Solutions of Higher Order Difference Equations Involving Discrete Exponential Function

Iruthayaraj S, John Borg S, Britto Antony Xavier G, Geethalakshmi S, Saraswathi D and Divya Bharathi S

***Abstract*—The main focus of this paper is to develop the solutions for the higher order difference equations with factorials and discrete exponential functions. Using these concept, we get the unique solution for the trigonometric exponential function for the initial valued problem. These results are verified using the numerical calculations.**

***Index Terms*—Difference equation, exponential function, trignometric exponential function, characteristic equations.**

I. Introduction

HIGHER-order difference equations play a vital role in discrete modeling, capturing the dynamics of systems that progress in distinct steps. Incorporating discrete exponential functions into these equations provides an effective framework for representing exponential growth and decay in such systems. The author [1] studies the finite-time stability of fractional delay difference equations with a discrete Mittag-Leffler kernel. First, a new generalized Gronwall inequality is established in the sense of the Atangana-Baleanu fractional difference sum operator. Then, using this inequality and the method of steps, the author derives finite-time stability criteria and illustrates the results with examples. The author[2] develops fundamental theorems for higher-order difference equations using q, $q^{(\alpha)}$, and h symmetric difference operators. While most existing works focus on summation forms, the author emphasizes closed-form solutions, which offer improved accuracy and computational efficiency. The results are validated through examples and graphical analysis to demonstrate the stability of the proposed methods.The author [3] introduce a generalized difference operator of the n^{th} kind, $\Delta_{\ell_1,\ell_2,...,\ell_n}$, extending traditional difference operators. They derive discrete analogues of classical results like Leibnizs rule and Newtons formula. Applications are given in summation formulas, polynomial factorials, and number theory.

The author [4] extend the theory of the generalized difference operator of the n^{th} kind and apply it to number-theoretic problems, deriving new summation formulas and relationships between generalized factorials and algebraic polynomials. They establish discrete analogues of classical identities and novel results in additive number theory. Numerous examples illustrate the applicability and effectiveness of these new operators. The difference operator is defined as $\Delta y(k) = y_{a+1} - y_a$ in the theory of difference equations, where $\mathbb{N}$ denotes the set of natural numbers. The potential outcomes on the generalized difference operator Δ_ℓ were proposed by numerous writers [5], [6], [7], [9], [12], [13], [14]. Basic features of Δ_ℓ, including the product and quotient rules of Δ_ℓ, were established by the generalized difference operator Δ_ℓ. Throughout this paper, we use the notation $\mathbb{N}_a$ and $\mathbb{N}_a^b$ as

$$\mathbb{N}_a = \{a, a+\ell, a+2\ell \cdots s-\ell, s, s+\ell, s+2\ell \cdots\}$$

$$\mathbb{N}_a^b = \{a, a+\ell, a+2\ell \cdots s-\ell, s, s+\ell, s+2\ell \cdots b\}$$

where $a, b \in \mathbb{R}$ and $b - a \in \mathbb{N}$.

The generalized Leibnitz theorem and Binomial theorem obtained by the generalized difference operators of the second, third, and n^{th} kinds, are represented as $\Delta_{\ell,m}$, $\Delta_{\ell_1,\ell_2,\ell_3}$, and $\Delta_{\ell_1,\ell_2,\ell_3,\cdots,\ell_n}$, respectively. Newton's formula, the inverse of the generalized operator $\Delta_{\ell_1,\ell_2,\ell_3,\cdots,\ell_n}$, and Stirling numbers of second kind are used to determine the sum of the general partial sums of the n^{th} powers of an arithmetic and arithmetical-geometric progressions, respectively. In addition to the delta operators, the authors in [8], [10], [11] developed certain difference operator such as alpha-delta operator. This paper focuses on constructing and solving higher-order difference equations involving discrete exponential functions, offering analytical tools and identities that enhance the understanding of discrete dynamical behaviors. We constructed some identities with delta trigonometric exponential functions in this research and used the higher order difference equations to solve the exponential functions.

Manuscript received May 13, 2025; revised July 11, 2025.

This work was supported in part by the Carreno Grant Fellowship (SHC/Fr.Carreno Research Grant/2023/03), Sacred Heart Fellowship (SHC/SHFellowship/2024/19), Don Bosco Research Grant (SHC/DB Grant/2024/03) and DST for the FIST Fund (SR/FST/College-2017/130(c)).

Iruthayaraj S is a PhD candidate at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India (e-mail: iruthayarajsh@gmail.com).

John Borg S is an Assistant Professor at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India (e-mail: sjborg@gmail.com).

Britto Antony Xavier G is an Associate Professor at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India (e-mail : brittoshc@gmail.com).

Geethalakshmi S is a PhD candidate at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India (e-mail: geethathiru126@gmail.com).

Saraswathi D is a PhD candidate at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India (e-mail: dswathisaranya@gmail.com).

Divya Bharathi S is a PhD candidate at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India (e-mail: divyabharathitacw@gmail.com).

II. Preliminaries

The fundamental definitions of the delta operator, falling factorial functions, sine and their cosine functions, and more are developed in this section.

Definition II.1. Assume $f : \mathbb{N}_a^b \to \mathbb{R}$ and if $\ell \in \mathbb{R}$ and $b > a$, the forward difference operator Δ_ℓ is then defined by

$$\Delta_\ell f(w) = f(w+\ell) - f(w) \text{ for } w \in \mathbb{N}_a^{b-\ell}. \tag{1}$$

Definition II.2. The falling polynomial factorial function for any positive integer 'n' can be defined as

$$w_\ell^{(n)} = w(w-\ell)(w-2\ell)\dots(w-n+\ell), \tag{2}$$

where $\ell \in \mathbb{R}$ and $w_\ell^{(0)} = 1$.

Lemma II.3. *For any real number ℓ and positive integer n, then we have $\Delta_\ell w_\ell^{(n)} = n\ell w_\ell^{(n-1)}$*

Proof: $\Delta_\ell w_\ell^{(n)} = (w+\ell)_\ell^{(n)} - w_\ell^{(n)}, \quad n = 1,2,3\cdots$ *Using the Definitions II.1 and II.2, we get the result.* ∎

Lemma II.4. *Let $f, g : \mathbb{N}_a^b \to \mathbb{R}$ and $\ell \in \mathbb{R}$. Then,*

$$\Delta_\ell\{f(w)g(w)\} = f(w)\Delta_\ell g(w) + g(w+\ell)\Delta_\ell f(w). \tag{3}$$

Proof: The proof completes by taking $f(w) = f(w)g(w)$ in (1) and then adding and subtracting the term $f(w)g(w+\ell)$. ∎

Theorem II.5. *Let $\ell, n > 0 \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then*

$$\Delta_\ell(w+\alpha)_\ell^{(n)} = n\ell(w+\alpha)_\ell^{(n-1)}, \tag{4}$$

$$\Delta_\ell(\alpha-w)_\ell^{(n)} = n\ell(\alpha-w)_\ell^{(n-1)} \tag{5}$$

Proof: Using Lemma II.3 we have

$$\begin{aligned}\Delta_\ell(w+\alpha)_\ell^{(n)} &= (w+\alpha+\ell)_\ell^{(n)} - (w+\alpha)_\ell^{(n)}\\ &= (w+\alpha+\ell)(w+\alpha)\cdots(w+\alpha-(n-2)\ell)\\ &\quad - [(w+\alpha)(w+\alpha-\ell)\cdots(w+\alpha-(n-1)\ell)]\\ &= n\ell(w+\alpha)_\ell^{(n-1)}\end{aligned}$$

$$\begin{aligned}\Delta_\ell(\alpha-w)_\ell^{(n)} &= (\alpha-w+\ell)_\ell^{(n)} - (\alpha-w)_\ell^{(n)}\\ &= (\alpha-w+\ell)(\alpha-w)\cdots(\alpha-w-(n-2)\ell)\\ &\quad - [(\alpha-w)(\alpha-w-\ell)\cdots(\alpha-w-(n-1)\ell)]\\ &= n\ell(\alpha-w)_\ell^{(n-1)}\end{aligned}$$

∎

Corollary II.6. *Let $\ell, n > 0 \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then*

$$\Delta_\ell^{-1}(w+\alpha)_\ell^{(n-1)} = \frac{(w+\alpha)_\ell^{(n)}}{n\ell} \tag{6}$$

$$\Delta_\ell^{-1}(\alpha-w)_\ell^{(n-1)} = \frac{(\alpha-w)_\ell^{(n)}}{n\ell} \tag{7}$$

Proof: By using Theorem II.5 we find (6),(7). ∎

III. Delta Exponential Function

In discrete calculus on $\mathbb{N}_a$, where the exponential function e^{jw}, $j \in \mathbb{R}$, where j is constant, and $x(w) = e^{jw}$ is the unique solution to the IVP, the Δ_ℓ exponential function is essential.

$$x' = jx,\ x(0) = 1.$$

The regressive function is,
$\Re = \{j : \mathbb{N}_a \to \mathbb{R}$ such that $1 + j(w) \neq 0$ for $w \in \mathbb{N}_a\}$.
The Δ_ℓ exponential function $j \in \mathbb{R}$, based at $s \in \mathbb{N}_a$, to be the solution exists $e_j(w,s)$, of the IVP

$$\Delta_\ell x(w) = j(w)x(w) \tag{8}$$

$$x(s) = 1 \tag{9}$$

Theorem III.1. *If $j \in \Re$ and $y(w) = ce_j(w,a)$, then its solution can be generalized by*

$$\Delta_\ell y(w) = j(w)y(w) \tag{10}$$

for $w \in \mathbb{N}_a$, where c is arbitrary constant.

Proof: Let $y(w) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+j(s+\tau\ell))$ *where* $w \in \mathbb{N}_s$, *then by Definition II.1, we have*

$$\Delta_\ell y(w) = \prod_{\tau=0}^{\frac{w+\ell-s}{\ell}-1}(1+j(s+\tau\ell)) - \prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+j(s+\tau\ell))$$

Using the Definition II.2, we get

$$\Delta_\ell y(w) = (1+j(s))(1+j(s+\ell))(1+j(s+2\ell))\cdots\left(1+j(s+\left(\tfrac{w-s}{\ell}-1\right)\ell)\right)$$
$$\left(1+j(s+\left(\tfrac{w+\ell-s}{\ell}-1\right)\ell)\right) - \prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+j(s+\tau\ell))$$

$$\Delta_\ell y(w) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+j(s+\tau\ell))\left(1+j(s+\left(\tfrac{w+\ell-s}{\ell}-1\right)\ell)\right)$$
$$- \prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+j(s+\tau\ell))$$

$$\Delta_\ell y(w) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+j(s+\tau\ell))\left(j(s+\left(\tfrac{w+\ell-s}{\ell}-1\right)\ell)-1\right)$$

which implies

$$\Delta_\ell y(w) = y(w)j(w).$$

For $w \in \mathbb{N}_a$, $y(w)$ is a general solution of equation (10).

Let $y(w) = \prod_{\tau=0}^{\frac{s-w}{\ell}-1}(1+j(w+\tau\ell))^{-1}$ *where* $w \in \mathbb{N}_a^{\frac{s-\ell-a}{\ell}}$

$$y(w) = (1+j(w))^{-1}(1+j(w+\ell))^{-1}\cdots(1+j(w+(\tfrac{s-w}{\ell}-1)\ell))^{-1}$$

Now, replacing w by $(w+\ell)$, we get the above equation as the form of

$$y(w+\ell) = \prod_{\tau=0}^{\frac{s-(w+\ell)}{\ell}-1}(1+j(w+\ell+\tau\ell))^{-1}$$

$$y(w+\ell) = (1+j(w+\ell))^{-1}(1+j(w+\ell+\ell))^{-1}\cdots$$
$$\left(1+j(w+\ell+\left(\tfrac{s-w-\ell}{\ell}-1\right)\ell)\right)^{-1}$$

Now, multiplying and dividing the term $(1+j(w))$, we obtain

$$y(w+\ell) = (1+j(w))(1+j(w))^{-1}(1+j(w+\ell))^{-1}\cdots$$
$$\left(1+j(w+\left(\tfrac{s-w}{\ell}-1\right)\ell)\right)^{-1}$$

which can be written in the form of
$y(w+\ell) = (1+j(w))y(w) = y(w) + j(w)y(w)$.
$\Delta_\ell y(w) = y(w+\ell) - y(w) = j(w)y(w)$. *Hence $y(w)$ is general solution of equation* (10) *for* $w \in \mathbb{N}_a^{\left(\frac{s-\ell-a}{\ell}\right)}$. ∎

Theorem III.2. *For $\oplus$ on $\Re$ by $j \oplus k := j + k + jk$, then $(\Re, \oplus)$ is an abelian group.*

Proof: We claim that $(\Re, \oplus)$ is an abelian group.

i) *Closure: If $j, k \in \Re$, then $1 + j(w) \neq 0$ and $1 + k(w) \neq 0$ for $w \in \mathbb{N}_a$, follows that*
$1 + (j \oplus k)(w) = 1 + [j(w) + k(w) + j(w)k(w)]$
$= [1 + j(w)][1 + k(w)] \neq 0$ *for* $w \in \mathbb{N}_a$.
Hence, $j \oplus k \in \Re$.
ii) *Associative: If $j, k, t \in \Re$, then* $(j \oplus k) \oplus t$
$= j + k + jk + t + jt + kt + jkt$
$= j \oplus (k \oplus t)$. *Hence,* $j \oplus (k \oplus t) \in \Re$.
iii) *Identity: Here the $0 \in \Re$ as $1 + 0 = 1 \neq 0$.*
Also, $0 \oplus j = 0 + j + 0.j = j$
for all $j \in \Re$. Hence, $0 \oplus j \in \Re$.
iv) *Inverse: If $j(w), k(w) \in \Re$, then* $1 + k(w)$
$= 1 + \dfrac{-j(w)}{1+j(w)} = \dfrac{1}{1+j(w)} \neq 0$, *for* $w \in \mathbb{N}_a$.
So, $j \oplus k = j \oplus \dfrac{-j}{1+j} = j + \dfrac{-j}{1+j} + \dfrac{-j^2}{1+j} = 0$
k is the additive inverse of j.

v) *Commutative: If* $j, k \in \Re$, *then* $j \oplus k = j + k + jk$
$= k \oplus j \in \Re$.

Hence, $(\Re, \oplus)$ *is abelian group on* $\Re$. ■

Theorem III.3. *If* $j, k \in \Re$ *and* $e_j(w, a) = e_k(w, a)$, *then* $j = k$ *for* $w \in \mathbb{N}_a$.

Proof: Let $j, k \in \Re$ *and* $j(w)e_j(w, a) = k(w)e_k(w, a)$ *where* $w \in \mathbb{N}_a$.
Hence the proof completes by $e_j(w, a) = e_k(w, a)$. ■

Definition III.4. The definition of the circle minus scalar multiplication $\ominus$ on $\Re^+$ is

$$\ominus j = \frac{-j}{1 + j}. \tag{11}$$

Definition III.5. The definition of the circle dot scalar multiplication $\odot$ on $\Re^+$ is

$$\alpha \odot j = (1 + j)^\alpha - 1. \tag{12}$$

Theorem III.6. $(\Re^+, \oplus, .)$ *is a vector space, as demonstrated by the set of positively regressive functions* $\Re^+$ *with* $\oplus$ *and scalar multiplication* $\odot$.

Proof: To prove $(\Re^+, \oplus)$ *is Abelian.*

i) *Closure: If* $j, k \in \Re^+$, *then* $j \oplus k = j + k + jk$
$\Rightarrow j \oplus k \in \Re^+$.

ii) *Associative: If* $j, k, t \in \Re^+$, *then* $(j \oplus k) \oplus t$
$= j \oplus k + t + (j \oplus k)t$
$= j + k + t + jk + (j + k + jk)t = j \oplus (k \oplus t)$.
This implies $(j \oplus k) \oplus t \in \Re^+$.

iii) *Identity: The zero function* $0 \in \Re^+$ *as* $1 + 0(t) > 0$ *for* $t \in N_a$, *then* $0 \oplus j$.
$= 0 + j + 0 = j \in \Re^+$.

iv) *Inverse: Let* $\ominus j$ *be the additive inverse of* j. *Therefore,*
$\ominus j = \dfrac{-j}{1 + j} \in \Re^+$ *and* $j \oplus [\ominus j] = j + \dfrac{-j}{1 + j} + \dfrac{-j^2}{1 + j}$
$= j + \dfrac{-j}{1 + j}[1 + j] = 0.$

v) *Commutative: If* $j, k \in \Re^+$, *then* $j \oplus k = j + k + jk$
$= k \oplus j \in \Re^+$.

Hence $(\Re^+, \oplus)$ *is an abelian group.*
Now, claim that $(\Re^+, \oplus, .)$ *is a vector space.*

i) *Let* $\alpha, \beta \in \mathbb{R}$ *and* $j \in \Re^+$, *then* $e_{(\alpha+\beta)\odot j}(w, a)$
$= e_j^{\alpha+\beta}(w, a) = e_j^{\alpha}(w, a)e_j^{\beta}(w, a)$
$= e_{\alpha\odot j}(w, a)e_{\beta\odot j}(w, a) = e_{(\alpha\odot j)\oplus(\beta\odot j)}(w, a)$.
Thus,
$$(\alpha + \beta) \odot j = (\alpha \odot j) \oplus (\beta \odot j)$$

ii) *Let* $\alpha \in \mathbb{R}$ *and* $j, k \in \Re^+$, *then* $e_{\alpha\odot(j\oplus k)}(w, a)$
$= e_{j\oplus k}^{\alpha}(w, a) = e_j^{\alpha}(w, a)e_k(w, a)$
$= e_{\alpha\odot j}(w, a)e_{\alpha\odot k}(w, a) = e_{\alpha\odot\oplus\alpha\odot k}(w, a)$.
Thus,
$$\alpha \odot (j \oplus k) = \alpha \odot j \oplus \alpha \odot k$$
Hence $(\Re^+, \oplus, .)$ *is distributive.*

iii) *Let* $e_{1\odot j}(w, a) = e_j^1(w, a)$, *then* $1 \odot j = j \in \Re^+$.

iv) *Here,* $e_{\alpha\odot(\beta\odot j)}(w, a) = e_{\beta\odot j}^{\alpha}(w, a) = e_{\alpha\beta\odot j}(w, a)$.
This gives
$$\alpha \odot (\beta \odot j) = \alpha\beta \odot j.$$

Hence $(\Re^+, \oplus, .)$ *is a vector space.* ■

Theorem III.7. *If* $j, k \in \Re$ *and* $t, s, w \in \mathbb{N}_a$, *then*

i) $e_0(w, s) = 1$ *and* $e_j(w, w) = 1$

ii) $e_j(w, s) \neq 0$ *for* $w \in \mathbb{N}_a$

iii) *if* $1 + j > 0$, *then* $e_j(w, s) > 0$

iv) $e_j^{\sigma}(w, s) = e_j(\sigma(w), s)$
$$= \left(1 + j(s + \left(\frac{\sigma(w) - s}{\ell} - 1\right)\ell)\right) e_j(w, s)$$

v) $e_j(w, s)e_j(s, t) = e_j(w, t)$

vi) $e_j(w, s)e_k(w, s) = e_{j\oplus k}(w, s)$

vii) $e_{\ominus j}(w, s) = \dfrac{1}{e_j(w, s)}$

viii) $\dfrac{e_j(w, s)}{e_k(w, s)} = e_{j\ominus k}(w, s)$

ix) $e_j(w, s) = \dfrac{1}{e_j(s, w)}$

Proof: Let $e_j(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1+j(s+\tau\ell))$ *for* $w \in \mathbb{N}_a$.

i) *The proof of (i) completes by taking* $j = 0$ *and* $s = w$.

ii) *Since* $1 + j(w) \neq 0$, *so that*
$e_j(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell)) \neq 0$ *where* $w \in \mathbb{N}_a$.

iii) *Since* $1 + j(w) > 0$, *so that*
$e_j(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell)) > 0$ *where* $w \in \mathbb{N}_a$.

iv) *Let* $e_j(\sigma(w), s) = \prod_{\tau=0}^{\frac{\sigma(w)-s}{\ell}-1} (1 + j(s + \tau\ell)), w \in \mathbb{N}_a$
$= (1+j(s))(1+j(s+\ell)) \ldots \left(1 + j(s + \left(\frac{\sigma(w)-s}{\ell} - 1\right)\ell)\right)$
$= \left(1 + j(s + \left(\frac{\sigma(w)-s}{\ell} - 1\right)\ell)\right) \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell))$
$e_j(\sigma(w), s) = \left(1 + j(s + \left(\frac{\sigma(w)-s}{\ell} - 1\right)\ell)\right) e_j(w, s)$

v) *Holds only* $t \leq s \leq w$.
Let $e_j(w, s)e_j(s, t) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell)) \prod_{\tau=0}^{\frac{s-t}{\ell}-1} (1 + j(t + \tau\ell))$
Since $e_j(w, s)$ *and* $e_j(s, t)$ *is commutative, the above equation can be written as*
$e_j(w, s)e_j(s, t) = e_j(s, t)e_j(w, s)$
$= \prod_{\tau=0}^{\frac{s-t}{\ell}-1} (1 + j(t + \tau\ell)) \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell))$
$= e_j(w, t)$.

vi) *Let* $e_j(w, s)e_k(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1+j(s+\tau\ell)) \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1+k(s+\tau\ell))$

$e_j(w, s)e_k(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell))(1 + k(s + \tau\ell))$

$e_j(w, s)e_k(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell) + k(s + \tau\ell) + j(s + \tau\ell)k(s + \tau\ell))$

By Theorem III.2, we obtain
$$e_j(w, s)e_k(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + j(s + \tau\ell) \oplus k(s + \tau\ell))$$
$$= e_{j\oplus k}(w, s).$$

vii) *Let* $e_{\ominus}(w, s) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1} (1 + \ominus j(s + \tau\ell))$
$$= \prod_{\tau=0}^{\frac{w-s}{\ell}-1} 1 - \frac{j(s + \tau\ell)}{1 + j(s + \tau\ell)}$$
$$e_{\ominus}(w, s) = \frac{1}{\prod_{\tau=0}^{\frac{w-s}{\ell}-1} \frac{1}{1 + j(s + \tau\ell)}}$$

$$= \frac{1}{\prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+j(s+\tau\ell))} = \frac{1}{e_j(w,s)}.$$

viii) *Let* $\frac{e_j(w,s)}{e_k(w,s)} = e_j(w,s)\left[\frac{1}{e_k(w,s)}\right]$. *By (vii), we have* $\frac{e_j(w,s)}{e_k(w,s)} = e_j(w,s)e_{\ominus j}(w,s) = e_{j\oplus[\ominus k]}(w,s)$
$= e_{j\ominus k}(w,s)$.

ix) *From (v), we can easily find* $e_j(w,s)e_j(s,w) = e_j(w,w) = 1$.
Therefore $e_j(w,s) = \frac{1}{e_j(s,w)}$.

■

Definition III.8. The hyperbolic sine and cosine functions are defined as follows, assuming $\pm p \in \Re$.

$$cosh_j(w,a) = \frac{e_j(w,a)+e_{-j}(w,a)}{2},$$
$$sinh_j(w,a) = \frac{e_j(w,a)-e_{-j}(w,a)}{2} \text{ for } w \in \mathbb{N}_a.$$

Theorem III.9. *Let* $\pm j \in \Re$ *and* $w \in \mathbb{N}_a$. *Next, the hyperbolic sine and cosine functions for delta are provided by*

$$\Delta_\ell cosh_j(w,a) = j(w)sinh_j(w,a) \quad (13)$$

$$\Delta_\ell sinh_j(w,a) = j(w)cosh_j(w,a). \quad (14)$$

Proof: The proof completes by using the Definition II.1 and Theorem III.1 by taking $y(w) = sinh_j(w,a)$ *and* $y(w) = cosh_j(w,a)$. ■

Definition III.10. Suppose that $\pm ij \in \Re$. The sine and cosine functions are defined as follows:

$$cos_j(w,a) = \frac{e_{ij}(w,a)+e_{-ij}(w,a)}{2},$$
$$sin_j(w,a) = \frac{e_{ij}(w,a)-e_{-ij}(w,a)}{2i} \text{ for } w \in \mathbb{N}_a.$$

Theorem III.11. *Let* $\pm ij \in \Re$ *and* $w \in \mathbb{N}_a$. *Then,*

$$\Delta_\ell cos_j(w,a) = -j(w)sin_j(w,a) \quad (15)$$

$$\Delta_\ell sin_j(w,a) = j(w)cos_j(w,a) \quad (16)$$

Proof: The proof is similar to Theorem III.9 by taking $y(w) = sin_j(w,a)$ *and* $y(w) = cos_j(w,a)$. ■

Theorem III.12. *Assume* $\pm j \in \Re$ *and* $w \in \mathbb{N}_a$ *then*

i) $cosh_j(a,a) = 1$ *and* $sinh_j(a,a) = 0$.
ii) $cosh_j^2(w,a) - sinh_j^2(w,a) = e_{-j^2}(w,a)$.
iii) $\Delta_\ell cosh_j(w,a) = j(w)sinh_j(w,a)$.
iv) $\Delta_\ell sinh_j(w,a) = j(w)cosh_j(w,a)$.
v) $cosh_{-j}(w,a) = cosh_j(w,a)$.
vi) $sinh_{-j}(w,a) = -sinh_j(w,a)$.
vii) $e_j(w,a) = cosh_j(w,a) + sinh_j(w,a)$.

Proof: Let $\pm j \in \Re$ *and by Definition III.8, we have*

i) $cosh_j(a,a) = \frac{e_j(a,a)+e_{-j}(a,a)}{2} = 1$,
$sinh_j(a,a) = \frac{e_j(a,a)-e_{-j}(a,a)}{2} = 0$.

ii) $cosh_j^2(w,a) - sinh_j^2(w,a)$
$= \left[\frac{e_j(w,a)+e_{-j}(a,a)}{2}\right]^2 - \left[\frac{e_j(w,a)-e_{-j}(a,a)}{2}\right]^2$,
which implies that
$\frac{1}{4}[4e_j(w,a)e_{-j}(w,a)] = e_{j\oplus[-j]}(w,a) = e_{-j^2}(w,a)$.

iii) $\Delta_\ell cosh_j(w,a) = \frac{\Delta_\ell e_j(w,a)+\Delta_\ell e_{-j}(w,a)}{2}$. *Then by Theorem III.1, we obtain* $\Delta_\ell cosh_j(w,a)$
$= \frac{j(w)e_j(w,a)-j(w)e_{-j}(w,a)}{2} = j(w)sinh_j(w,a)$.

iv) *The proof is similar to (iii).*

v) $cosh_{-j}(w,a) = \frac{e_{-j}(w,a)+e_j(w,a)}{2} = cosh_j(w,a)$.

vi) $sinh_{-j}(w,a) = -\frac{e_j(w,a)-e_{-j}(w,a)}{2}$
$= -sinh_j(w,a)$.

vii) $cosh_j(w,a) + sinh_j(w,a)$
$= \frac{e_j(w,a)+e_{-j}(w,a)+e_j(w,a)-e_{-j}(w,a)}{2}$
$= e_j(w,a)$.

■

Theorem III.13. *Assume* $\pm ij \in \Re$ *and* $w \in \mathbb{N}_a$ *then*

i) $cos_j(a,a) = 1$ *and* $sin_j(a,a) = 0$.
ii) $cos_j^2(w,a) + sin_j^2(w,a) = e_{j^2}(w,a)$.
iii) $\Delta_\ell cos_j(w,a) = -j(w)sin_j(w,a)$.
iv) $\Delta_\ell sin_j(w,a) = j(w)cos_j(w,a)$.
v) $cos_{-j}(w,a) = cos_j(w,a)$.
vi) $sin_{-j}(w,a) = -sin_j(w,a)$.
vii) $e_{ij}(w,a) = cos_j(w,a) + isin_j(w,a)$.

Proof: Then the proof is similar to Theorem III.12 using the Definition III.10. ■

Theorem III.14. *Assume* $\pm j \in \Re$ *for* $w \in \mathbb{N}_a$, *then*

i) $sin_{ij}(w,a) = isinh_j(w,a)$.
ii) $cos_{ij}(w,a) = cosh_j(w,a)$.
iii) $sinh_{ij}(w,a) = isin_j(w,a)$.
iv) $cosh_{ij}(w,a) = cos_j(w,a)$.

Proof: Using the Definition III.8 and Definition III.10, we get

i) $sin_{ij}(w,a) = \frac{e_{-j}(w,a)-e_j(w,a)}{2i} = isinh_j(w,a)$.
ii) $cos_{ij}(w,a) = \frac{e_j(w,a)+e_{-j}(w,a)}{2} = cosh_j(w,a)$.
iii) $sinh_{ij}(w,a) = \frac{i(e_{ij}(w,a)-e_{-ij}(w,a))}{2i}$
$= isin_j(w,a)$.
iv) $cosh_{ij}(w,a) = \frac{e_{ij}(w,a)+e_{-ij}(w,a)}{2} = cos_j(w,a)$.

■

A. Higher Order Difference equations

The non homogeneous higher order difference equations is defined by

$$\Delta_\ell^2 y(w) + j(w)\Delta_\ell y(w) + k(w)y(w) = f(w), w \in \mathbb{N}_a \quad (17)$$

We assume that $j(w) \neq k(w)+1$ for $w \in \mathbb{N}_a$.

$$\Delta_\ell^2 y(w) + j\Delta_\ell y(w) + ky(w) = 0, w \in \mathbb{N}_a \quad (18)$$

where we assume the constants $j,k \in \mathbb{R}$ satisfy $j \neq 1+k$.

Theorem III.15. *Assume that* $j,k,f : \mathbb{N}_a \to \mathbb{R}$, $j(w) \neq 1+k(w)$ *and* $A,B \in \mathbb{R}$, *then the solution of the IVP will be of the form*
$\Delta_\ell^2 y(w) + j(w)\Delta_\ell y(w) + k(w)y(w) = f(w)$,

$$y(w_0) = A, y(w_0+\ell) = B \quad (19)$$

has a unique solution $y(w)$ *on* $w \in \mathbb{N}_a$.

Proof: *From Definition II.1, we find*
$\Delta_\ell y(w) = y(w+\ell) - y(w)$ *and also*
$\Delta_\ell^2 y(w) = y(w+2\ell) - 2y(w+\ell) + y(w)$. *Substituting the* $\Delta_\ell y(w)$ *and* $\Delta_\ell^2 y(w)$ *values in equation* (17), *we obtain*

$$y(w+2\ell) = [2-j(w)]y(w+\ell) - [1-j(w)+k(w)]y(w) + f(w) \quad (20)$$

and $j(w) \neq 1 + k(w)$, $w \in \mathbb{N}_a$.
Now, equation (20) *can be rewritten in the form of*

$$y(w) = \frac{2-j(w)}{1-j(w)+k(w)}y(w+\ell) - \frac{1}{1-j(w)+k(w)}y(w+2\ell) - \frac{f(w)}{1-j(w)+k(w)}. \quad (21)$$

If we take $w = w_0$ *in* (20), *then equation* (17) *holds at* $w = w_0$ *if and only if*

$$y(w_0+2\ell) = [2-j(w_0)]B - [1-j(w_0)+k(w_0)]A + f(w_0).$$

Thus the solution is reached at $w + 2\ell$. *However, using* (20) *evaluated at* $w = w_0 + \ell$, *we can see that* $y(w_0+\ell)$ *and* $y(w_0+2\ell)$ *determines the values of* $w_0 + 3\ell$. *We can determine that IVP* (19) *is unique on* $\mathbb{N}_{w_0}$. *In contrast, if* $w_0 > a$, *then* $w = w_0 - \ell$ *with* (21) *is used. Therefore*

$$y(w_0+\ell) = \frac{1}{1-j(w_0-\ell)+k(w_0-\ell)}[(2-j(w_0-\ell))A - B - f(w_0-\ell)]$$

The IVP (19) *solution is thus uniquely found at* $w_0 - \ell$. *Proceeding in this manner* (19) *is uniquely determined on* $\mathbb{N}_a^{w_0}$. ■

Theorem III.16. *Let* $j \neq 1+k$. *If the characteristic equation of* (18) *is* $\lambda^2 + j\lambda + k = 0$ *and* $\lambda_1 \neq \lambda_2$ *are distinct solutions of* (18), *then the general solution is*

$$y(w) = c_1 e_{\lambda_1}(w,a) + c_2 e_{\lambda_2}(w,a) \quad (22)$$

Proof: *Let us assume that*
$\Delta_\ell^2 y(w) + j\Delta_\ell y(w) + ky(w) = 0$ *and* $w \in \mathbb{N}_a$.
Taking $y(w) = \prod_{\tau=0}^{\frac{w-s}{\ell}-1}(1+\lambda(\tau))$ *where* $\lambda(\tau), w \in \mathbb{N}_a$, *then*

$$y(w) = (1+\lambda)^{\left(\frac{w-s}{\ell}\right)}. \quad (23)$$

Both sides of the equation (23) *can be solved by applying the* Δ_ℓ *operator, yielding*

$$\Delta_\ell y(w) = (1+\lambda)^{\left(\frac{w+\ell-s}{\ell}\right)} - (1+\lambda)^{\left(\frac{w-s}{\ell}\right)} = (1+\lambda)^{\left(\frac{w-s}{\ell}\right)}(\lambda). \quad (24)$$

Returning to the equation (24) *and using the* Δ_ℓ *operator on both sides, we obtain*

$$\Delta_\ell^2 y(w) = (1+\lambda)^{\left(\frac{w+\ell-s}{\ell}\right)}(\lambda) - (1+\lambda)^{\left(\frac{w-s}{\ell}\right)}(\lambda) = (1+\lambda)^{\left(\frac{w-s}{\ell}\right)}(\lambda^2). \quad (25)$$

From equation (24) *and equation* (25), *we obtain*

$$\lambda^2 + j\lambda + k = 0. \quad (26)$$

Taking $\lambda = m$ *in equation* (26), *we get*

$$m^2 + jm + k = 0. \quad (27)$$

If m_1, m_2 *are roots of* (27), *then* $\lambda_1 = m_1$ *and* $\lambda_2 = m_2$ *are the roots of* (26).

Hence, $y(w) = c_1 e_{\lambda_1}(w,a) + c_2 e_{\lambda_2}(w,a)$ *is the general solution of* (18). ■

Theorem III.17. *In the event when the characteristic values are* $\lambda = \alpha \pm i\beta$, $\beta > 0$ *and* $\alpha \neq -1$, *then equation* (18)*'s general solution is provided by*

$$y(w) = c_1 e_\alpha(w,a) cos_\gamma(w,a) + c_2 e_\alpha(w,a) sin_\gamma(w,a), \quad (28)$$

where $\gamma = \dfrac{\beta}{1+\alpha}$.

Proof: *Taking* $j = -2\alpha$, $k = \alpha^2 + \beta^2$ *and* $1 + k - j \neq 0$ *in equation* (18), *we get*

$$\Delta_\ell^2 y(w) - 2\alpha\Delta_\ell y(w) + (\alpha^2+\beta^2)y(w) = 0. \quad (29)$$

The characteristic equation of (29) *is*

$$\lambda^2 - 2\alpha\lambda + (\alpha^2+\beta^2) = 0. \quad (30)$$

The roots of the equation (30) *is* $\lambda = \alpha \pm i\beta$.
Let $y(w) = e_{\alpha+i\beta}(w,a)$ *is complex valued solution using* $\alpha + i\beta = \alpha \oplus i\dfrac{\beta}{1+\alpha} = \alpha + i\gamma$, *we get* $y(w) = e_{\alpha+i\beta}(w,a)$
$= e_{\alpha\oplus i\gamma}(w,a) = e_\alpha(w,a)e_{i\gamma}(w,a)$.
By Euler's formula, $y(w) = e_\alpha(w,a)e_{i\gamma}(w,a)$
$= e_\alpha(w,a)[cos_\gamma(w,a) + isin_\gamma(w,a)]$. *Thus*

$$y(w) = y_1(w) + iy_2(w).$$

To prove $y_1(w) = e_\alpha(w,a)cos_\gamma(w,a)$ *is a general solution of equation* (29), *so that*

$$y_1(w) = e_\alpha(w,a)\left[\frac{e_{i\gamma}(w,a) + e_{-i\gamma}(w,a)}{2}\right] = \frac{e_{\alpha\oplus i\gamma}(w,a) + e_{\alpha\oplus -i\gamma}(w,a)}{2}.$$

$$\Delta_\ell y_1(w) = \frac{(\alpha+i\beta)e_{\alpha+i\beta}(w,a) + (\alpha-i\beta)e_{\alpha-i\beta}(w,a)}{2} \quad (31)$$

$$\Delta_\ell^2 y_1(w) = \frac{(\alpha+i\beta)^2 e_{\alpha+i\beta}(w,a) + (\alpha-i\beta)^2 e_{\alpha-i\beta}(w,a)}{2} \quad (32)$$

$$[\Delta_\ell^2 - 2\alpha\Delta_\ell + (\alpha^2+\beta^2)cos_\gamma(w,a)]e_\alpha(w,a) = e_{i\beta}(w,a)\frac{1}{2}[(\alpha+i\beta)^2 - 2\alpha(\alpha+i\beta) + \alpha^2+\beta^2] + e_{-i\beta}(w,a)\frac{1}{2}[(\alpha-i\beta)^2 - 2\alpha(\alpha-i\beta) + \alpha^2+\beta^2],$$

which gives

$$e_{i\beta}[(\alpha+i\beta)((\alpha+i\beta)-2\alpha) + \alpha^2+\beta^2] + e_{-i\beta}[(\alpha-i\beta)((\alpha-i\beta)-2\alpha) + \alpha^2+\beta^2] = 0.$$

So, $y_1(w)$ *is a solution of* (29). *Similarly, we can find* $y_2(w)$ *is solution of* (29).
Hence $y(w)$ *is general solution of equation* (29). ■

Example III.18. *For* $w \in \mathbb{N}_a$, *solve the difference equation*

$$\Delta_\ell^2 y(w) - 2\Delta_\ell y(w) + 2y(w) = 0. \quad (33)$$

The characteristic equation is, $\lambda^2 - 2\lambda + 2 = 0$. *Consequently, the traits that are present are* $\lambda = 1 \pm i$. *Now, by using the Theorem* (III.17), *we get*
$y(w) = c_1 e_1(w,a)cos_{\frac{1}{2}}(w,a) + c_2 e_1(w,a)sin_{\frac{1}{2}}(w,a)$ *is a general solution of* (33) *on* $\mathbb{N}_a$.

Theorem III.19. *If* $-1 \pm i\beta$, *is the characteristic value of* (18), *and* $\beta > 0$, $w \in \mathbb{N}_a$ *is the general solution of* (18), *then*

$$y(w) = c_1\beta^{\left(\frac{w-a}{\ell}\right)}cos\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right) + c_2\beta^{\left(\frac{w-a}{\ell}\right)}sin\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right)$$

Proof: By Theorem(III.17)*, we assume* $-1 \pm i\beta$ *is a characteristic root of* (18)*.*
Let $y(w) = e_{-1+i\beta}(w,a)$ *is a complex valued solution of* (18)*, then*

$$y(w) = e_{-1+i\beta}(w,a) = (i\beta)^{w-a} = \left(e^{i\frac{\pi}{2}}\beta\right)^{w-a}$$

The above equation can be written as
$e_{-1+i\beta}(w,a) = \beta^{\left(\frac{w-a}{\ell}\right)} cos\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right) + i\beta^{\left(\frac{w-a}{\ell}\right)} sin\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right)$

It follows that, $y_1(w) = \beta^{\left(\frac{w-a}{\ell}\right)} cos\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right)$, $y_2(w) = \beta^{\left(\frac{w-a}{\ell}\right)} sin\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right)$

are the solutions of equation (18)*.*
Given that these solutions on $\mathbb{N}_a$ *are linearly independent,*

$y(w) = c_1\beta^{\left(\frac{w-a}{\ell}\right)} cos\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right) + c_2\beta^{\left(\frac{w-a}{\ell}\right)} sin\left(\frac{\pi}{2}\frac{(w-a)}{\ell}\right)$
is a general solution of (18)*.* ■

Example III.20. *For* $w \in \mathbb{N}_0$*, resolve the linear difference equation for delta,*

$$\Delta_\ell^2 y(w) + 2\Delta_\ell y(w) + 5y(w) = 0. \quad (34)$$

The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$*,*
So, $\lambda = -1 \pm 2i$*.*
$y(w) = c_1 2^w cos\left(\frac{\pi}{2}w\right) + c_2 2^w sin\left(\frac{\pi}{2}w\right)$ *is a general solution on* $\mathbb{N}_0$*.*

Theorem III.21. *Assume* $\lambda_1 = \lambda_2 = t$ *and* $j \neq 1 + k$ *is the characteristic equation* (18)*'s double root. Then, the general solution is*

$$y(w) = c_1 e_w(w,a) + c_2(w-a)e_w(w,a). \quad (35)$$

Proof: If $\lambda_1 = t$ *is the characteristic value, then the solution of* (18) *is*

$$y_1(w) = e_t(w,a). \quad (36)$$

If $\lambda_1 = \lambda_2 = t$*, then the characteristic equation for* (18) *is*

$$(\lambda - t)^2 = \lambda^2 - 2t\lambda + t^2 = 0. \quad (37)$$

To prove $y_2(w) = (w-a)e_t(w,a)$ *is solution of equation* (18)*.*
Applying the Δ_ℓ *on* $y_2(w)$*, we arrive that*
$\Delta_\ell y_2(w) = \Delta_\ell[(w-a)e_t(w,a)] = \Delta_\ell[e_t(w,a)(w-a)]$.
Now, using the uv method, one can easily find

$$\Delta_\ell y_2(w) = e_t(w,a)\Delta_\ell(w,a) + (w+\ell-a)\Delta_\ell e_t(w,a)$$
$$= \ell e_t(w,a) + (w+\ell-a)te_t(w,a). \quad (38)$$

$$\Delta_\ell^2 y_2(w) = \ell\Delta_\ell e_t(w,a) + t\Delta_\ell(w+\ell-a)e_t(w,a)$$
$$= e_t(w,a)[2t\ell + (w+2\ell-a)t^2]. \quad (39)$$

From equations (38) *and* (39)*, we have*
$[\Delta_\ell^2 - 2t\Delta_\ell + t^2]y_2(w)$
$= (2t\ell + (t+2\ell-a)t^2 - 2t(\ell+(w+\ell-a)t) + t^2(w-a)) = 0.$
So, $y_1(w)$ *and* $y_2(w)$ *are the solutions of equation* (18)*.*
Hence the proof completes. ■

IV. CONCLUSION

In this research work, we explores the solutions of higher-order difference equations involving discrete exponential functions, emphasizing their effectiveness in modeling discrete dynamical systems. By employing the delta operator, we developed new identities and applied them to trigonometric exponential functions, enriching the theoretical framework. Several illustrative examples confirm the validity and applicability of the proposed methods. Overall, this study contributes to the advancement of discrete analysis by combining delta operators with exponential functions for solving complex difference equations.

REFERENCES

[1] Chen, Yuting, Xiaoyan Li, and Song Liu. "Finite-time stability of ABC type fractional delay difference equations." Chaos, Solitons & Fractals 152 (2021): 111430.

[2] Zhao, Weidong, et al. "Symmetric difference operator in quantum calculus." Symmetry 14.7 (2022): 1317.

[3] Manuel, M. Maria Susai, G. Britto Antony Xavier, and E. Thandapani. "Theory of generalised difference operator and its applications." Far East Journal of Mathematical Sciences 20.2 (2006): 163.

[4] Pugalarasu, R., et al. "Theory of generalized difference operator of n-th kind and its applications in number theory (part II)." International Journal of Pure and Applied Mathematics 64.1 (2010): 121-132.

[5] Britto Antony Xavier. G, Sathya. S and Vasanthakumar. S. U, *m-Series Of The Generalized Difference Equation To Circular Functions* International Journal of Mathematical Archive-4(7) (2013), 200-209.

[6] Britto Antony Xavier. G, Vasanthakumar. S.U, and Govindan. B, *Discrete beta-factorial function and its series in terms of closed form solution of generalized difference equation* International journal of Mathematical Archive-4(10), Oct.-2013

[7] Ferreira. R. A. C and Torres. D. F. M, *Fractional h-difference equations arising from the calculus of variations, Applicable Analysis and Discrete Mathematics*, 5(1) (2011), 110-121.

[8] Jerzy Popenda and Blazej Szmanda, *On the Oscillation of Solutions of Certain Difference Equations*, Demonstratio Mathematica, XVII(1), (1984), 153 - 164.

[9] Maria Susai Manuel. M, B. A. Xavier. G, Chandrasekar. V and Pugalarasu. R, *Theory and application of the Generalized Difference Operator of the* n^{th} *kind(Part I)*, Demonstratio Mathematica, 45(1) (2012),95-106,2012.

[10] Maria Susai Manuel. M, Chandrasekar V and Britto Antony Xavier. G, *Solutions and Applications of Certain Class of* α*-Difference Equations*, International Journal of Applied Mathematics, 24(6) (2011), 943-954.

[11] Maria Susai Manuel. M, Chandrasekar. V and Britto Antony Xavier. G, *Theory of Generalized* α*-Difference Operator and its Applications in Number Theory*, Advances in Differential Equations and Control Processes, 9(2) (2012), 141-155.

[12] Miller. K.S and Ross. B, *Fractional Difference Calculus in Univalent Functions, Fractional Calculus and Their Applications (Koriyama, 1988)*, pp.139-152, Horwood, Chichester, UK, 1989.

[13] Maria Susai Manuel. M, Britto Antony Xavier. G and Chandrasekar. V, *Theory of Generalized Difference Operator of the Second Kind and Its Applications in Number Theory*, Far East Journal of Mathematical Sciences, 20(2) (2009), 163 - 171.

[14] Saber N. Elaydi Saber, *An Introduction to Difference Equations*, 2/e, Springer Verlag, 1999.