} \in Z$, $\complement \in \Psi$ and $L \geq 0$ such that $\delta(œ, ȷ) \geq \vartheta(œ, ȷ)$ implies

$$\zeta_{sm}(d_{nebm}(\mathcal{T}œ, \mathcal{T}ȷ), \complement(M_{\vartheta}(œ, ȷ)) + LN_{\vartheta}(œ, ȷ)) \geq 0, \quad (4)$$

where $M_{\vartheta}(œ, ȷ) = max\{d_{nebm}(œ, ȷ), \frac{d_{nebm}(œ, \mathcal{T}œ) d_{nebm}(ȷ, \mathcal{T}ȷ)}{1 + d_{nebm}(œ, ȷ)},$
$\frac{d_{nebm}(œ, \mathcal{T}ȷ) d_{nebm}(ȷ, \mathcal{T}œ)}{1 + d_{nebm}(œ, ȷ)}, \frac{d_{nebm}(œ, \mathcal{T}œ)(1 + d_{nebm}(ȷ, \mathcal{T}œ)]}{1 + d_{nebm}(œ, ȷ)},$
$\frac{d_{nebm}(ȷ, \mathcal{T}œ)[1 + d_{nebm}(œ, \mathcal{T}œ)]}{1 + d_{nebm}(œ, ȷ)}\}$

and

$$N(œ, ȷ) = min\{d_{nebm}(œ, \mathcal{T}ȷ), d_{nebm}(ȷ, \mathcal{T}œ), \frac{d_{nebm}(œ, \mathcal{T}ȷ)[1 + d_{nebm}(ȷ, \mathcal{T}œ)]}{1 + d_{nebm}(œ, ȷ)}\}.$$

Theorem II.1: *Consider a new extended b metric space $(\mho, d_{nebm})$ associated with the function $\vartheta : \mho \times \mho \times \mho \to [1, \infty)$. Assume that there exists a sequence $\{\Upsilon_n\}$ with $\Upsilon_n > 1$ such that $\vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) < \Upsilon_n$ for all $n \in \mathcal{N}$ and $m > n$. Additionally, assume that the mapping $\mathcal{T} : \mho \to \mho$ is a generalized $\delta - \vartheta$ extended Z contraction satisfies the following conditions :*

1) The transformation $\mathcal{T}$ is a triangular δ-orbital admissible with respect to ϑ
2) $\exists$ $\Theta_1 \in \mho$ such that $\delta(\Theta_1, \mathcal{T}\Theta_1) \geq \vartheta(\Theta_1, \mathcal{T}\Theta_1)$, and
3) $\mathcal{T}$ is $\delta - \vartheta$ continuous mapping.

Under these conditions, it follows that $\mathcal{T}$ admits a fixed point $\Theta^* \in X$. Further, for any $\Theta_1 \in \mho$ $\{\mathcal{T}^n\Theta_1\}$ is converges to Θ^* with respect to the extended b-metric d_{nebm}.

Proof: Consider an initial point $\Theta_1 \in \mho$. From condition (ii) of our premises, *i.e.*, $\delta(\Theta_1, \mathcal{T}\Theta_1) \geq \vartheta(\Theta_1, \mathcal{T}\Theta_1)$, define a sequence $\{\Theta_n\}$ in $\mho$ by

$$\Theta_{n+1} = \mathcal{T}\Theta_n \quad (5)$$

for all $n \in \mathcal{N}$. Suppose that $\Theta_{n_0} = \Theta_{n_0+1}$ then Θ_{n_0} follows as a fixed point of $\mathcal{T}$. Hence, Suppose that $\Theta_n \neq \Theta_{n+1}$ for all $n \in \mathcal{N}$.

In view of Lemma I.12, we have

$$\delta(\Theta_n, \Theta_{n+1}) \geq \vartheta(\Theta_n, \Theta_{n+1}). \quad (6)$$

for all $n \in \mathcal{N}$.

Utilizing condition (4) when $œ = \Theta_{n-1}$ and $ȷ = \Theta_n$, we have

$$\zeta_{sm}(d_{nebm}(\Theta_n, \Theta_{n+1}), M_{\vartheta}(\Theta_{n-1}, \Theta_n) + LN_{\vartheta}(\Theta_{n-1}, \Theta_n)) \geq 0, \quad (7)$$

where

$$\begin{aligned} &M_{\vartheta}(\Theta_{n-1}, \Theta_n) \\ &= max\{d_{nebm}(\Theta_{n-1}, \mathcal{T}\Theta_{n-1}), \\ &\frac{d_{nebm}(\Theta_{n-1}, \mathcal{T}\Theta_{n-1}) d_{nebm}(\Theta_n, \mathcal{T}\Theta_n)}{1 + d_{nebm}(\Theta_n, \Theta_{n-1})}, \\ &\frac{d_{nebm}(\Theta_{n-1}, \mathcal{T}\Theta_n) d_{nebm}(\Theta_n, \mathcal{T}\Theta_{n-1})}{1 + d_{nebm}(\Theta_n, \Theta_{n-1})}, \\ &\frac{d_{nebm}(\Theta_{n-1}, \mathcal{T}\Theta_{n-1})[1 + d_{nebm}(\Theta_n, \mathcal{T}\Theta_{n-1})]}{1 + d_{nebm}(\Theta_n, \Theta_{n-1})}, \\ &\frac{d_{nebm}(\Theta_n, \mathcal{T}\Theta_{n-1})[1 + d_{nebm}(\Theta_{n-1}, \mathcal{T}\Theta_{n-1})]}{1 + d_{nebm}(\Theta_n, \Theta_{n-1})}\} \\ &\leq max\{d_{nebm}(\Theta_n, \Theta_{n-1}), d_{nebm}(\Theta_n, \Theta_{n+1})\} \quad (8) \end{aligned}$$

and $N_{\mathcal{T}}(\Theta_{n-1}, \Theta_n) =$

$$min\{d_{nebm}(\Theta_n,\mathcal{T}\Theta_{n-1}),d_{nebm}(\Theta_{n-1},\mathcal{T}\Theta_n), \frac{d_{nebm}(\Theta_{n-1},\mathcal{T}\Theta_n)[1+d_{nebm}(\Theta_n,\mathcal{T}\Theta_{n-1})]}{1+d_{nebm}(\Theta_n,\Theta_{n-1})}\}=0. \quad (9)$$

Suppose that $d_{nebm}(\Theta_n,\Theta_{n-1})<d_{nebm}(\Theta_n,\Theta_{n+1})$, then from (7), (8) and (9), it follows that

$$0\leq\zeta_{sm}(d_{nebm}(\Theta_n,\Theta_{n+1}),\complement(d_{nebm}(\Theta_n,\Theta_{n+1})) < \complement(d_{nebm}(\Theta_n,\Theta_{n+1})-d_{nebm}(\Theta_n,\Theta_{n+1})).$$

Consequently,

$$d_{nebm}(\Theta_n,\Theta_{n+1})\leq\complement(d_{nebm}(\Theta_n,\Theta_{n+1})) < d_{nebm}(\Theta_n,\Theta_{n+1}),$$

a contradiction. Therefore

$$d_{nebm}(\Theta_n,\Theta_{n-1})>d_{nebm}(\Theta_n,\Theta_{n+1}). \quad (10)$$

Similarly, we can prove that

$$d_{nebm}(\Theta_{n-1},\Theta_{n-2})>d_{nebm}(\Theta_n,\Theta_{n-1}). \quad (11)$$

Hence from (10) and (11), we conclude that

$$d_{nebm}(\Theta_n,\Theta_{n+1})>d_{nebm}(\Theta_{n+1},\Theta_{n+2})$$

for all $n\in N$.

Thus from (4), we have,

$$0\leq\zeta_{sm}(d_{nebm}(\Theta_{n+1},\Theta_n),\complement(d_{nebm}(\Theta_n,\Theta_{n-1})) < \complement(d_{nebm}(\Theta_n,\Theta_{n-1}))-d_{nebm}(\Theta_n,\Theta_{n+1})$$

which implies

$$d_{nebm}(\Theta_{n+1},\Theta_n)\leq\complement(d_{nebm}(\Theta_n,\Theta_{n-1})) \leq\complement^n(d_{nebm}(\Theta_0,\Theta_1)). \quad (12)$$

On letting $n\to\infty$ and in light of property of $\complement$, it follows that

$$lim_{n\to\infty}d_{nebm}(\Theta_n,\Theta_{n+1})=0\ . \quad (13)$$

From Lemma I.7 together with condition (13), it follows that the sequence $\{\Theta_n\}$ is a Cauchy sequence in the space $\mho$. In view of $\mho$ is a complete extended b metric space, we may therefore can choose $\Theta\in\mho$ such that

$$lim_{n\to\infty}d_{nebm}(\Theta_n,\Theta)=0. \quad (14)$$

By the property of $\mathcal{T}$ is continuous, it follows that

$$lim_{n\to\infty}d_{nebm}(\mathcal{T}\Theta_n,\mathcal{T}\Theta)=0 \quad (15)$$

this implies $\Theta=\mathcal{T}\Theta$.

Hence the theorem.

Theorem II.3: Let $(\mho,\delta)$ be a new extend b metric space equipped with a function $\vartheta:\mho\times\mho\times\mho\to[1,\infty)$ and consider a sequence $\{p_n\}$ such that $p_n>1$ for all $n\in\mathcal{N}$, with the property that $\vartheta(\Theta_n,\Theta_{n+1},\Theta_m)<p_n$. Also, assume that $\mathcal{T}:\mho\to\mho$ is generalized $\delta-\vartheta$ extended Z contraction satisfying conditions (i), (ii) of Theorem II.2 along with

(iii) a sequence $\{\Theta_n\}$ in $\mho$ is such that $\delta(\Theta_n,\Theta_{n+1})\geq\vartheta(\Theta_n,\Theta_{n+1})$ for all $n\in\mathcal{N}$ and $\Theta_n\to\Theta$ as $n\to\infty$, then there exist a sub sequences $\{\Theta_{n_k}\}$ of $\{\Theta_n\}$ such that $\delta(\Theta_{n_k},\Theta^*)\geq\vartheta(\Theta_{n_k},\Theta^*)$ for all $k\in\mathcal{N}$.

Then $\mathcal{T}$ has a fixed point $\Theta^*\in\mho$ and $\{\mathcal{T}^n\Theta_1\}$ is converges to Θ^* is a fixed point of $\mathcal{T}$.

Proof. Proceeding in the manner of the proof of Theorem II.2, we construct a sequence $\{\Theta_n\}$ by the iteration $\Theta_{n+1}=\mathcal{T}\Theta_n$, and establish that it converges to $\Theta^*\in\mho$.

Furthermore, it holds $\delta(\Theta_n,\Theta_{n+1})\geq\vartheta(\Theta_n,\Theta_{n+1})$, for all $n\in N$. According to our assumption (iii), there exist a sub sequence $\{\Theta_{n_k}\}$ of $\{\Theta_n\}$ such that $\delta(\Theta_{n_k},\Theta^*)\geq\vartheta(\Theta_{n_k},\Theta^*)$ for all $k\in\mathcal{N}$. Utilizing condition (4) when $\jmath=\Theta^*$ and $œ=\Theta_{n_k}$, we have

$$0\leq\zeta_{sm}(d_{nebm}(\mathcal{T}\Theta_{n_k},\mathcal{T}\Theta^*),\complement(M_\vartheta(\Theta_{n_k},\Theta^*)) +LN_\vartheta(\Theta_{n_k},\Theta^*)) < \complement(M_\vartheta(\Theta_{n_k},\Theta^*))+LN_\vartheta(\Theta_{n_k},\Theta^*)-d_{nebm}(\mathcal{T}\Theta_{n_k},\mathcal{T}\Theta^*),$$

which implies

$$d_{nebm}(\mathcal{T}\Theta_{n_k},\mathcal{T}\Theta^*)<\complement(M_\vartheta(\Theta_{n_k},\Theta^*))+LN_\vartheta(\Theta_{n_k},\Theta^*) \quad (16)$$

$$M_\vartheta(\Theta_{n_k},\Theta^*) =max\{d_{nebm}(\Theta_{n_k},\Theta^*),\frac{d_{nebm}(\Theta_{n_k},\mathcal{T}\Theta_{n(k)})d_{nebm}(\Theta^*,\mathcal{T}\Theta^*)}{1+d_{nebm}(\Theta_{n_k},\Theta^*)}, \frac{d_{nebm}(\Theta^*,\mathcal{T}\Theta_{n_k})d_{nebm}(\Theta_{n_k},\mathcal{T}\Theta^*)}{1+d_{nebm}(\Theta_{n_k},\Theta^*)}, \frac{d_{nebm}(\Theta^*,\mathcal{T}\Theta_{n_k})[1+d_{nebm}(\Theta_{n_k},\mathcal{T}\Theta_{n(k)})]}{1+d_{nebm}(\Theta_{n_k},\Theta^*)}, \frac{d_{nebm}(\Theta_{n_k},\mathcal{T}\Theta_{n(k)})[1+d_{nebm}(\Theta^*,\mathcal{T}\Theta_{n_k})]}{1+d_{nebm}(\Theta_{n_k},\Theta^*)}\} \quad (17)$$

and

$$N_\vartheta(\Theta_{n_k},\Theta^*)=min\{d_{nebm}(\Theta_{n_k},\mathcal{T}\Theta^*),d_{nebm}(\Theta^*,\mathcal{T}\Theta_{n_k}), \frac{d_{nebm}(\Theta_{n_k},\mathcal{T}\Theta^*)[1+d_{nebm}(\Theta^*,\mathcal{T}\Theta_{n_k})]}{1+d_{nebm}(\Theta_{n_k},\Theta^*)}\}. \quad (18)$$

Letting limsup as $k\to\infty$, in the inequalities (17) and (18), we attain

$$limsup_{k\to\infty}M_\vartheta(\Theta_{n_k},\Theta^*)=0. \quad (19)$$

$$limsup_{k\to\infty}N_\vartheta(\Theta_{n_k},\Theta^*)=0. \quad (20)$$

Thus from (16), (19) and (20), we get

$$limsup_{k\to\infty}d_{nebm}(\mathcal{T}\Theta_{n_k},\mathcal{T}\Theta^*) \leq limsup_{k\to\infty}\complement(d_{nebm}(\Theta_{n_k},\mathcal{T}\Theta^*))$$

which implies $d_{nebm}(\Theta^*,\mathcal{T}\Theta^*)=0$. Hence $\mathcal{T}\Theta^*=\Theta^*$.

Theorem II.4: Along with the hypotheses of Theorem II.3, assume the following:

(Ł) $\ni\jmath\neq œ\in\mho,\exists\ x\in\mho$ such that $\delta(\jmath,x)\geq\vartheta(\jmath,x)$, $\delta(œ,x)\geq\vartheta(œ,x)$ and $\delta(x,\mathcal{T}x)\geq\vartheta(x,\mathcal{T}x)$, under these conditions $\mathcal{T}$ admits a unique fixed point.

Proof. Assume that u^* , v^* be two fixed points of $\mathcal{T}$ with $u^*\neq v^*$.

Hence, by our presumption, $\exists\ x\in\mho$ such that $\delta(\jmath,x)\geq\vartheta(\jmath,x)$, $\delta(œ,x)\geq\vartheta(œ,x)$ and $\delta(x,\mathcal{T}x)\geq\vartheta(x,\mathcal{T}x)$.

Using Theorem II.2, it follows that $\{\mathcal{T}^nx\}$ converges to a fixed point z^*(say).

In view of $\mathcal{T}$ is triangular δ-orbital admissible map with respect to ϑ, we have

$\delta(x,\mathcal{T}^nx)\geq\vartheta(x,\mathcal{T}^nx)$ and hence

$$\delta(u^*,\mathcal{T}^nx)\geq\vartheta(u^*,\mathcal{T}^nx)\ and\ \delta(v^*,\mathcal{T}^nx)\geq\vartheta(v^*,\mathcal{T}^nx) \quad (21)$$

Now, $d_{nebm}(u^*,\mathcal{T}^nx)\leq M_\vartheta(d_{nebm}(u^*,\mathcal{T}^nx))$

$$\leq max\{d_{nebm}(u^*,\mathcal{T}^nx),\frac{d_{nebm}(u^*,\mathcal{T}u^*)d_{nebm}(\mathcal{T}^nx,\mathcal{T}^{n+1}x)}{1+d_{nebm}(u^*,\mathcal{T}^nx)}, \frac{d_{nebm}(u^*,\mathcal{T}^{n+1}u)d_{nebm}(\mathcal{T}^nx,\mathcal{T}u^*)}{1+d_{nebm}(u^*,\mathcal{T}^nx)}, \frac{d_{nebm}(\mathcal{T}^nu,\mathcal{T}u^*)[1+d_{nebm}(\mathcal{T}^nx,\mathcal{T}^{n+1}x)]}{1+d_{nebm}(u^*,\mathcal{T}^nx)}, \frac{d_{nebm}(\mathcal{T}^nx,\mathcal{T}^{n+1}x)[1+d_{nebm}(\mathcal{T}^nu,\mathcal{T}u^*)]}{1+d_{nebm}(u^*,\mathcal{T}^nx)}\}.$$

Therefore

$$limsup_{n\to\infty} M_{\vartheta}(d_{nebm}(u^*, \mathcal{T}^n x)) = d_{nebm}(u^*, z^*).$$

Also

$$N_{\vartheta}(d_{nebm}(u^*, \mathcal{T}^n x))$$
$$= min\{d_{nebm}(u^*, \mathcal{T}^{n+1}x), d_{nebm}(\mathcal{T}^n x, \mathcal{T}u^*), \frac{d_{nebm}(\mathcal{T}^n x, \mathcal{T}^{n+1}x)[1+d_{nebm}(\mathcal{T}^n u, \mathcal{T}u^*)]}{1+d_{nebm}(u^*, \mathcal{T}^n x)}\}.$$

Thus

$$limsup_{n\to\infty} N_{\vartheta}(d_{nebm}(u^*, \mathcal{T}^n x)) = 0.$$

We now show that $u^* = z^*$.

Now from (4) and (21), we have

$$0 \le \zeta_{sm}(d_{nebm}(\mathcal{T}u^*, \mathcal{T}^{n+1}x)), \mathsf{C}(M_{\vartheta}(d_{nebm}(u^*, \mathcal{T}^n x)))$$

which implies

$$d_{nebm}(u^*, \mathcal{T}^{n+1}x) \le \mathsf{C}(M_{\vartheta}(d_{nebm}(u^*, \mathcal{T}^n x))).$$

Letting limtsupt as $n \to \infty$, we have

$$\begin{aligned} d_{nebm}(z^*, u^*) &\le limsup_{n\to\infty}\mathsf{C}(M_{\vartheta}(d_{nebm}(z^*, \mathcal{T}^n x))) \\ &\le \mathsf{C}(d_{nebm}(z^*, u^*)) \\ &< d_{nebm}(z^*, u^*), \end{aligned}$$

this leads to a contradiction. Therefore $z^* = u^*$.

In a similar manner, we can prove that $v^* = z^*$. Thus, it follows that $u^* = v^*$.

Thus $\mathcal{T}$ admits a unique fixed point.

III. Corollaries and Examples

Corollary III.1: Let $\mho$ a new extended b metric space equipped with the functional $\vartheta : \mho \times \mho \times \mho \to [1, \infty)$. Assume that there exists a sequence $\{\Upsilon_n\}$; $\Upsilon_n > 1$, such that $\vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) < \Upsilon_n$ holds for all $n \in \mathcal{N}$ and $m > n$. Assume that $\mathcal{T} : \mho \to \mho$ is a generalized δ extended Z contraction i.e., for all $\oe, \jmath \in \mho$

$$\zeta_{sm}(d_{nebm}(\mathcal{T}\oe, \mathcal{T}\jmath), \mathsf{C}(M_{\vartheta}(\oe, \jmath))) \ge 0 \tag{22}$$

where $\mathsf{C} \in \Phi$ and $M_{\vartheta}(\oe, \jmath)$ is defined as in Definition II.1. Furthermore, suppose that

1) $\mathcal{T}$ is a triangular δ-orbital admissible mapping with respect to ϑ.
2) there exists $\Theta_1 \in \mho$ such that $\delta(\Theta_1, \mathcal{T}\Theta_1) \ge \vartheta(\Theta_1, \mathcal{T}\Theta_1)$,
3) either $\mathcal{T}$ is continuous

or

4) if $\{\Theta_n\}$ is a sequence in $\mho$ such that $\delta(\Theta_n, \Theta_{n+1}) \ge 1$, holds for all $n \in N$ and $\Theta_n \to \Theta$ as $n \to \infty$, then there exist a subsequence $\{\Theta_{n_k}\}$ of $\{\Theta_n\}$ with $\delta(\Theta_{n_k}, \Theta^*) \ge 1$.

Then $\mathcal{T}$ has a fixed point $\Theta^* \in \mho$ and $\{\mathcal{T}^n\Theta_1\}$ is converges to Θ^*.

Moreover, for all $\jmath, \oe \in Fix(\mathcal{T})$, we have $\delta(\jmath, \oe) \ge 1$, where $Fix(\mathcal{T})$ denotes the set of fixed points of $\mathcal{T}$, then $\mathcal{T}$ has a unique fixed point.

Proof. Proof follows by choosing $L = 0$ and $\vartheta(\jmath, \oe) = 1$ in Theorem II.2,Theorem II.3 and Theorem II.4 respectively.

Corollary III.2: Consider a new extended b metric space $\mho$ with a functional $\vartheta : \mho \times \mho \times \mho \to [1, \infty)$ and $\delta, \vartheta : \mho \times \mho \to [0, \infty)$ be two mappings. Further, suppose that there exists a sequence $\{\Upsilon_n\}$; $\Upsilon_n > 1$, for all $n \in N$ such that $\vartheta(\Theta_n, \Theta_{n+1}, \Theta_m) < \Upsilon_n$ for all $m > n$. Consider, $\mathcal{T} : \mho \to \mho$ is a $\delta - \vartheta$ extended Z contraction i.e.,

$$d_{nebm}(\mathcal{T}\oe, \mathcal{T}\jmath) \le \mathsf{C}(M_{\vartheta}(\oe, \jmath)) + LN_{\vartheta}(\oe, \jmath) \tag{23}$$

where $\mathsf{C} \in \Phi$, $L \ge 0$, $M_{\vartheta}(\oe, \jmath)$ and $N(\oe, \jmath)$ are defined as in Definition II.1. Further, suppose that

1) $\mathcal{T}$ is a triangular δ-orbital admissible mapping with respect to ϑ
2) there exists $\Theta_1 \in \mho$ such that $\delta(\Theta_1, \mathcal{T}\Theta_1) \ge \vartheta(\Theta_1, \mathcal{T}\Theta_1)$, and
3) $\mathcal{T}$ is $\delta - \vartheta$ continuous mapping

or

4) $\{\Theta_n\}$ is a sequence in $\mho$ such that $\delta(\Theta_n, \Theta_{n+1}) \ge 1$ for all $n \in N$ and $\Theta_n \to \Theta$ as $n \to \infty$, then there exist a subsequence $\{\Theta_{n_k}\}$ of $\{\Theta_n\}$ such that $\delta(\Theta_{n_k}, \Theta^*) \ge 1$.

Then $\mathcal{T}$ has a fixed point $\Theta^* \in \mho$ and $\{\mathcal{T}^n\Theta_1\}$ is converges to Θ^*.

Proof. Proof follows by choosing $d(t, s) = s - t$ in Theorem II.2, Theorem II.3 and Theorem II.4.

Example III.3: Let $\mho = [0, 1]$, we define $\vartheta : \mho \times \mho \times \mho \to [0, \infty)$ by

$$\vartheta(\oe, \jmath, d) = \frac{1}{8} + \sup_{t\in[0,1]} \frac{\jmath(t)+\oe(t)+d(t)}{1+\jmath(t)+\oe(t)+d(t)}$$

and $d_{nebm} : \mho \times \mho \to [0, \infty)$ by

$$d_{nebm}(\oe, \jmath) = \begin{cases} 0 & if\ \oe = \jmath \\ (\oe - \jmath)^2 & if\ \oe \ne \jmath. \end{cases}$$

Clearly, $d_{nebm}(\oe, \jmath)$ forms a new extended b-metric space with respect to ϑ.

Also, suppose that $\mathsf{C}(t) = \frac{t}{4}$.

It is easy to see that C is increasing and

$$\mathsf{C}^n(t)\Pi_{i=1}^{j}\vartheta(\Theta_i, \Theta_{i+1}\Theta_m) = t(\frac{1}{4})^n(\frac{7}{8})^n = t(\frac{7}{32})^n < \infty.$$

Hence C is a b-comparison function.

We now define $\mathcal{T} : \mho \to \mho$ by

$$\mathcal{T}\jmath = \begin{cases} \frac{\jmath+1}{4}\ if\ \jmath \in [0, \frac{1}{2}] \\ 1 - \frac{\jmath}{2}\ if\ \jmath \in (\frac{1}{2}, 1]. \end{cases}$$

Further, suppose that $\delta, \vartheta : \mho \times \mho \to [0, \infty)$ by

$$\delta(\jmath, \oe) = \begin{cases} 5 + e^{\jmath\oe} & if\ \jmath, \oe \in [0, \frac{1}{2}] \\ 0 & otherwise, \end{cases}$$

and

$$\vartheta(\jmath, \oe) = \begin{cases} 2 + e^{\jmath\oe} & if\ \jmath, \oe \in [0, \frac{1}{2}] \\ 3 & otherwise. \end{cases}$$

When $\oe \in [0, \frac{1}{2}]$, we have $\delta(\mathcal{T}\oe, \mathcal{T}\mathcal{T}\oe) \ge \vartheta(\mathcal{T}\oe, \mathcal{T}\mathcal{T}\oe)$.

Hence $\mathcal{T}$ is δ-orbital admissible with respect to ϑ.

Suppose that $\delta(\oe, \jmath) \ge \vartheta(\oe, \jmath)$ and $\delta(\oe, \mathcal{T}\jmath) \ge \vartheta(\oe, \mathcal{T}\jmath)$, then $\jmath, \oe, \mathcal{T}\jmath \in [0, \frac{1}{2}]$ which implies that $\delta(\oe, \mathcal{T}\jmath) \ge$

$\vartheta(œ, \mathcal{T}\jmath)$. Hence $\mathcal{T}$ is triangular δ orbital admissible with respect to ϑ.

Consider a sequence $\Theta_n \in N$ such that $\Theta_n \to \Theta^*$ as $n \to \infty$ and $\delta(\Theta_n, \Theta_{n+1}) \geq \vartheta(\Theta_n, \Theta_{n+1})$ for all $n \in \mathcal{N}$, then $\{\Theta_n\} \subseteq [0, \frac{1}{2}]$ for all $n \in \mathcal{N}$.

Then $\lim_{n\to\infty} \mathcal{T}\Theta_n = \lim_{n\to\infty} \frac{\Theta_n+1}{4} = \lim_{n\to\infty} \frac{\Theta_n}{4} + \frac{1}{4} = \frac{\Theta^*}{4} + \frac{1}{4} = \mathcal{T}\jmath$,

hence $\mathcal{T}$ is $\delta - \vartheta$ continuous .

We now verify the inequality (2.1.1) with $d : [0, \infty) \times [0, \infty)$ by $d(t, s) = \frac{s}{2} - t$ and $L = 0$.

$d(d_{nebm}(\mathcal{T}œ, \mathcal{T}\jmath), \theta(M_\vartheta(d_{nebm}(œ, \jmath))))$
$= d(d_{nebm}(\frac{œ+1}{4}, \frac{\jmath+1}{4}), \frac{1}{4}M_\vartheta(d_{nebm}(œ, \jmath)))$
$= \frac{1}{8}Md_{nebm}(œ, \jmath) - \frac{1}{16}d_{nebm}(œ, \jmath)$
$\geq \frac{d_b(œ,\jmath)}{8} - \frac{1}{16}d_{nebm}(œ, \jmath) = \frac{d_{nebm}(œ,\jmath)}{16} \geq 0.$

Hence $\mathcal{T}$ satisfies all the hypotheses of Theorem II.2 with $œ = \frac{1}{3}$ and $\frac{2}{3}$ are the fixed points of $\mathcal{T}$.

Here we observe that condition (H) fails to hold at $œ = \frac{3}{4}$ and $\jmath = 1$, then there is no x such that $\delta(\frac{3}{4}, x) \geq \vartheta(\frac{3}{4}, x)$ and $\delta(1, x) \geq \vartheta(1, x)$.

Example III.4: Let $\mho = [0, \frac{6}{5}]$, we define by $\vartheta : \mho \times \mho \times \mho \to [0, \infty)$ by $\vartheta(œ, \jmath, \vartheta) = œ + \jmath + \vartheta + 1.$ and $d_{nebm} : \mho \times \mho \to [0, \infty)$ by

$$d_{nebm}(œ, \jmath) = \begin{cases} 0 & if\ œ = \jmath \\ (œ + \jmath)^3 & if œ \neq \jmath. \end{cases}$$

Clearly, $d_{nebm}(œ, \jmath)$ forms an extended b-metric space with respect to ϑ.

suppose that $\complement(t) = \frac{5t}{43}$.

It is easy to see that θ is decreasing and

$$\complement^n(t)\Pi_{i=1}^{j}\vartheta(\Theta_i, \Theta_{i+1}, \Theta_m) = t(\frac{5}{43})^n(\frac{23}{5})^n = t(\frac{23}{45})^n < \infty.$$

Hence θ is an extended comparison function.

We now define that $\mathcal{T} : \mho \to \mho$ by

$$\mathcal{T}\jmath = \begin{cases} \frac{\jmath}{4}\ if\ \jmath \in [0, \frac{1}{2}] \\ \jmath - \frac{1}{5}\ if\ \jmath \in (\frac{1}{2}, \frac{6}{5}]. \end{cases}$$

Further, suppose that $\delta, \vartheta : \mho \times \mho \to [0, \infty)$ by

$$\delta(\jmath, œ) = \begin{cases} e^{\jmath+œ} & if\ \jmath, œ \in [0, \frac{1}{2}] \\ 2 & if \jmath \in [\frac{1}{2}, 1], œ = 0 \\ 0 & otherwise, \end{cases}$$

and

$$\vartheta(\jmath, œ) = \begin{cases} e^{\frac{\jmath+œ}{2}} & if\ \jmath, œ \in [0, \frac{1}{2}] \\ 1 & if \jmath \in [\frac{1}{2}, 1], œ = 0 \\ 3 & otherwise. \end{cases}$$

When $œ, \jmath \in [0, \frac{1}{2}]$, we have $\delta(\mathcal{T}œ, \mathcal{T}\mathcal{T}œ) \geq \vartheta(\mathcal{T}œ, \mathcal{T}\mathcal{T}œ)$.

Hence $\mathcal{T}$- is δ-orbital admissible with respect to ϑ.

Suppose that $\delta(œ, \jmath) \geq \vartheta(œ, \jmath)$ and $\delta(œ, \mathcal{T}\jmath) \geq \vartheta(œ, \mathcal{T}\jmath)$, then $\jmath, œ, \mathcal{T}\jmath \in [0, \frac{1}{2}]$ which implies that $\delta(œ, \mathcal{T}\jmath) \geq \vartheta(œ, \mathcal{T}\jmath)$. Hence $\mathcal{T}$ is triangular δ orbital admissible with respect to ϑ. Also, $\mathcal{T}$ is $\delta - \vartheta$ continuous .

We now verify the inequality (4) with $d : [0, \infty) \times [0, \infty)$ by $d(t, s) = \frac{2s}{3} - t$ and $L = 0$.

$d(d_{nebm}(\mathcal{T}œ, \mathcal{T}\jmath), \complement(M_\vartheta(d_{nebm}(œ, \jmath))))$
$= d(d_{nebm}(\frac{œ}{4}, \frac{\jmath}{4}), \frac{5}{43}M_\vartheta(d_b(œ, \jmath)))$
$= \frac{10}{129}Md_{nebm}(œ, \jmath) - \frac{1}{64}d_b(œ, \jmath)$
$\geq \frac{10d_{nebm}(œ,\jmath)}{129} - \frac{1}{64}d_{nebm}(œ, \jmath)$
$= \frac{511d_{nebm}(œ,\jmath)}{8256} \geq 0.$

Hence $\mathcal{T}$ satisfies the inequality (4). Also, since for any $œ \neq \jmath \in \mho$, we have $\delta(\jmath, 0) \geq \vartheta(\jmath, 0)$, $\delta(œ, 0) \geq \vartheta(œ, 0)$ and $\delta(0, \mathcal{T}0) \geq \vartheta(0, \mathcal{T}0)$, $\mathcal{T}$ satisfies condition (H). Hence $\mathcal{T}$ satisfies all the hypotheses of Theorem II.2 and $\jmath = 0$ is the unique fixed point of $\mathcal{T}$.

Example III.5: Let $\mho = [0, 2]$, we define by $\zeta_{sm} : \mho \times \mho \times \mho \to [0, \infty)$ by

$\vartheta(œ, \jmath, \vartheta) = \jmath + œ + \vartheta + 1.$

and $d_{nebm} : \mho \times \mho \to [0, \infty)$ by

$$d_{nebm}(œ, \jmath) = \begin{cases} 0 & if\ œ = \jmath \\ (œ + \jmath)^2 & if œ \neq \jmath. \end{cases}$$

Clearly, $d_{nebm}(œ, \jmath)$ forms an extended b-metric space with respect to ϑ.

suppose that $\complement(t) = \frac{t}{21}$.

It is easy to see that θ is increasing and

$$\complement^n(t)\Pi_{i=1}^{j}\vartheta(\Theta_i, \Theta_{i+1}, \Theta_m) = t(\frac{1}{21})^n(7)^n = t(\frac{1}{3})^n < \infty.$$

Hence $\complement$ is b comparison function.

We now define that $\mathcal{T} : \mho \to \mho$ by

$$\mathcal{T}\jmath = \begin{cases} 2\ if\ \jmath \in [0, \frac{1}{4}] \\ \jmath + \frac{1}{2}\ if\ \jmath \in (\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{2}\ \ if\ \ \jmath = \frac{1}{2} \\ \jmath + \frac{1}{4}\ \ if\ \ \jmath \in (\frac{1}{2}, 2] \end{cases}$$

Further, suppose that $\delta, \vartheta : \mho \times \mho \to [0, \infty)$ by

$$\delta(\jmath, œ) = \begin{cases} e^{\jmath+œ} & if\ \jmath, œ \in [0, \frac{1}{2}] \\ 2 & if \jmath \in (\frac{1}{2}, 1], œ = \frac{1}{2} \\ 0 & otherwise, \end{cases}$$

and

$$\vartheta(\jmath, œ) = \begin{cases} e^{\frac{\jmath+œ}{2}} & if\ \jmath, œ \in [0, \frac{1}{2}] \\ 1 & if \jmath \in (\frac{1}{2}, 1], œ = \frac{1}{2} \\ 3 & otherwise. \end{cases}$$

When $œ \in [0, \frac{1}{2}]$, we have $\delta(\mathcal{T}œ, \mathcal{T}\mathcal{T}œ) \geq \vartheta(\mathcal{T}œ, \mathcal{T}\mathcal{T}œ)$.

Hence $\mathcal{T}$- is δ-orbital admissible with respect to ϑ.

Suppose that $\delta(œ, \jmath) \geq \vartheta(œ, \jmath)$ and $\delta(œ, \mathcal{T}\jmath) \geq \vartheta(œ, \mathcal{T}\jmath)$, then $\jmath, œ, \mathcal{T}\jmath \in [0, \frac{1}{2}]$ which implies that $\delta(œ, \mathcal{T}\jmath) \geq \vartheta(œ, \mathcal{T}\jmath)$. Hence $\mathcal{T}$ is triangular δ orbital admissible with respect to ϑ. Let $d : [0, \infty) \times [0, \infty)$ by $d(t, s) = \frac{s}{2} - t$ Also, $\mathcal{T}$ is $\delta - \vartheta$ continuous.

We verify the inequality (4)

Case(i) When $\jmath \in [0, \frac{1}{4}]$ and $œ \in (\frac{1}{4}, \frac{1}{2})$ then

$d(d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe),\mathtt{C}(M_\vartheta(d_{nebm}(\jmath,\oe)))+LN(\jmath,\oe))$
$=\frac{1}{2}[\frac{(\jmath+2)^2[1+(\oe+2)^2]}{1+(\jmath+\oe)^2)}+L(\oe+\jmath+\frac{1}{2})^2$

Case(ii) When $\oe,\ \jmath\in(\frac{1}{4},\frac{1}{2})$ then
$d(d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe),\mathtt{C}(M_\vartheta(d_{nebm}(\jmath,\oe)))+LN(\jmath,\oe))$
$=\frac{1}{2}[\frac{(\oe+\jmath+\frac{1}{2})^2[1+(2\oe+\frac{1}{2})^2]}{1+(\jmath+\oe)^2)}+L(\oe+\jmath+\frac{1}{2})^2)]-(\oe+\jmath)^2$,
Hence $\mathcal{T}$ satisfies the inequality (4) for any $L\geq 0$.

Case(iii) When $\oe=\frac{1}{2},\ \jmath\in(\frac{1}{4},\frac{1}{2})$ or $\jmath=\frac{1}{2},\ \jmath\in(\frac{1}{4},\frac{1}{2})$ then
$d(d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe),\mathtt{C}(M_\vartheta(d_{nebm}(\jmath,\oe)))+LN(\jmath,\oe))$
$=\frac{1}{2}[\frac{(2+\frac{1}{2})^2(\jmath+\frac{1}{2})^2]}{1+(\frac{1}{2}+\jmath)^2)}+L(\jmath+\frac{1}{2})^2)-(2+\frac{1}{2})^2$,
Hence $\mathcal{T}$ satisfies the inequality (4) for any $L\geq 47$.

Case(iv) When $\jmath=\frac{1}{2},\ \oe\in(\frac{1}{4},\frac{1}{2})$ or $\oe=\frac{1}{2},\ \jmath\in(\frac{1}{4},\frac{1}{2})$, with out loss of generality suppose that $\oe\neq\frac{1}{2}$ then

$d(d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe),\mathtt{C}(M_\vartheta(d_{nebm}(\jmath,\oe)))+LN_\vartheta(\jmath,\oe))$
$=\frac{1}{2}(\frac{(\jmath+1)^2)(\jmath+\frac{1}{2})^2)}{1+(\frac{1}{2}+\jmath)^2)}+L(\jmath+\frac{1}{2})^2)-(2+\frac{1}{2})^2$,
Hence $\mathcal{T}$ satisfies the inequality (4) for any $L\geq\frac{23}{5}$.

Case(V) When $\oe,\jmath=\frac{1}{2}$ or $\jmath,\oe\in[0,\frac{1}{4}]$ then we have $d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe)=0$, hence
$d(d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe),\mathtt{C}(M_\vartheta(d_{nebm}(\jmath,\oe)))+LN(\jmath,\oe))\geq 0$

Hence $\mathcal{T}$ satisfies the inequality (4).

Also, since for any $\jmath\neq\oe\in\mho$, we have $\delta(\jmath,\frac{1}{2})\geq\vartheta(\jmath,\frac{1}{2})$, $\delta(\oe,\frac{1}{2})\geq\vartheta(\oe,\frac{1}{2})$ and $\delta(\frac{1}{2},\mathcal{T}\frac{1}{2})\geq\vartheta(\frac{1}{2},\mathcal{T}\frac{1}{2})$, $\mathcal{T}$ satisfies condition Ł. Hence $\mathcal{T}$ satisfies all the hypotheses of Theorem II.2 and $\jmath=\frac{1}{2}$ is the unique fixed point of $\mathcal{T}$.

Also, when $\jmath=\frac{1}{2}$ and $\oe=\frac{1}{4}$, $d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe)=\frac{25}{4}$ and $M_\vartheta(\jmath,\oe)=\frac{81}{16}$, by virtue of conditions on φ, there does not exist any φ such that equation (3) is satisfied hence $\mathcal{T}$ is not a nonlinear contraction. Hence we conclude that Theorem I.18, cannot be applied to this example. Hence, we can conclude that our results are more general than the results due to Chifu et. al.,[4] and Aydi et. al. [2].

IV. Solving a System of Linear Equations

In this segment, we will make an attempt to find unique solution of system of linear equations and we apply this method to solve a RLC Circuit. Consider the following system of equations.

$\theta_{11}\jmath_1+\theta_{12}\jmath_2+.....+\theta_{1n}\jmath_n=\rho_1$
$\theta_{21}\jmath_1+\theta_{21}\jmath_2+.....+\theta_{2n}\jmath_n=\rho_2$
$\theta_{31}\jmath_1+\theta_{32}\jmath_2+.....+\theta_{3n}\jmath_n=\rho_3$
....
....
....
$\theta_{n1}\jmath_1+\theta_{n2}\jmath_2+.....+\theta_{nn}\jmath_n=\rho_n$ (24)

consisting of n linear equations in n variables, where $\jmath\in\mho$, where $[\theta_{pq}]$ is a coefficient matrix and let $\mho=R^n$.

We define $d_{nebm}:\mho\times\mho\to\mathcal{R}$ by

$d_{nebm}(\jmath,\oe)=sup_{t\in[0,1]}|\jmath(t)-\oe(t)|^2$,

$\vartheta(\jmath(t),\oe(t),d(t))=2+\sup\frac{\jmath(t)+\oe(t)+d(t)}{1+\jmath(t)+\oe(t)+d(t)}$.

Then $\mho$ is complete new extended b-metric space.

Example IV.1: Let $(\mho,d_{nebm})$ be new extended b metric space with the function $\vartheta:\mho\times\mho\times\mho\to[1,\infty)$. Then system of linear equations defined by (24) has a unique solution if $\sum_{i=1,j=1}^{n}|\theta_{ij}|<0.5$.

Proof: Let $\delta,\vartheta:\mho\times\mho\to[0,\infty)$. We define a linear mapping $\mathcal{T}:\mho\to\mho$ when $\delta(\jmath,\oe)\geq\vartheta(\jmath,\oe)$ such that

$\mathcal{T}\jmath=\Gamma\jmath+M$ where $\jmath=(\jmath_1,\jmath_2,.....,\jmath_n)\in\mathcal{R}^n$ and $M=(\rho_1,\rho_2,....\rho_n)\in\mathcal{R}^n$ and

$$\Gamma=\begin{pmatrix}\theta 11 & .. & \theta 1n\\ \theta 21 & .. & \theta 2n\\ . & . & .\\ . & . & .\\ \theta n1 & .. & \theta nn\end{pmatrix}$$

We now prove that $\mathcal{T}$ is contraction mapping. Consider

$d_{nebm}(\mathcal{T}\jmath,\mathcal{T}\oe)$
$=2max_{1\leq i\leq n}|(\Sigma_j^n\theta_{ij}\jmath_j+\rho_j)-(\Sigma_j^n\theta ij\oe_j+\rho_j)|^2$
$=2max_{1\leq i\leq n}|(\Sigma_j^n\theta ij(\jmath_j-\oe_j)|^2$
$\leq 2max_{1\leq i\leq n}(\Sigma_j^n|\theta ij|^2|\jmath_j-\oe_j|^2)$
$\leq 2max_{1\leq i\leq n}\Sigma_j^n|\theta ij|^2|\times max_{1\leq i\leq n}|\jmath_j-\oe_j|^2$
$\leq 0.5max_{1\leq i\leq n}|\jmath_j-\oe_j|^2$
$\leq 0.5(2max_{1\leq i\leq n}|\jmath_j-\oe_j|^2)$
$\leq 0.5d_{nebm}(\jmath,\oe)$
$\leq 0.5d_{nebm}M_\vartheta(\jmath,\oe)$
$\leq\mathtt{C}(d_{nebm}M_\vartheta(\jmath,\oe))$,
where $\mathtt{C}(t)=\frac{1}{2}<1$ and

$$\mathtt{C}^n(t)\Pi_{i=1}^{j}\mathtt{C}(\Theta_i,\Theta_{i+1},\Theta_m)<(\frac{1}{5})^n3^n=t(\frac{5}{3})^n<\infty.$$

Hence by Corollary III.2, the system of equations has a unique solution.

To demonstrate this, we consider the following example.

Consider the following LCR Circuit with $V_1=0.1$ and $V_2=0.1$.

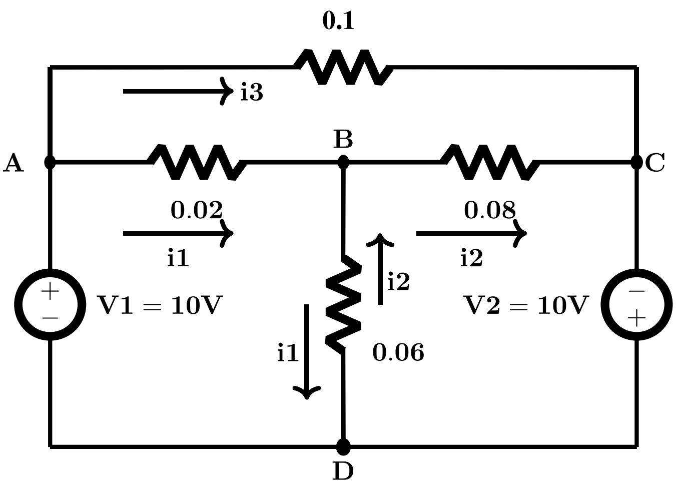

Figure 1: Electronic Circuit

We know that from Kirchhoff voltage law "the sum of all voltage in a closed loop is zero" in this example, we observe it in three loops (Figure 1).

From the first loop ABCD we have

$$(0.02+0.06)i_1-0.06i_2-0.02i_3=10$$

$$\Rightarrow 0.08i_1 - 0.06i_2 - 0.02i_3 = 10 \quad (25)$$

From the second loop BCDB we have

$$-0.06i_1 + (0.08 + 0.06)i_2) - 0.08i_3 = 10$$
$$\Rightarrow -0.06i_1 + 0.14i_2 - 0.08i_3 = 10 \quad (26)$$

From the third loop ABCA we have

$$-0.02i_1 - 0.08i_2 + (0.1 + 0.08 + 0.02)i_3) = 0$$
$$\Rightarrow -0.02i_1 - 0.08i_2 + 0.02i_3 = 0 \quad (27)$$

So we have below system of linear equations

$$\begin{aligned} 0.08i_1 - 0.06i_2 - 0.02i_3 &= 10 \\ -0.06i_1 + 0.14i_2 - 0.08i_3 &= 10 \\ -0.02i_1 - 0.08i_2 + 0.2i_3 &= 0 \end{aligned} \quad (28)$$

Now on taking the sum of the coefficients I_{ij}, where $i = 1, 2, 3$ and $j = 1, 2, 3$, we have

$$\sum_1^j |I_{1j}| = 0.08 + 0.06 + .02 = 0.16 < 0.5$$
$$\sum_2^j |I_{2j}| = 0.06 + 0.14 + .08 = 0.28 < 0.5$$
$$\sum_3^j |I_{3j}| = 0.02 + 0.08 + 0.02 = 0.3 < 0.5$$

Hence, from Theorem IV.1, system of equations (28) has a unique solution.

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