On Partition Energy of Lexicographic Product of Regular Graphs

D. P. Pushpa, S. V. Roopa, M. A. Sriraj

***Abstract*—The partition energy $P_k(G)$ is defined as the sum of absolute values of k-partition eigenvalues of G. In this paper, we consider two regular graphs G and H with n and m vertices respectively and construct lexicographic product $G[H]$ and $G[H_1]$, where $H_1 = H\nabla K_1$. We investigate the spectra and hence energy with respect to a partition P_n of these graphs and their generalized complements, which relate to the 1-partition energies of the factor graphs H providing a deeper understanding of their combined spectral characteristics. Further more, we construct graphs which are equienergetic with respect to partition P_n.**

***Index Terms*—k-partition eigenvalues, k-partition energy, Lexicographic product, equienergetic Graphs and block circulant matrix.**

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I. Introduction

Spectral graph theory focuses on exploration of relationship between the structure of a graph, eigenvalues and eigenvectors of matrices associated with them such as adjacency matrix, Laplacian matrix, distance matrix, color matrix, Harary matrix, partition matrix, Albertson matrix etc. These matrices encode important properties of the graph and their spectral properties reveal insights like clustering, expansion, connectivity of graphs.

For spectra and energy with respect to above matrices of various graph structures, one can refer to [1], [4], [5], [6], [8], [12] and [19].

In the study of graph energy, various graph operation and transformation such as graph products, partitions and complements play significant roles. They enrich the study of graph energy by unveiling relationships between structural modification and spectral properties.

Graph partitions play a crucial role in spectral clustering, where eigenvalues of partitioned graphs help identify structural patterns. An application of graph partition in medical field to partition RNAs and classification of frame-shifting elements in viruses can be found in [13].

Let $G(V, E)$ be a graph. Let $\{V_1, V_2, \ldots, V_k\}$ be non-empty disjoint subsets of V. Then $P_k = \{V_1, V_2, \ldots, V_k\}$ is called a partition of vertex set V [18].

In [19], the authors have introduced partition energy of a graph as follows. Let $G(V, E)$ be a graph of order n and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of V. Then the partition matrix $P_k(G) = [a_{ij}]$ is a unique square symmetric matrix with zero diagonal defined as follows:

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent where } v_i,\, v_j \in V_r, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent where} \\ & \quad v_i, v_j \in V_r, \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent between the sets} \\ & V_r \text{ and } V_s \text{ for } r \neq s \text{ where } v_i \in V_r \text{ and} \\ & \quad v_j \in V_s, \\ 0 & \text{otherwise} \end{cases}$$

This partition matrix determines the partition of vertex set of graph G uniquely. The eigenvalues of this matrix are called partition eigenvalues of G. Further, k-partition energy of a graph G denoted by $E_{P_k}(G)$ is defined as the sum of the absolute values of k-partition eigenvalues of G, where eigenvalues of $P_k(G)$ are k-partition eigenvalues of G.

The complement of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

In literature, we can see different types of complements such as complement of a graph $G(V, E)$ with respect to a subset $S \subseteq V$ called partial complement and complement of a graph with respect to a partition P_k called generalized complements.

More on partial complements of a graph and associated energy concepts can be found in [2] and [9].

If P_k is a partition of the vertex set of G, we can also determine partition energy of two types of complements of given graph called k-complement and $k(i)$- complement graph introduced by E. Sampathkumar in [17]. If the vertex set of a graph G of order n is partitioned into n sets then the partition energy matches with the usual energy of a graph. So partition energy may be considered as a generalization of energy of a graph introduced by I. Gutman in [10]. More information on partition energy can be found in [16], [20], [21], [22], [23] and [24].

Graph products combine two graphs to form a new graph, and analyzing the energy of the resulting graph can reveal insights in to the spectral properties of the original graph. Various graph products are introduced in [11] and one notable product among them is the lexicographic product which is defined as follows.

Definition I.1 ([11]). *Let G and H be two graphs with n and m vertices respectively. Then lexicographic product of two graphs G and H is formed by taking one copy of G and n copies of H and joining any two vertices (u, v) and (x, y) if and only if, either u is adjacent to x in G or $u = x$ and v is adjacent to y. This graph is represented by $G[H]$ and is also called as composition of graphs.*

Definition I.2 ([14]). *Two non-isomorphic graphs G_1 and*

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G_2 of same order are said to be equienergetic if $E(G_1) = E(G_2)$.

Some definitions and results which are essential for our computation are as follows.

Definition I.3 ([7]). *Let $A_1, A_2, \ldots, A_m$ be square matrices of order n. A block circulant matrix of type (m, n) (of order mn) is an $mn \times mn$ matrix of the form*

$$bcirc(A_1, A_2, \ldots, A_m) = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ A_m & A_1 & \cdots & A_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}.$$

If each A_i for $1 \leqslant i \leqslant n$ is circulant then we call the above matrix as block circulant with circulant blocks.

Theorem I.4 ([19]). *If G is a r-regular graph with n vertices and $3r-n+1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of $P_1(G)$, then 1-partition eigenvalues of its $1(i)$-complement $\overline{(G)_{1(i)}}$ are $2n-3r-2, -\lambda_2 - 1, -\lambda_3 - 1, \ldots, -\lambda_n - 1$.*

Theorem I.5 ([3]). *Let C be an A-factor block circulant. Then*

$$C = V_A P(D_A) V_A^{-1},$$

where V_A is block Vandermonde matrix and $P(z)$ is the representor of C. Moreover, the set of A-factor circulants coincides with the set of matrices of the form

$$V_A diag[M_1, M_2, \ldots, M_m] V_A^{-1},$$

that is, $P(D_A) = diag[M_1, M_2, \ldots, M_m]$ for a matrix polynomial

$$P(z) = C_1 + C_2 z + \cdots + C_m z^{m-1} \text{ if and only if}$$
$$[C_1 C_2 \ldots C_m] V_A = [M_1 M_2 \ldots M_m].$$

The following result is a consequence of the Theorem I.5.

Corollary I.6 ([3]). *The factor circulant C can also be expressed as*

$$C = \Re F_{mn}^* P(K\Omega) F_{mn} \Re^{-1}$$

where F_{mn} is a block Fourier matrix, $\Omega = diag[I, \omega I, \omega^2 I, \ldots, \omega^{m-1} I]$ $(\omega = \exp(\frac{2\pi i}{m})), K$ is the principal m^{th} root of the non-singular matrix A and $\Re = diag[IKK^2 \ldots K^{m-1}]$. In particular if C is a block circulant then it can be represented as

$$C = F_{mn}^* P(\Omega) F_{mn}.$$

II. Lexicographic product of two regular graphs

In this section, we determine the n-partition energy of lexicographic product of two regular graphs G and H denoted by $G[H]$, where G is circulant and also their generalized complements with respect to a partition P_n. Further we establish relationship not only between spectra of $G[H]$ and its generalized complements but also relationship between spectra of $G[H]$ and its generalized complements with 1-partition spectra of H and $\overline{(H)_{1(i)}}$.

Now we give a brief description of the vertex partition P_n of lexicographic product of two graphs considered throughout this paper. Let G and H be two graphs with vertex sets $\{u_1, u_2, \ldots, u_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ respectively. Then the lexicographic product $G[H]$ will have mn vertices of the form $v_{ij} = (u_i, w_j)$. Let $P_n = \{V_1, V_2, \ldots, V_n\}$ be a partition of the vertex set of $G[H]$ where $V_i = \{v_{ij}\}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. With this kind of partition, we observe that the graph induced by vertices of V_i is isomorphic to H.

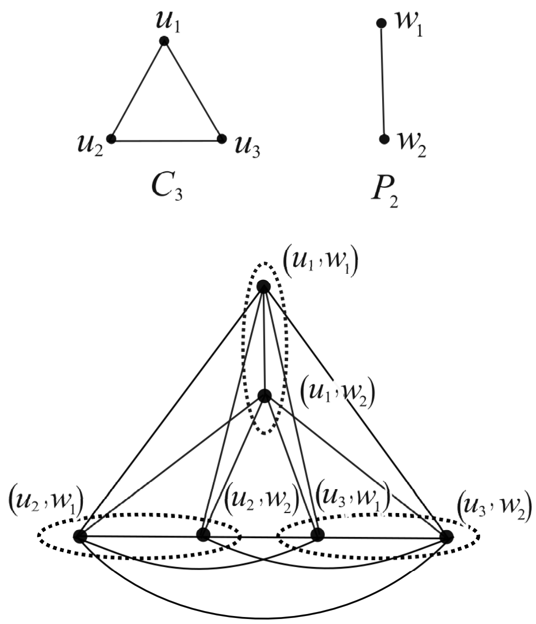

Fig. 1: $C_3[P_2]$

For example in Figure 1, the lexicographic product of C_3 and P_2 that is $C_3[P_2]$ is shown above. Its vertex partition is given by P_3=$\{V_1, V_2, V_3\}$, where V_1=$\{(u_1, w_1), (u_1, w_2)\}$=$\{v_{11}, v_{12}\}$, V_2=$\{(u_2, w_1), (u_2, w_2)\}$=$\{v_{21}, v_{22}\}$ and V_3=$\{(u_3, w_1), (u_3, w_2)\}$=$\{v_{31}, v_{32}\}$. Here we can observe that the graphs within each partition are isomorphic to P_2.

In [24], the authors have discussed partition energy of lexicographic product of two graphs like $C_m[K_n]$, $K_m[C_n], C_m[C_n], S_m[K_n]$ and $K_{m\times 2}[K_n]$. Motivated by this, we are considering lexicographic product of two graphs G and H where G is r_1-regular circulant and H is r_2-regular respectively and discuss partition energy of $G[H]$ and its two types of complements obtained with respect to the partition considered as above in terms of 1-partition energy of H and $\overline{(H)_{1(i)}}$ respectively.

Theorem II.1. *Let H be a r-regular graph with m vertices and $J_{m\times m}$ is a matrix of ones, then the characteristic polynomial of $C = P_1(H) + uJ_{m\times m}$ and $D = P_1(\overline{(H)_{1(i)}}) + uJ_{m\times m}$ are*

$$(i)\ \phi_C(\lambda) = \frac{\phi_{P_1(H)}(\lambda)}{\lambda - (3r - m + 1)}[\lambda - (3r - m + 1 + um)]$$

$$(ii)\ \phi_D(\beta) = \frac{\phi_{P_1(\overline{(H)_{1(i)}})}(\beta)}{\beta - (2m - 3r - 2)}[\beta - (2m - 3r - 2 + um)]$$

where u is a constant.

Proof. (i) Let $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ be the eigenvalues of $P_1(H)$. Since H is regular, the 1-partition matrix $P_1(H)$ will have the row sum $\lambda_0 = 3r - m + 1$ as an eigenvalue.

It can be observed that the corresponding eigenvector is $J_{m\times 1}$. Also the row sum in $J_{m\times m}$ is m which is same for all the rows. This implies that m is an eigenvalue of $J_{m\times m}$ with the corresponding eigenvector $J_{m\times 1}$ and its remaining eigenvalues are zeros.

Consider

$$\begin{aligned}(C)J_{m\times 1} &= (P_1(H)+uJ_{m\times m})J_{m\times 1}\\ &= P_1(H)J_{m\times 1}+(uJ_{m\times m})J_{m\times 1}\\ &= (3r-m+1)J_{m\times 1}+umJ_{m\times 1}\\ &= (3r-m+1+um)J_{m\times 1}\end{aligned}$$

which implies that $3r-m+1+um$ is an eigenvalue of $P_1(H)+uJ$. Let $X_i=(x_{(i1)},x_{(i2)},\ldots,x_{(im)})^T$ be the eigenvector of $P_1(H)$ corresponding to the eigenvalue λ_i for $i=1,2,\ldots,m-1$. This implies that

$$x_{(i1)}+x_{(i2)}+\cdots+x_{(im)}=0. \quad (1)$$

It can be observed from equation (1) that X_i is also is an eigenvector of $J_{m\times m}$ corresponding to the zero eigenvalues.

Therefore for $i=1,2,\ldots,m-1$

$$(P_1(H)+uJ_{m\times m})X_i=\lambda_i X_i.$$

Hence

$$\phi_C(\lambda)=\frac{\phi_{P_1(H)}(\lambda)}{\lambda-(3r-m+1)}[\lambda-(3r-m+1+um)].$$

(ii) With similar argument as above,

$$\phi_D(\beta)=\frac{\phi_{P_1(\overline{(H)_{1(i)}})}(\beta)}{\beta-(2m-3r-2)}[\beta-(2m-3r-2+um)]. \quad \square$$

Theorem II.2. *Let $S=G[H]$, where G is a r_1-regular circulant graph with n vertices and H is a r_2-regular graph with m vertices. If $P_n=\{V_1,V_2,\ldots,V_n\}$ where $V_i=\{v_{i1},v_{i2},\ldots,v_{im}\}$ for $i=1,2,\ldots,n$ is a partition of vertex set of S, then*

$$(i)\ E_{P_n}(S)=nE_{P_1}(H)-n|3r_2-m+1|+\sum_{t=0}^{n-1}|3r_2-m+1+\alpha_t m|.$$

$$(ii)\ E_{P_n}(\overline{(S)_n})=nE_{P_1}(H)-n|3r_2-m+1|+|3r_2-m+1+(n-r_1-1)m|+\sum_{t=1}^{n-1}|3r_2-m+1+(-1-\alpha_t)m|.$$

$$(iii)\ E_{P_n}(\overline{(S)_{n(i)}})=nE_{P_1}(\overline{(H)_{1(i)}})-n|2m-3r_2-2|+\sum_{t=0}^{n-1}|2m-3r_2-2+\alpha_t m|.$$

where α_t are adjacency eigenvalues of G.

Proof. (i) Since G and H are r_1 and r_2 regular respectively, it follows from the construction of lexicographic product that $S=G[H]$ is also regular of degree r_1+r_2. Noted that the matrix of S with respect to the given partition P_n is a block circulant matrix which can be represented as

$$P_n(S)=bcirc(P_1(H),H_2,H_3,\ldots,H_n)_{n\times n}$$

where $P_1(H)$ is the 1-partition matrix of H and remaining H_i's are either zero matrices or matrices with all of whose entries are ones. All these matrices are of order m.

Since H is r_2-regular, let $\lambda_0=3r_2-m+1,\lambda_1,\ldots,\lambda_{m-1}$ be the 1-partition eigenvalues of H. Let us take $H_2=J_{m\times m}$ and H_3 as zero matrix. Since G is r_1-regular, their will be r_1 numbers of $J_{m\times m}$ matrices and $(n-r_1-1)$ numbers of zero matrices. We know that $diag(3r_2-m+1,\lambda_1,\ldots,\lambda_{m-1})$ and $diag(m,0,\ldots,0)$ are the matrices of eigenvalues of $P_1(H)$ and J respectively. Since $G[H]$ is block circulant, from Corollary I.6, the diagonal form is

$$diag(P_1(H)+\alpha_0 J,P_1(H)+\alpha_1 J,\ldots,P_1(H)+\alpha_{n-1}J)$$

where α_t for $t=0,1,\ldots,n-1$ are adjacency eigenvalues of G. Applying Theorem II.1 to each of $P_1(H)+uJ$, we get

$$\phi_{P_n(S)}(\lambda)=\left(\frac{\phi_{P_1(H)}(\lambda)}{\lambda-(3r_2-m+1)}\right)^n\times\prod_{t=0}^{n-1}[\lambda-(3r_2-m+1+\alpha_t m)].$$

Hence,

$$E_{P_n}(S)=nE_{P_1}(H)-n|3r_2-m+1|+\sum_{t=0}^{n-1}|3r_2-m+1+\alpha_t m|.$$

(ii) The matrix of $(\overline{(S)_n})$ is

$$P_n(\overline{(S)_n})=bcirc(P_1(H),\overline{H_2},\overline{H_3},\ldots,\overline{H_n})_{n\times n}$$

Since G is r_1-regular, their will be r_1 numbers of $\overline{H_2}$ matrices and $(n-r_1-1)$ number of $\overline{H_3}$ matrices. Here $\overline{H_2}$ is zero matrix, and $\overline{H_3}=J_{m\times m}$. The diagonal form of $P_n(\overline{(S)_n})$ is

$$diag(P_1(H)+(n-r_1-1)J,\ P_1(H)+(-\alpha_1-1)J,\ldots,P_1(H)+(-\alpha_{n-1}-1)J)$$

Applying Theorem II.1, we get

$$\phi_{P_n(\overline{(S)_n})}(\lambda)=\left(\frac{\phi_{P_1(H)}(\lambda)}{\lambda-(3r_2-m+1)}\right)^n\times[\lambda-(3r_2-m+1+(n-r_1-1)m)]\times\prod_{t=1}^{n-1}[\lambda-(3r_2-m+1+(-1-\alpha_t)m)].$$

Hence,

$$E_{P_n}(\overline{(S)_n})=nE_{P_1}(H)-n|3r_2-m+1|+|3r_2-m+1+(n-r_1-1)m|+\sum_{t=1}^{n-1}|3r_2-m+1+(-1-\alpha_t)m|.$$

(iii) The matrix $\overline{(S)_{n(i)}}$ is

$$P_n(\overline{(S)_{n(i)}})=bcirc(P_1(\overline{(H)_{1(i)}}),H_2,\ldots,H_n)_{n\times n}$$

we know that the matrix of eigenvalues of $J_{m\times m}$ is $diag(m,0,\ldots,0)$ and of $P_1(\overline{(H)_{1(i)}})$ is $diag(2m-3r_2-2,-\lambda_1-1,-\lambda_2-1,\ldots,-\lambda_{m-1}-1)$.

We can diagonalise $P_n(\overline{(S)_{n(i)}})$ as

$$diag(P_1(\overline{(H)_{1(i)}}) + r_1 J, \quad P_1(\overline{(H)_{1(i)}}) + \alpha_1 J, \ldots, P_1(\overline{(H)_{1(i)}}) + \alpha_{n-1} J),$$

From (ii) of Theorem II.1, we get

$$\phi_{P_n(\overline{(S)_{n(i)}})}(\beta) = \left(\frac{\phi_{P_1(\overline{(H)_{1(i)}})}(\beta)}{\beta - (2m - 3r_2 - 2)} \right)^n \times \prod_{t=0}^{n-1} \left[\beta - (2m - 3r_2 - 2 + \alpha_t m) \right].$$

Hence,

$$E_{P_n}(\overline{(S)_{n(i)}}) = nE_{P_1}(\overline{(H)_{1(i)}}) - n|2m - 3r_2 - 2| + \sum_{t=0}^{n-1} |2m - 3r_2 - 2 + \alpha_t m|. \qquad \square$$

Note:

1. As the graph S contains n copies of H, it can be observed that the eigenvalues $\lambda_1, \ldots, \lambda_{m-1}$ of $P_1(H)$ are eigenvalues of $P_n(S)$ and $P_n(\overline{(S)_n})$ repeated n times.
2. As $P_n(\overline{(S)_{n(i)}})$ contains $P_1(\overline{(H)_{1(i)}})$, it can be observed that $-\lambda_1 - 1, \ldots, -\lambda_{m-1} - 1$ are eigenvalues of $P_n(\overline{(S)_{n(i)}})$ repeated n times.

III. Construction of some equienergetic graphs

In this section, we consider two graphs H_1 and H_2 of same order and same degree which are 1-partition equienergetic. We consider a graph G which is r_1-regular circulant and construct new graphs $G[H_1]$ and $G[H_2]$ which are also equienergetic.

Definition III.1. *Two non-isomorphic graphs G_1 and G_2 of same order are said to be partition equienergetic if $E_{P_n}(G_1) = E_{P_n}(G_2)$ with respect to P_n.*

Let $V = \{v_1, v_2, \ldots, v_{12}\}$ and $U = \{u_1, u_2, \ldots, u_{12}\}$ be the vertex sets of two graphs G_1 and G_2 respectively. We now construct new graphs H_1 and H_2 using graph operations between G_1 and G_2 as follows.

The graph H_1 is obtained by inserting the edges $v_i u_i$ for $i = 1, 2, \ldots, 12$ as shown in the schematic diagram Fig. 2.

Similarly, The graph H_2 is obtained by inserting the edges $v_i u_i$ for $i = 1, 2, \ldots, 12$ and also the edges $v_i u_{i+1}, v_{i+1} u_i$ for $i = 1, 3, 5, 9, 11$ as shown in the schematic diagram Fig. 3.

In the graph H_1, let us choose the component graphs $G_1 = G_2 = K_{12}$. Then the 1-partition spectra of H_1 is

$$\begin{cases} 13 & \text{once} \\ 1 & 11 \ \text{times} \\ 31 & \text{once} \\ -5 & 11 \ \text{times} \end{cases}$$

and $E_{P_1}(H_1) = 110$.

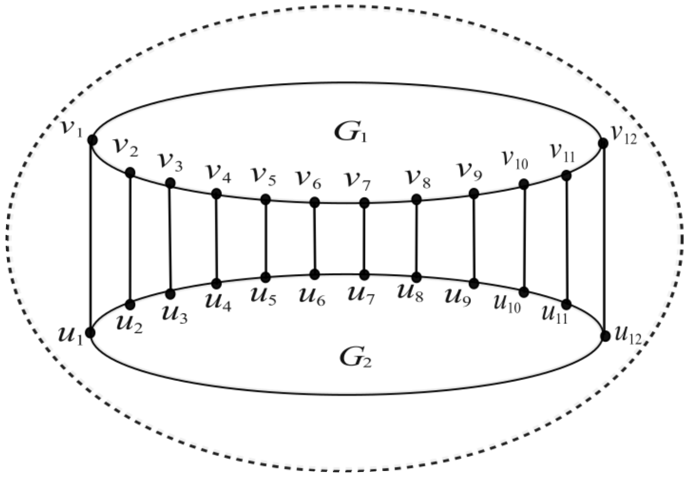

Fig. 2: H_1

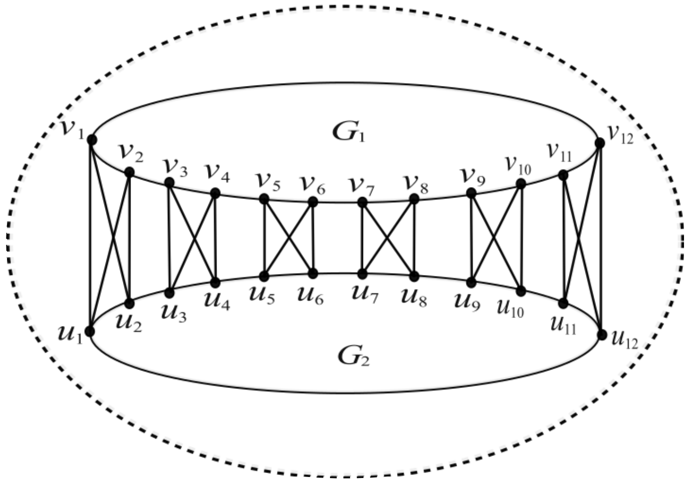

Fig. 3: H_2

In the graph H_2, let us choose the component graphs $G_1 = G_2 = K_{6\times 2}$. Then the 1-partition spectra of H_2 is

$$\begin{cases} 13 & \text{once} \\ 1 & 17 \ \text{times} \\ 25 & \text{once} \\ -11 & 5 \ \text{times.} \end{cases}$$

Thus, $E_{P_1}(H_2) = 110$.

Hence, H_1 and H_2 are non cospectral 1-partition equienergetic.

The above example confirms the existence of 1-partition equienergetic graphs which leads to the following theorem.

Theorem III.2. *Let G be a r_1-regular circulant graph with n vertices. If H_1 and H_2 are r_2-regular graphs with m vertices which are 1-partition equienergetic then*

(i) *$G[H_1]$ and $G[H_2]$ are equienergetic with respect to P_n.*

(ii) *$\overline{(G[H_1])_n}$ and $\overline{(G[H_2])_n}$ are equienergetic with respect to P_n.*

Proof. Given H_1 and H_2 are 1-partition equienergetic.

$$\therefore E_{P_1}(H_1) = E_{P_1}(H_2) \tag{2}$$

Using equation (2), in (i) and (ii) of Theorem II.2, it follows that $\underline{G[H_1]}$ and $\underline{G[H_2]}$ are equienergetic with respect to P_n. Also $\overline{(G[H_1])_n}$ and $\overline{(G[H_2])_n}$ are equienergetic with respect to P_n. □

IV. Lexicographic product of regular graph with semi regular graph

In this section, we consider lexicographic product $G[H_1]$ of an r_1-regular circulant graph G of order n with a semi regular graph $H_1 = H\nabla K_1$, where H is any r_2-regular graph of order m and discuss its partition energy with respect to P_n and its generalized complements.

Following theorem is useful to find the spectra of $G[H_1]$.

Theorem IV.1. *[15] Let $C = P_1(H_1) + uJ_{m\times 1}$ where u is a constant and $J_{m\times 1}$ is a matrix in which all entries are ones and $D = P_1(\overline{(H_1)_{1(i)}}) + uJ_{m\times 1}$, then the characteristic polynomials of C and D are respectively,*

(i) $$\frac{\phi_{P_1(H)}(\lambda)}{(\lambda - 3r_2 + m - 1)}[\lambda^2 + \lambda(m - 3r_2 - u(m+1) - 1) + u(3r_2 - 5m + 1) - 4m].$$

(ii) $$\frac{\phi_{P_1(\overline{(H)_{1(i)}})}(\lambda)}{(\lambda - 2m + 3r_2 + 2)} \times [\lambda^2 + \lambda(3r_2 + 2(1-m) - u(m+1)) + u(4m - 3r_2 - 2) - m].$$

In the following theorem, we consider lexicographic product of an r_1-regular graph G with the semi regular graph H_1 defined as above and obtain energy of $G[H_1]$ and its generalized complements with respect to P_n.

Theorem IV.2. *If $P_n = \{V_1, V_2, \ldots, V_n\}$ where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{i(m+1)}\}$ for $i = 1, 2, \ldots, n$ is a partition of vertex set of $G[H_1]$, where G is an r_1-regular graph with n vertices, then*

$$\begin{aligned}(i)\quad E_{P_n}(G[H_1]) &= nE_{P_1}(H) - n|3r_2 - m + 1| \\ &+ \sum_{t=0}^{n-1}[|\beta_t + \gamma_t| + |\beta_t - \gamma_t|] \\ (ii)\quad E_{P_n}\overline{(G[H_1])_n} &= nE_{P_1}(H) - n|3r_2 - m + 1| \\ &+ |\eta + \zeta| + |\eta - \zeta| \\ &+ \sum_{t=1}^{n-1}[|\delta_t + \rho_t| + |\delta_t - \rho_t|] \\ (iii)\quad E_{P_n}\overline{((G[H_1])_{n(i)})} &= nE_{P_1}(\overline{(H)_{1(i)}}) \\ &- n|2m - 3r_2 - 2| \\ &+ \sum_{t=0}^{n-1}[|\mu_t + \nu_t| + |\mu_t - \nu_t|]\end{aligned}$$

where

$$\beta_t = \frac{-(m - 3r_2 - \alpha_t(m+1) - 1)}{2},$$

$$\gamma_t = \frac{\sqrt{\begin{array}{l}(m - 3r_2 - \alpha_t(m+1) - 1)^2 \\ -4[\alpha_t(3r_2 - 5m + 1) - 4m]\end{array}}}{2},$$

$$\eta = \frac{-(m - 3r_2 - (n - r_1 - 1)(m+1) - 1)}{2},$$

$$\zeta = \frac{\sqrt{\begin{array}{l}(m - 3r_2 - (n - r_1 - 1)(m+1) - 1)^2 \\ -4[(n - r_1 - 1)(3r_2 - 5m + 1) - 4m]\end{array}}}{2},$$

$$\delta_t = \frac{-(m - 3r_2 - (-1 - \alpha_t)(m+1) - 1)}{2},$$

$$\rho_t = \frac{\sqrt{\begin{array}{l}(m - 3r_2 - (-1 - \alpha_t)(m+1) - 1)^2 \\ -4[(-1 - \alpha_t)(3r_2 - 5m + 1) - 4m]\end{array}}}{2},$$

$$\mu_t = \frac{-(3r_2 + 2(1-m) - \alpha_t(m+1))}{2},$$

$$\nu_t = \frac{\sqrt{\begin{array}{l}(3r_2 + 2(1-m) - \alpha_t(m+1))^2 \\ -4[\alpha_t(4m - 3r_2 - 2) - m]\end{array}}}{2}.$$

Proof. (i) The matrix $P_n(G[H_1])$ is a block circulant matrix of order $n(m+1)$ which is denoted by

$$P_n(G[H_1]) = bcirc(P_1(H_1), H_2, \ldots, H_n)$$

where $P_1(H_1) = P_1(H\nabla K_1)$ and remaining $H_i's$ are either zero matrices or matrices with all of whose entries are ones. All these matrices are of order $m+1$.

Since H is r_2-regular, $\lambda_0 = 3r_2 - m + 1, \lambda_1, \ldots, \lambda_{m-1}$ be the 1-partition eigenvalues of H. Let us take $H_2 = J$ and H_3 as zero matrix.

Since G is r_1-regular, their will be r_1 numbers of J matrices and $(n - r_1 - 1)$ numbers of zero matrices. Thus the diagonal form of $P_n(G[H_1])$ is

$$diag(P_1(H_1) + \alpha_0 J, P_1(H_1) + \alpha_1 J, \ldots, P_1(H_1) + \alpha_{n-1} J)$$

where $\alpha_t = \sum_{k=1}^{n} a_k \; e^{\frac{2\pi i t(k-1)}{n}}$ for $t = 0, 1, \ldots, n-1$ are adjacency eigenvalues of G and $a_1, a_2, \ldots, a_m$ are first row elements of G. Let $A_t = P_1(H_1) + \alpha_t J$ for $t = 0, 1, 2, \ldots, n-1$. Using (i) of Theorem IV.1, we get the spectra of $P_n(G[H_1])$ as follows

$$\begin{cases}\lambda_i \text{ for } i = 1, 2, \ldots, m-1 & n \text{ times} \\ \beta_t \pm \gamma_t, \; 0 \le t \le n-1 & \text{once,}\end{cases}$$

where λ_i are 1-partition eigenvalues of H,

$$\beta_t = \frac{-(m-3r_2-\alpha_t(m+1)-1)}{2} \text{ and}$$

$$\gamma_t = \frac{\sqrt{\begin{array}{c}(m-3r_2-\alpha_t(m+1)-1)^2\\ -4[\alpha_t(3r_2-5m+1)-4m]\end{array}}}{2}.$$

Hence,

$$E_{P_n}(G[H_1]) = nE_{P_1}(H) - n|3r_2-m+1| + \sum_{t=0}^{n-1}[|\beta_t+\gamma_t|+|\beta_t-\gamma_t|].$$

(ii) The diagonal form of $P_n\overline{(G[H_1])_n}$ is

$$diag(P_1(H_1)+(n-r_1-1)J, P_1(H_1)+(-1-\alpha_1)J,\dots,P_1(H_1)+(-1-\alpha_{n-1})J)$$

with similar discussion as above, we get the spectra of $P_n\overline{(G[H_1])_n}$ as

$$\begin{cases}\lambda_i \text{ for } i=1,2,\dots,m-1 & n \text{ times}\\ \eta\pm\zeta & \text{once}\\ \delta_t\pm\rho_t, \quad 1\le t\le n-1 & \text{once,}\end{cases}$$

where λ_i are 1-partition eigenvalues of H,

$$\eta = \frac{-(m-3r_2-(n-r_1-1)(m+1)-1)}{2},$$

$$\zeta = \frac{\sqrt{\begin{array}{c}(m-3r_2-(n-r_1-1)(m+1)-1)^2\\ -4[(n-r_1-1)(3r_2-5m+1)-4m]\end{array}}}{2},$$

$$\delta_t = \frac{-(m-3r_2-(-1-\alpha_t)(m+1)-1)}{2} \text{ and}$$

$$\rho_t = \frac{\sqrt{\begin{array}{c}(m-3r_2-(-1-\alpha_t)(m+1)-1)^2\\ -4[(-1-\alpha_t)(3r_2-5m+1)-4m]\end{array}}}{2}.$$

Hence,

$$E_{P_n}\overline{(G[H_1])_n} = nE_{P_1}(H) - n|3r_2-m+1| + |\eta+\zeta|+|\eta-\zeta| + \sum_{t=1}^{n-1}[|\delta_t+\rho_t|+|\delta_t-\rho_t|].$$

(iii) The diagonal form of $P_n\overline{((G[H_1])_{n(i)})}$ is

$$diag(P_1(\overline{(H_1)_{1(i)}})+r_1J, P_1(\overline{(H_1)_{1(i)}})+\alpha_1J,\dots,P_1(\overline{(H_1)_{1(i)}})+\alpha_{n-1}J)$$

using (ii) of Theorem IV.1, and proceeding as above, we get the spectra of $P_n\overline{((G[H_1])_{n(i)})}$ as

$$\begin{cases}-1-\lambda_i \text{ for } i=1,2,\dots,m-1 & n \text{ times}\\ \mu_t\pm\nu_t, \quad 0\le t\le n-1 & \text{once,}\end{cases}$$

where, $-1-\lambda_i$ are 1-partition eigenvalues of $\overline{(H)_{1(i)}}$,

$$\mu_t = \frac{-(3r_2+2(1-m)-\alpha_t(m+1))}{2} \text{ and}$$

$$\nu_t = \frac{\sqrt{\begin{array}{c}(3r_2+2(1-m)-\alpha_t(m+1))^2\\ -4[\alpha_t(4m-3r_2-2)-m]\end{array}}}{2}.$$

Hence,

$$E_{P_n}\overline{((G[H_1])_{n(i)})} = nE_{P_1}(\overline{(H)_{1(i)}}) - n|2m-3r_2-2| + \sum_{t=0}^{n-1}[|\mu_t+\nu_t|+|\mu_t-\nu_t|]. \quad \square$$

Note. If we choose $H = C_m$ then $H_1 = W_{1,m}$ which is a wheel graph. So by putting $r_2 = 2$ in the above Theorem IV.2, we can obtain n-partition energy $G[W_{1,m}]$ and its generalized complements.

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