The Third-Order Solution of AB-Equation Using the Homotopy Perturbation Method

Mashuri, Renny, Niken Larasati, Agus Sugandha, Budi Pratikno, Jajang

***Abstract*— This paper discusses the solution of AB equation using the homotopy perturbation method. The AB equation is a wave equation that improves the KdV wave equation. Unlike the KdV wave equation whose dispersion relation is non-exact, this AB equation has an exact dispersion relation. The AB equation is solved using the homotopy perturbation method up to the third order. The generated waves are monochromatic. The results show that in the first-order solution, the relation between the frequency and the wave number, referred to as the dispersion relation was obtained. In the second-order solution, we found that the frequency of the wave was two times larger than in the first-order solution. Nevertheless, it had less amplitude than in the first-order solution. Finally, the homotopy perturbation method also provided a good solution because of the absence of the need to specify the embedding parameters used.**

Keywords: AB equation, KdV equation, Homotopy Perturbation method, monochromatic wave.

I. INTRODUCTION

It is always interesting to study water wave equation. This is because not only is the earth covered by a large part of sea area, but the equations that form sea water waves are also increasingly varied. An example of this is how the KdV wave equation keeps on being improved. Originally, it is a wave equation for waves with a long enough wavelength and a low enough depth proposed for the first time by Korteweigh and De Vries in 1895[1]. Since some of the solutions of this equation, namely the soliton, are so interesting, this equation has always been improved to produce equations that are more realistic and consistent with natural phenomena. Therefore, this equation has undergone several modifications such as those made by Groesen [2]. Furthermore, the equation was also improved by Groesen and Andonowati. The improvements were made by modifying the dispersive relation in the equation in such a way that the dispersive relation becomes exact and by using pseudo-differential operators in the equation so as to provide a much better description, especially for infinitesimal waves [3]. The latest equation proposed by Grosen and Andonowati is called AB equation. The AB equation in this paper is solved using the homotopy perturbation method to help solve its nonlinear terms and its pseudo differential operator. This paper is structured as follows. The research method is discussed in section 2. Then the mathematical model used is discussed in section 3. Furthermore, the solution of the AB equation is discussed in section 4. Finally, the paper is concluded with a conclusion.

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II. RESEARCH METHOD

As its name suggests, the homotopy perturbation method is a combination of its two constituent methods, namely the homotopy and perturbation methods. This method was first proposed by He [4]. The method has been applied in the Duffing equation at an error value of less than 5.8% (see [5]). In addition, the homotopy perturbation method was also applied to the pendulum equation and the error value was not greater than 1.5% [6]. Both studies involved mathematical models in the form of nonlinear ordinary differential equations. This is only logical since problems in nature are also often modeled with nonlinear partial differential equations. This method has also been applied to nonlinear partial differential equations and the results show that the homotopy perturbation method is highly effective and simple [7]. MohyudDin and Noor [8] also apply the homotopy perturbation method to nonlinear differential equations and find that this method are simpler than the adomian decomposition method [9].

In this paper, we used the homotopy perturbation method to solve the wave model as described by the AB-equation.

The basic idea of homotopy perturbation method for solving nonlinear differential equations is given as follows.

Consider the following differential equations.

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

With a boundary condition

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \qquad r \in \Gamma \tag{2}$$

where A is the general differential operator, $f(r)$ is known as the known function, u is the function to be determined, B is the boundary operator, Γ is the boundary of the domain (Ω), and $\frac{\partial u}{\partial t}$ denotes the differential along the normal to Γ.

In general, operator A is divided into two, namely L and N, where L is the linear operator, and N is the nonlinear operator. Therefore, Equation (1) can be written as follows.

$$L(u)+N(u)-f(r)=0, \qquad r\in\Omega \quad (3)$$

In the case of the nonlinear Equation (3), since it does not include small parameters, we can construct the homotopy equation as follows.

$$H(v,p)=(1-p)[L(v)-L(u_0]+p[A(v)-f(r)]=0 \quad (4)$$

$$v(r,p):\Omega\times[0,1]\to R$$

In Equation (4), $p\in[0,1]$ is the embedding parameter and u_0 is the first approximation that satisfies the boundary conditions.

From Equation (4), we have

$$H(v,0)=L(v)-L(u_0)=0$$

$$H(v,1)=A(v)-f(r)=0.$$

The process of moving p from zero to one represents moving of $v(r,p)$ from u_0 to u_r. In topology, this is called deformation and $L(v)-L(u_0)$, $A(v)-f(r)$ are the homotopy.

We introduced the Embedding parameters p in a more natural way, i.e., not affected by artificial factors.

Furthermore, we can consider p as a small parameter for $0\le p\le 1$.

Therefore, it is reasonable to assume that the solution of Equation (4) can be expressed as

$$v=v_0+pv_1+p^2\ v_2+\ldots \quad (5)$$

Hence, the approximate solution of Equation (1) can be obtained as

$$\lim_{p\to 1} v=v_0+v_1+v_2+\ldots$$

III. MATHEMATICAL MODEL

The mathematical model used in this study was the AB equation. The model was an improvement of the KdV equation and could be interpreted as a higher-order KdV equation for wave above finite depth and in certain approximation it became the KdV equation. The nonlinear terms of the model were also improved to include the effects of short wave interactions. The model could be used for all wave lengths and any depth (see [10]). As an improvement of the KdV equation, the disperse relation contained in the AB equation was an exact disperse relation. The AB equation is given as follows.

$$\partial_t\eta=-\sqrt{g}A[\eta+\frac{1}{2}A(\eta A\eta)-\frac{1}{4}(A\eta)^2+\frac{1}{2}B(\eta B\eta)+\frac{1}{4}(B\eta)^2] \quad (6)$$

With η representing wave elevation, $A=\frac{\partial_x C}{\sqrt{g}}$ and $B=\sqrt{g}\ C^{-1}$ being pseudo differential operator with symbol $\hat{C}(k)=\frac{\Omega(k)}{k}$, $\Omega(k)=c_0k\sqrt{\frac{\tanh(kh_0)}{kh_0}}$, $c_0=\sqrt{gh_0}$ with g and h_0 being acceleration of gravitation and the water depth respectively. In this paper, the AB equation was solved using the homotopy perturbation method. For further writing, we write the symbol $\hat{C}$ with C only.

IV. RESULTS AND DISCUSSION

Consider the AB equation (6). Using the homotopy perturbation method, the equation was built into a homotopy equation as follows.

$$H(\eta,\ p)=(1-p)\left[\partial_t\eta+\sqrt{g}A\eta\right]+p[\partial_t\eta+\sqrt{g}A\eta+\ldots+\frac{1}{2}A(\eta A\eta)-\frac{1}{4}(A\eta)^2+\frac{1}{2}B(\eta B\eta)+\frac{1}{4}(B\eta)^2\ ]=0 \quad (7)$$

Next, assume the elevation $\eta(x,t)$ to be a power series in p.

$$\eta=\eta^{(0)}+p\eta^{(1)}+p^2\eta^{(2)}+p^3\eta^{(3)}+\ldots \quad (8)$$

Substituting the series form (8) into the homotopy function (7), we obtained

At order p^0 (order one), the following equation is obtained.

$$\partial_t\eta^{(0)}+\sqrt{g}A\eta^{(0)}=0 \quad (9)$$

The monochromatic waves taken and generated at the generating source were given in the form of

$$\eta^{(0)}=ae^{i(kx-\omega t)}+c.c \quad (10)$$

With $c.c$ being the complex conjugate of the complex function at the initial term. Furthermore, we obtained

$$\partial_t\eta^{(0)}+\sqrt{g}\partial_x\frac{\hat{C}}{\sqrt{g}}\eta^{(0)}=\partial_t\eta^{(0)}+\partial_x\hat{C}\eta^{(0)}=0$$

$-i\omega e^{i\theta}+i\Omega(k)e^{i\theta}=0$ With $\theta=(kx-\omega t)$ Thus, at order one the following relationship is obtained.

$$\omega=\Omega(k)=c_0k\sqrt{\frac{\tanh(kh_0)}{kh_0}} \quad (11)$$

which represented the exact linear dispersion relation [2].

The relation between frequencies ω for various wave numbers k and for $\omega 1=\Omega(k), \omega 2=\Omega(2k), \omega 3=\Omega(3k)$ is given in Fig 1. Fig 1 shows that the larger the wave number k, the larger the frequency ω will be.

The wave profile of The first-order solution of AB can be seen in Fig 2.

Then, at order p^1 we obtained

$$[\partial_t\eta^{(1)}-\partial_t\eta^{(0)}-\sqrt{g}A\eta^{(0)}+\sqrt{g}A\eta^{(1)}]+[\partial_t\eta^{(0)}+\sqrt{g}A\eta^{(0)}+\ldots+\frac{1}{2}A(\eta^{(0)}A\eta^{(0)})-\frac{1}{4}(A\eta^{(0)})^2+\frac{1}{2}B(\eta^{(0)}B\eta^{(0)})+\frac{1}{4}(B\eta^{(0)})^2]=0 \quad (12)$$

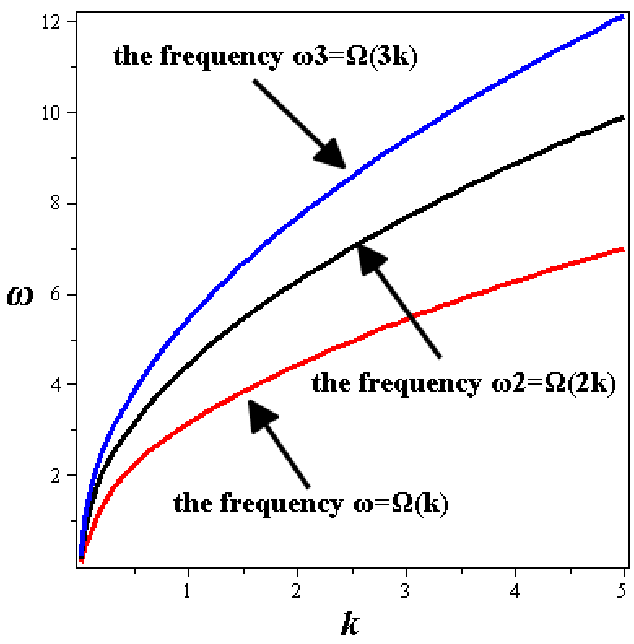

Fig 1. The relation between frequencies ω for various wave numbers k

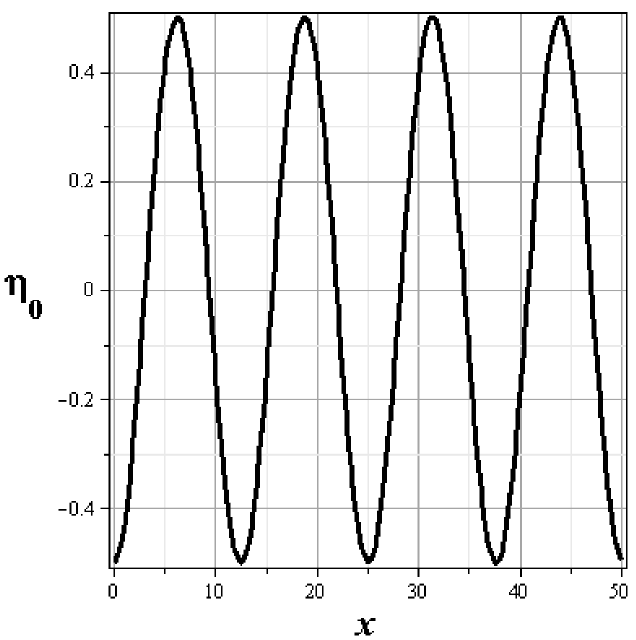

Fig 2. The first-order solutions of AB equation at time t = 10s.

Furthermore, we obtained

$$\left[\partial_t\eta^{(1)} + i\omega a e^{i\theta} - i\Omega(k) a e^{i\theta} + \sqrt{g}A\eta^{(1)}\right] + \left[-i\omega a e^{i\theta} + \dots\right.$$

$$+ i\Omega(k) a e^{i\theta} - \frac{1}{2}\frac{\Omega(2k)\Omega(k)}{g}(a^2 e^{2i\theta}) + \frac{1}{4}\left(\frac{\Omega^2(k)}{g} a^2 e^{2i\theta}\right) + \dots$$

$$\left. + g\frac{k^2}{\Omega(2k)\Omega(k)} a^2 e^{2i\theta} + \frac{1}{4} g \frac{k^2}{\Omega^2(k)} a^2 e^{2i\theta}\right] = 0.$$

$$[\partial_t\eta^{(1)} + \sqrt{g}A\eta^{(1)} = [\frac{1}{2}\frac{\Omega(2k)\Omega(k)}{g}(a^2 e^{2i\theta}) - \frac{1}{4}\left(\frac{\Omega^2(k)}{g} a^2 e^{2i\theta}\right)$$

$$-g\frac{k^2}{\Omega(2k)\Omega(k)} a^2 e^{2i\theta} - \frac{1}{4} g \frac{k^2}{\Omega^2(k)} a^2 e^{2i\theta}] =$$

$$\frac{2\Omega(2k)\Omega(k)-\Omega^2(k)}{4g} a^2 e^{2i\theta} - gk^2 \frac{4k^2\Omega(k)+\Omega(2k)}{4\Omega(2k)\Omega^2(k)} gk^2 a^2 e^{2i\theta}$$

$$[\partial_t\eta^{(1)} + \sqrt{g}A\eta^{(1)} = \left[\frac{2\Omega(2k)\Omega(k)-\Omega^2(k)}{4g} - \frac{4k^2\Omega(k)+\Omega(2k)}{4\Omega(2k)\Omega^2(k)} gk^2\right] a^2 e^{2i\theta} \quad (13)$$

From the last equation (13), we could construct the solution in the form of

$$\eta^{(1)} = \alpha e^{2i\theta} + c.c \quad (14)$$

Then, we obtained

$$[\partial_t\eta^{(1)} + \sqrt{g}A\eta^{(1)} = \left[\frac{2\Omega(2k)\Omega(k)-\Omega^2(k)}{4g} - \frac{4k^2\Omega(k)+\Omega(2k)}{4\Omega(2k)\Omega^2(k)} gk^2\right] a^2 e^{2i\theta}$$

$$-\alpha 2i\Omega(k)e^{2i\theta} + \alpha i\Omega(2k)e^{2i\theta} =$$

$$\left[\frac{2\Omega(2k)\Omega(k)-\Omega^2(k)}{4g} - \frac{4k^2\Omega(k)+\Omega(2k)}{4\Omega(2k)\Omega^2(k)} gk^2\right] a^2 e^{2i\theta}$$

$$\alpha = \frac{\left[\frac{2\Omega(2k)\Omega(k)-\Omega^2(k)}{4g} - \frac{4k^2\Omega(k)+\Omega(2k)}{4\Omega(2k)\Omega^2(k)} gk^2\right]}{i(\Omega(2k)-2\Omega(k))} a^2$$

Where α represented the coefficient of the second-order solution (order p^1). Fig 3 shows the coefficient α at the order to wave number k.

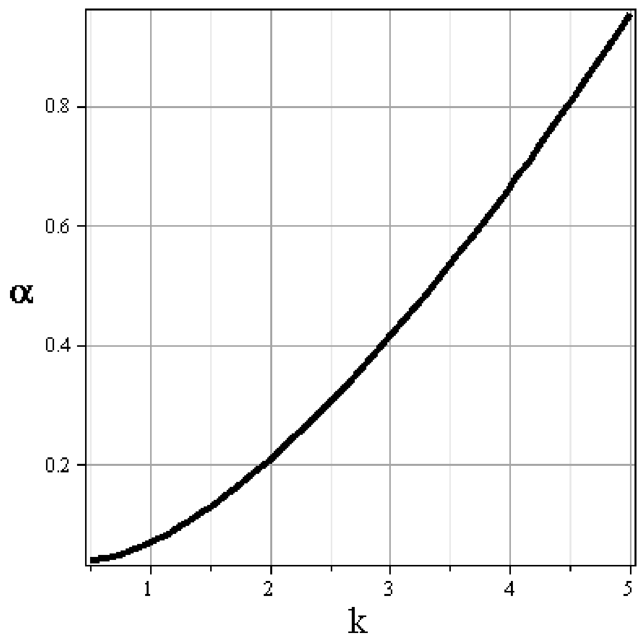

Fig 3. The relation between coefficients α for various wave numbers k.

The second-order solutions of the AB equation at time t = 10s, amplitude a = 0.5m, acceleration of gravitation g = 9.8 m/s^2, wave number k = 0.5 and the waterdepth $h_0 = 5$m, is given in Fig 4. in Fig 4 the wave with frequency $2\omega = \Omega(2k)$, the wave travels faster than the wave in the Fig 2, because the speed depends on the frequency. however, the amplitude is smaller than the solution on the first order, due to the use of the perturbation method.

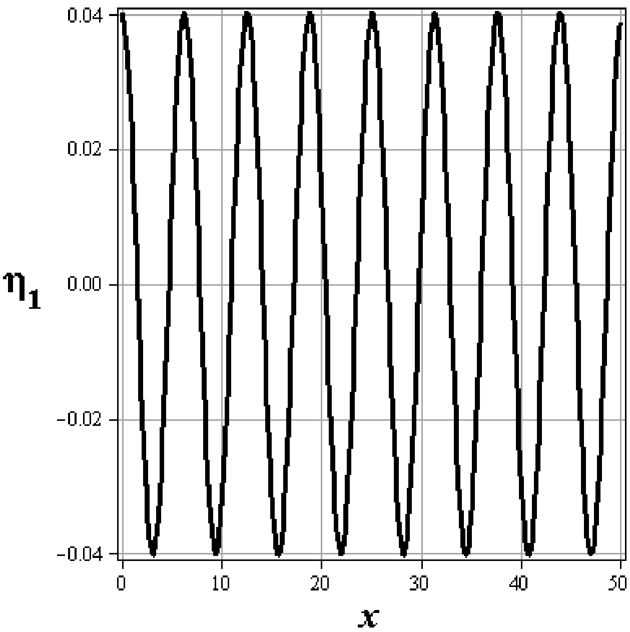

Fig 4. The second-order solutions of AB equation at time t=10s.

Furthermore, at order p^2, we obtained

$$[\partial_t \eta^{(2)} + \sqrt{g} A\eta^{(2)}] - [\partial_t \eta^{(1)} + \sqrt{g} A\eta^{(1)}] + [\partial_t \eta^{(1)} + \sqrt{g} A\eta^{(1)} + \ldots + \\ + \frac{1}{2} A(\eta^{(0)} A\eta^{(1)} + \eta^{(1)} A\eta^{(0)}) - \frac{2}{4} A\eta^{(0)} A\eta^{(1)} + \\ [\frac{1}{2} B(\eta^{(0)} B\eta^{(1)}) + \frac{1}{2} B(\eta^{(1)} B\eta^{(0)}) + \frac{2}{4} B\eta^{(0)} B\eta^{(1)}]] = 0 \tag{15}$$

By substituting (9) and (14) to the equation (9), we obtained

$$\partial_t \eta^{(2)} + \sqrt{g} A\eta^{(2)}] + \frac{1}{2} A(ae^{i\theta} A\alpha e^{2i\theta} + \alpha e^{2i\theta} Aae^{i\theta}) - \\ - \frac{2}{4} Aae^{i\theta} A\alpha e^{2i\theta} + [\frac{1}{2} B(ae^{i\theta} B\alpha e^{2i\theta}) + \frac{1}{2} B(\alpha e^{2i\theta} Bae^{i\theta}) + \ldots + \\ + \frac{2}{4} Bae^{i\theta} B\alpha e^{2i\theta}] = 0$$

$$[\partial_t \eta^{(2)} + \sqrt{g} A\eta^{(2)}] = \begin{bmatrix} (\frac{\Omega(2k)\Omega(3k) - \Omega(k)\Omega(2k)}{2g}) \\ -g\frac{3k^2}{\Omega(2k)\Omega(3k)} - \frac{3}{2}(\frac{gk^2}{\Omega(k)\Omega(3k)}) - \frac{gk^2}{\Omega(k)\Omega(2k)} \end{bmatrix} a\alpha e^{3i\theta} \tag{16}$$

Furthermore, we assumed the solution of (16) as

$$\eta^{(2)} = \beta e^{3i\theta} + c.c, \tag{17}$$

hence, we obtain

$$[-i3\omega\beta e^{3i\theta} + i\Omega(3k)\beta e^{3i\theta}] = \begin{bmatrix} (\frac{\Omega(2k)\Omega(3k) - \Omega(k)\Omega(2k}{2g}) \\ -g\frac{3k^2}{\Omega(2k)\Omega(3k)} - \frac{3}{2}(\frac{gk^2}{\Omega(k)\Omega(3k)}) - \frac{gk^2}{\Omega(k)\Omega(2k)} \end{bmatrix} a\alpha e^{3i\theta}$$

With the coefficient β being

$$\beta = \frac{\begin{bmatrix} (\frac{\Omega(2k)\Omega(3k) - \Omega(k)\Omega(2k)}{2g}) \\ -g\frac{3k^2}{\Omega(2k)\Omega(3k)} - \frac{3}{2}(\frac{gk^2}{\Omega(k)\Omega(3k)}) - \frac{gk^2}{\Omega(k)\Omega(2k)} \end{bmatrix}}{(3\Omega(k) - \Omega(3k))} ia\alpha \tag{18}$$

β represent the coefficient of the third-order solution (order p^2) of AB equation. Fig 5 shows the coefficient β at the order to wave number k.

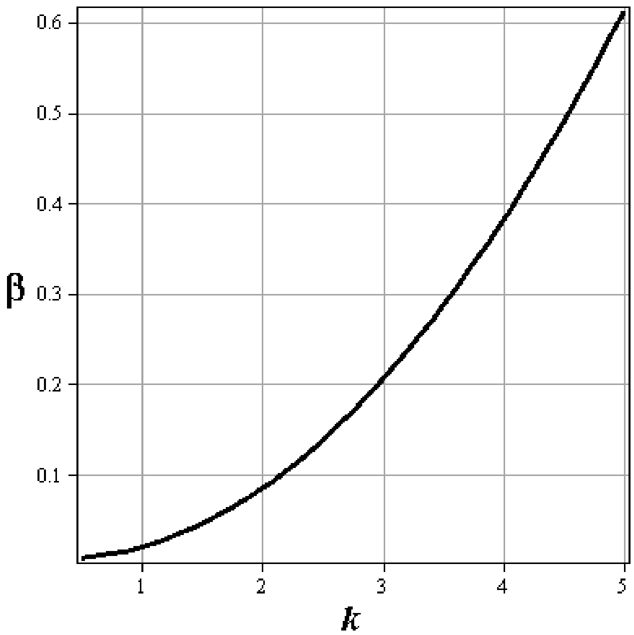

Fig 5. The coefficient of the third-order solution β for various wave numbers k

The third-order solution of AB equation at time t = 10 is given in the Fig 6.

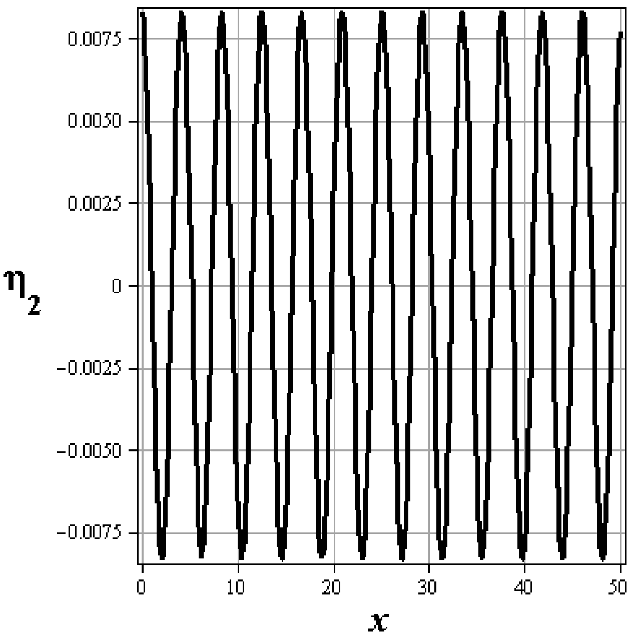

Fig 6. The third-order solution of AB equation at time t = 10s.

Comparison between wave profile of the first order solution and the second order solution is given in Fig 7.

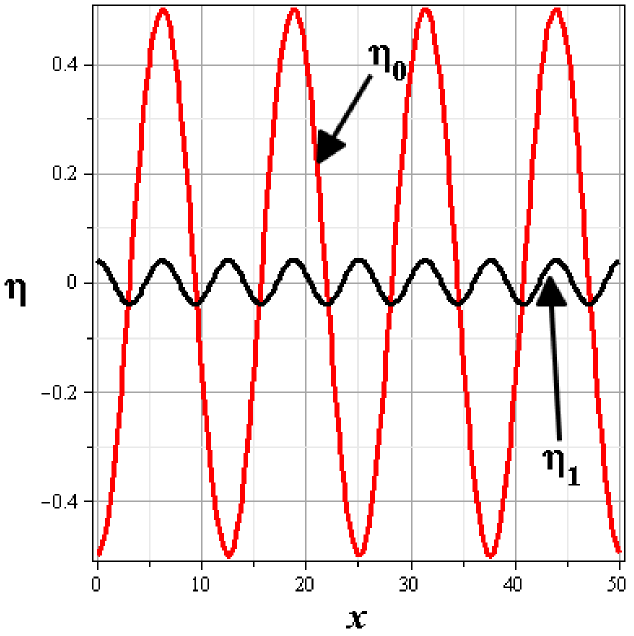

Fig 7. Comparison between waves profiles of the first order solution and the second order solution.

While, the comparison between waves profiles of the second order solution and the third order solution is given in Fig 8.

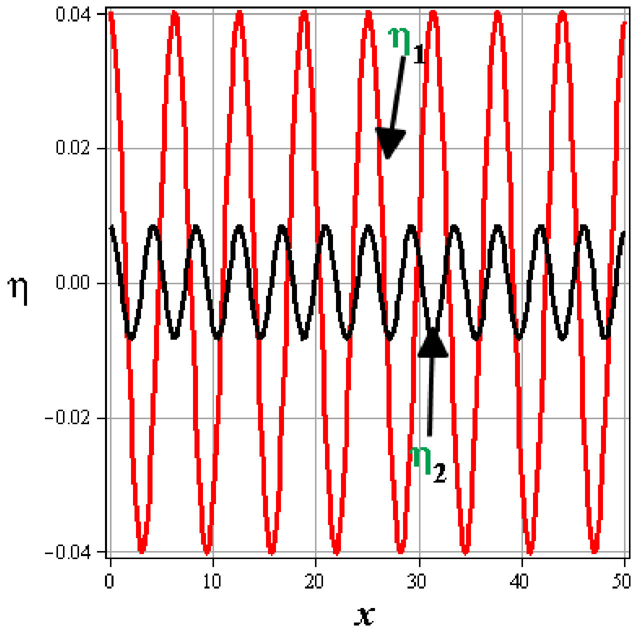

Fig 8. Comparison between waves profiles of the second order solution and the third solution.

From the Fig 7 and Fig 8, we obtained that the first order solution larger than the second and the second order solution larger than the third order solution and so on. This shows that the perturbation method has been satisfied.

The wave profile of the total solution which is given as $\eta = \eta_0 + \eta_1 + \eta_2$, at amplitude a = 0.5m, acceleration of gravitation g = $9.8\, m/s^2$, wave number k = 0.5 and waterdepth h_0 = 5m, at time t=10s , is given in Fig 9.

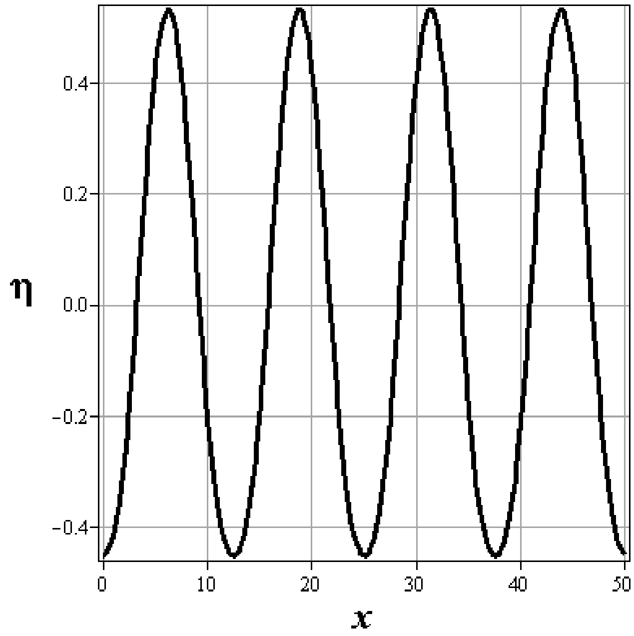

Fig 9. The wave profile of the total solution of AB equation.

The comparison between waves profiles of the leading solution(η_0) and the total solution is given in Fig 10.

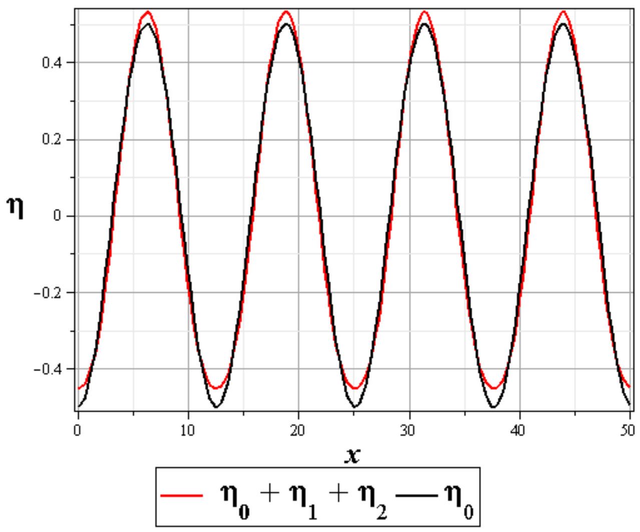

Fig10. The comparison between waves profiles of the leading solution (η_0) and the total solution (η)

from Fig 10 we obtained that the influence of the second order, the third order and so on to the first order on the increase in amplitude is very small only about 0.028m at time t = 10s

V. CONCLUSION

In this paper, we discussed the solution of AB equation, a water wave equation enhanced from the KdV equation. The solution of the AB equation was obtained by using the homotopy perturbation method. This method itself was a combination of two methods, namely the homotopy and perturbation methods. Using the method in the first-order solution, we obtained the dispersion relation of AB equation we call it $\omega = \Omega(k)$. In the second order solution we obtained the frequency $\omega 2 = \Omega(2k)$ and so on. The effect of this difference causes the solution in different waveforms. In the second-order solution the wave travels faster than in the first-order solution, in the third-order solution wave travels faster than the second-order and so on. The influence of the second order, the third order and so on to the first order on the increase in amplitude is very small only about 0.028m at time t = 10s Using the method also allowed the solution to be obtained by taking the limit of elevation for embedding parameter to one.

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