Operator Preserving Optimum Method for Solving Multiobjective Optimization Problems

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***Abstract*—In this work, we introduce an algorithm leveraging an optimality-preserving operator to address multi-objective optimization challenges. Following a comprehensive review of prior research, we formalize the methodology through an operator designed to retain optimal solutions in multi-objective settings without relying on scalarization techniques. Under explicitly stated conditions, we establish that all solutions produced by our framework correspond to Pareto optimal outcomes for the original problem. To validate the algorithm's efficacy, we assess its performance using benchmark problems widely recognized in the literature. Quantitative metrics are employed to measure its behavior, and the results are benchmarked against those of a prominent evolutionary algorithm, NSGA-II, serving as a reference standard.**

***Index Terms*—Multiobjective Method, Penalty function, Alienor , Pareto front, OPO.**

I. Introduction

MULTI-OBJECTIVE optimization consists of simultaneously optimizing several objective functions. Thus, for such multi-objective optimization problems, no unique solution simultaneously optimizes all objective functions. The goal in addressing these problems across disciplines such as economics [23], computer science [26], physics [25], transportation [29], and social choice theory [3], [7], among others, is to identify high-quality compromise or Pareto optimal solutions.

Several methods have been proposed in the literature, which are grouped into two main categories : exact methods [19], [20] that have a theoretical foundation on the optimality of solutions, and stochastic methods [1], [2], [25]most of which do not have a theoretical foundation.

Among the various existing methods, a substantial number are based on scalarization techniques, which aim to transform multi-objective optimization problems into equivalent single-objective formulations. An illustrative case is the Multiobjective Metaheuristic based on the Aliénor method (MOMA) [11], which combines the Optimum Preserving Operator (OPO) and the Aliénor transformation to simplify the problem into a one-variable decision model. This method was later extended and enhanced by Kounhinir et al. [14], [15], [18] led to the development of MOMO-Plus, an enhanced variant utilizing an optimized version of OPO denoted as OPO*. This refined approach has been applied across diverse operational research domains, including fuzzy optimization problems [4], [5], followed by transportation challenges [15]. The computational efficacy of MOMO-Plus was rigorously validated in [17].

In this study, we present a new non-scalar approach to tackle multi-objective optimization problems. The main contributions and distinctive features of this research are as follows:

- design and implementation of an innovative non-scalar technique employing an operator to tackle multi-objective optimization problems;
- comprehensive theoretical analysis examining the convergence properties of solutions produced by the proposed technique;
- empirical comparison of the proposed approach against established methods found in the literature.

In the remainder of the article, we will cover the preliminaries in Section 2, where the basic concepts, properties, and definitions regarding multi-objective optimization will be presented. Section 3 outlines the principal outcomes of this study, first introducing the method ”Multi-objective Operator Preserving Optimum” (M-OPO) and the algorithm for numerical resolution, as well as the theoretical convergence results of M-OPO. We subsequently illustrate the application of the M-OPO algorithm to structural optimization problems, supported by a comprehensive numerical convergence analysis. Finally, Section 4 provides concluding remarks and outlines potential directions for future research.

II. Preliminaries

A. Basic concepts

Consider the multi-objective optimization problem formulated as follows:

$$\min F(x) = \Big(f_1(x), \dots, f_m(x)\Big),\ m \geq 2$$
$$\text{s.t.} \begin{cases} g_j(x) \leqslant 0,\ j = 1, ..., p \\ x \in \mathbb{R}^n\ ; \end{cases} \tag{1}$$

where:

- $x = (x_1, \dots, x_n)$ represents the vector of n decision variables;
- f_i, $i = \overline{1, m}$, represent the objective functions;
- g_j, $j = \overline{1, p}$, are the constraint functions governing the optimization of f_i, $i = \overline{1, m}$.

We denote $\mathbf{D} = \{\ x \in \mathbb{R}^n : g_j(x) \leq 0;\ j = \overline{1, p}\}$. as the feasible domain of problem (1).

By the following definitions, we characterize an optimal and weakly optimal solution in the Pareto sense.

Definition 1 ([18], [22]). A point $x^* \in \mathbf{D}$ is said to be a weakly efficient or weakly optimal solution in the Pareto

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sense of the problem (1) if and only if there does not exist another $x \in \mathbf{D}$ such that:

$$f_i(x) < f_i(x^*), \quad \forall i = \overline{1, m}.$$

Definition 2 ([5], [18]). A point $x^* \in \mathbf{D}$ is referred to as an efficient (or Pareto optimal) solution to the problem (1) if no $x \in \mathbf{D}$ exists such that $f_i(x^*) \geq f_i(x)$ for all $i \in I = \{1, \ldots, m\}$, with at least one index $k \in \{1, \cdots, m\}$ satisfying $f_k(x^*) > f_k(x)$.

From Definitions 1 and 2, we characterize an optimal and weakly optimal Pareto solution by the following lemma.

Lemma 3 ([20]).

1) *A point $x^* \in \mathbf{D}$ is said to be a Pareto optimal solution of problem (1) if and only if there exists no $x \in \mathbf{D}$ such that all of the following conditions hold:*
   a) $\max\limits_{i=1,\ldots,m} \{f_i(x) - f_i(x^*)\} > 0$;
   b) $\min\limits_{i=1,\ldots,m} \{f_i(x) - f_i(x^*)\} \geq 0$;
2) *A point $x^* \in \mathbf{D}$ is a weakly Pareto optimal solution of problem ((1)) if and only if for all $x \in \mathbf{D}$, we have* $\max\limits_{i=1,\ldots,m} \{f_i(x) - f_i(x^*)\} \geq 0$.

B. Alienor transformation

If we consider a function with n continuous variables defined on $\mathbb{R}^n$, a reducing transformation (called Alienor transformation) allows expressing all variables as a function of a single variable ϱ

$$x_i = h_i(\varrho), \ \varrho \in \mathbb{R}_+.$$

This transformation was developed by Yves Cherruault, Arthur Guillez, and Blaise Somé [18], enabling the reduction of any multivariate function to a single-variable function through the use of the Archimedean spiral.

Definition 4 ([12]). Let f be a function of n variables. A reductive transformation associated with f is defined as any mapping that converts f into a function depending on a single specified variable.

The reducing transformation is based on the introduction of a curve that fills the space in the sense of alpha-density.

Definition 5 ([5]). A subset $\mathsf{S} \in \mathbb{R}^n$ is said to be α-dense in $\mathbb{R}^n$ if

$$\forall K \in \mathbb{R}^n, \ \exists M' \in \mathsf{S} \ \textit{such that} \ d(K, K') \leq \alpha.$$

Thus, if we consider a function with n variables $f(x_1, x_2, ..., x_n)$, continuous and defined on $\mathbb{R}^n$, a reducing transformation allows expressing all variables as a function of a single variable ϱ by using $x_i = h_i(\varrho), \forall i = 1, \cdots, n$. Therefore $f(x_1, x_2, ..., x_n)$ becomes $f(h_1(\varrho), h_2(\varrho), ..., h_n(\varrho))$. This reducing transformation has several variants, the one we are interested in is the Konfé-Cherruault transformation [11]. It's defined as $h(\varrho) = \big(h_1(\varrho), h_2(\varrho), ..., h_n(\varrho)\big)$ where $h_i(\varrho), i = 1, \cdots, n$ are defined as follows:

$$h_i(\varrho) = \frac{1}{2}\big[(ub_i - lb_i)\cos(\omega_i \varrho + \varphi_i) + ub_i + lb_i\big] = x_i; \quad (2)$$

where $(\omega_i)_{i=\overline{1,n}}$ and $(\varphi_i)_{i=\overline{1,n}}$ are slowly increasing sequences, $x_i \in [lb_i, ub_i]$ and $\varrho \in [0, \varrho_{\max}]$ with $\varrho^1 = \dfrac{2\pi - \varphi_1}{\omega_1}$ and $\varrho_{\max} = \dfrac{(ub_1 - lb_1)\varrho^1 + (ub_1 + lb_i)}{2}$.

C. Optimum-preserving operator and Optimum-preserving operator*

In 2003, G. Mora, Y. Cherruault, and A. Benabidallah [16] proposed a new operator called Optimum-Preserving Operators (OPO), which allows for the elimination of minima and obtaining a global minimum in a very short time, even with n variables.

Proposition 6 ([16]). *Let ϱ_0 be an arbitrary point in $\mathbb{I} = [0, +\infty[$, A a constant, and H a Heaviside function. Let f be a Lipschitz function with constant L and $\Phi : \mathbb{R} \to (0, \infty)$, a bijective function of class $\mathbb{C}^1$, with $\Phi^{'} \neq 0$, then the operator T_f^ϵ defined by:*

$$T_f^\epsilon(\varrho) = \Phi^{-1}[\epsilon + \Phi(f(\varrho) - f(\varrho_0) + A)] + \frac{1}{\epsilon}\varrho H[f(\varrho) - f(\varrho_0)]$$

with $0 < \epsilon < 1$ is an Optimum-Preserving Operator (O.P.O).

where H is a Heaviside function that takes the value 0 for $x < 0$ and 1 for $x \geq 0$.

Definition 7 ([8], [12]). Let f be a continuous function from $\mathbb{I}$ to $\mathbb{R}$ and r a real number. The level set, denoted by $N_f(r)$, is defined as $N_f(r) = \big\{\varrho \in \mathbb{I} : f(\varrho) \leq r\big\}$.

Theorem 8 ([8]).

1) *The operator T_f^ϵ converges to f, up to an additive constant, within the level set $N_f(f(\varrho_0))$.*
2) *Furthermore, all extrema of T_f^ϵ lie within $N_f(f(\varrho_0))$.*

In March 2004 [11], a much simpler operator called Optimization-Preserving-Operators* (OPO*) was proposed by B. Konfé.

Definition 9 ([11]). Let $f^* : \mathbb{I} \to \mathbb{R}$ denote the objective function under consideration. Suppose that f^* is Lipschitz and satisfies the growth condition at infinity. Let ϱ_0 be an arbitrary element of $\mathbb{I}$.

The application $T : \mathbb{F} \to T_{f^*} \in \mathcal{C}^{(0)}(\mathbb{I})$ defined by:

$$T_{f^*_{\varrho_0}}(\varrho) = \frac{1}{2}\big[f^*(\varrho) - f^*(\varrho_0) + \mid f^*(\varrho) - f^*(\varrho_0) \mid\big]$$

is called the new Optimum-Preserving Operator, denoted as OPO*.

III. Main results

A. Operator Preserving Optimum for multiobjective cases

Let us set $\mathbb{I} = [0, +\infty[$ and consider the following problem:

$$\min_{\varrho \in \mathbb{I}} F_i(\varrho) = (f_1^*(\varrho), f_2^*(\varrho), ..., f_m^*(\varrho)) \quad (3)$$

If $\forall \ i = 1, ..., m$, f_i^* is Lipschitz continuous and satisfies the growth conditions at infinity, then M-OPO for problem (3) is defined as follows:

$$\begin{array}{rccl} T: & \mathcal{F} & \to & \big[\mathcal{C}^{(0)}(\mathbb{I})\big]^m \\ & F & \to & T_{f^*}(\varrho) = \big(T_{f_1^*}(\varrho), \ldots, T_{f_m^*}(\varrho)\big) \end{array} \quad (4)$$

where $\mathcal{C}^{(0)}(\mathbb{I})$ is the set of continuous functions wich $\mathcal{F} \subset \left[\mathcal{C}^{(0)}(\mathbb{I})\right]^m$ and $T_{f_i^*}(\varrho) = \frac{1}{2}\left[f_i^*(\varrho) - f_i^*(\varrho_0) + \mid f_i^*(\varrho) - f_i^*(\varrho_0) \mid \right], \forall\ i$. As by hypothesis f_i^*, $\forall\ i = 1, ..., m$, satisfy the conditions of infinite growth, we have the following results.

Proposition 10. *Let $\varrho_0 \in \mathbb{I}$, we have:*

$$\min_{x\in\mathbb{I}} T_{f_i^*}(\varrho) = 0,\ \forall\ i = 1, \ldots, m.$$

Proof: It is established that, for every $i \in \{1, 2, \cdots, m\}$

$$T_{f_i^*}(\varrho) = \begin{cases} 0 & \text{if } f_i^*(\varrho) \leq f_i^*(\varrho_0) \\ f_i^*(\varrho) - f_i^*(\varrho_0) & \text{if } f_i^*(\varrho) > f_i^*(\varrho_0) \end{cases}$$

Using the definition, $T_{f_i^*}(\varrho) \geq 0$, $i = 1, ..., m$, $\forall \varrho \in \mathbb{I}$. Since $T_{f_i^*}(\varrho_0) = 0 \leq T_{f_i^*}(\varrho)$ in this case $\max\limits_{i=1,...,m}\{f_i^*(\varrho) - f_i^*(\varrho_0)\} > 0$ then ϱ_0 is a Pareto optimal solution of the system. Furthermore, $T_{f_i^*}(\varrho) = 0$ if $f_i^*(\varrho) \leq f_i^*(\varrho_0)$ then $\min\limits_{\varrho\in\mathbb{I}} T_{f_i^*}(\varrho) = 0$. Consequently, $\min\limits_{\varrho\in\mathbb{I}} T_{f_i^*}(\varrho) = \min\limits_{\varrho\in\mathbb{I}} T_{f_i^*}(\varrho_0) = 0$. ■

Proposition 11. *$T_{f_i^*}$ and f_i^* have at least one common Pareto optimal solution $\forall\ i = 1, ..., m$.*

Proof: Suppose that ϱ^* is a Pareto optimal solution of f_i^*, but not a solution of $T_{f_i^*}$. Since ϱ^* is a Pareto optimal solution of f_i^*, there necessarily exists an element $\varrho_0 \in \mathbb{I}$ such that:

$$f_i^*(\varrho^*) < f_i^*(\varrho_0), \quad \forall\, i = 1, \ldots, m.$$

This is equivalent to the following statements:

$$\begin{aligned}
&\Rightarrow f_i^*(\varrho^*) - f_i^*(\varrho_0) < 0,\ \forall i;\\
&\Rightarrow f_i^*(\varrho^*) - f_i^*(\varrho_0) + \mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid = 0,\ \forall i;\\
&\Rightarrow \frac{1}{2}\left[f_i^*(\varrho^*) - f_i^*(\varrho_0) + \mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid\right] = 0,\ \forall i;\\
&\Rightarrow \frac{1}{2}\left[f_i^*(\varrho^*) - f_i^*(\varrho_0) + \mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid\right] = 0, \forall i;\\
&\Rightarrow T_{f_i^*}(\varrho^*) = 0, \forall i;\\
&\Rightarrow \min_{\varrho^*\in\mathbb{I}} T_{f_i^*}(\varrho^*) = 0, \forall\ i.
\end{aligned}$$

Thus ϱ^* is a solution of $T_{f_i^*}$, $\forall\ i = 1, ..., m$. Now suppose that ϱ^* is a solution of $T_{f_i^*}$ but not a Pareto optimal solution of f_i^*. ϱ^* is a solution of $T_{f_i^*}$ is equivalent to $\min\limits_{\varrho^*\in\mathbb{I}} T_{f_i^*}(\varrho^*) = 0$, $\forall\ i = 1, ..., m$. That can be rewritten as the following lines:

$$\begin{aligned}
&\Rightarrow \frac{1}{2}\left[f_i^*(\varrho^*) - f_i^*(\varrho_0) + \mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid\right] = 0,\ \forall i\\
&\Rightarrow f_i^*(\varrho^*) - f_i^*(\varrho_0) + \mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid = 0,\ \forall i\\
&\Rightarrow f_i^*(\varrho^*) - f_i^*(\varrho_0) = - \mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid,\ \forall i\\
&\Rightarrow f_i^*(\varrho^*) - f_i^*(\varrho_0) < 0,\ \forall i\\
&\Rightarrow f_i^*(\varrho_0) - f_i^*(\varrho^*) > 0,\ \forall i\\
&\Rightarrow \max_{i=1,...,m}\{f_i^*(\varrho_0) - f_i^*(\varrho^*)\} > 0.
\end{aligned}$$

From Lemma 3, ϱ^* is a Pareto optimal solution of f_i^*. ■

Theorem 12. *Let ϱ_0 be an arbitrary point in $\mathbb{I}$. Let S be the set of solutions of $T_{f_i^*}(\varrho) = 0$, $\forall\ i = 1, ..., m$. For any fixed $\varrho_0 \in \mathbb{I}$, if $S = \{\varrho^*\}$, then ϱ^* is a Pareto optimal solution of f_i^*.*

Proof: Let $\varrho^* \in S$ such that $T_{f_i^*}(\varrho) = 0$ and let $\varrho_0 \in \mathbb{I}$ be an arbitrary point. By definition, we have $\varrho^* \in \mathbf{D}$. Assume, for contradiction's sake, that ϱ^* fails to be a Pareto optimal solution of the problem.Then, there exists $\varrho_\varepsilon \in \mathbf{D}$ such that $T_{f_i^*}(x_\varepsilon) = 0$ and $f_i^*(\varrho_\varepsilon) < f_i^*(\varrho^*)$, $\forall\ i = 1, ..., m$. According to Lemma 2.3, we have $\min\limits_{i=1,...,m}\{f_i^*(\varrho_\varepsilon) - f_i^*(\varrho^*)\} \leq 0$. Since by hypothesis $T_{f_i^*}(\varrho^*) = 0$ and $T_{f_i^*}(\varrho_\varepsilon) = 0$, $\forall\ i = 1, ..., m$ we have $f_i^*(\varrho^*) - f_i^*(\varrho_0) = - \mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid$, $\forall\ i = 1, ..., m$ and $f_i^*(\varrho_\varepsilon) - f_i^*(\varrho_0) = - \mid f_i^*(\varrho_\varepsilon) - f_i^*(\varrho_0) \mid$, $\forall\, i = 1, ..., m$ According to the previous equations, we have the following result: $f_i^*(\varrho_\varepsilon) - f_i^*(\varrho^*) = \mid f_i^*(\varrho^*) - f_i(\varrho_0) \mid - \mid f_i^*(\varrho_\varepsilon) - f_i^*(\varrho_0) \mid$ which is the negative quantity. And then, we obtain $\mid f_i^*(\varrho^*) - f_i^*(\varrho_0) \mid - \mid f_i^*(\varrho_\varepsilon) - f_i^*(\varrho_0) \mid < 0$; $\forall\, i = 1, ..., m$. Therefore, we have $\min\limits_{i=1,...,m}\{f_i^*(\varrho^*) - f_i^*(\varrho_\varepsilon)\} \leq 0$. This contradicts equation hypothesis. ■

B. Multiobjective Operator Preserving Optimum method (M-OPO)

1) Principle of M-OPO: The Multi-Objective Optimum Preserving Operator (M-OPO) consists of directly transforming a multi-objective optimization problem with constraints by penalizing the objectives to obtain a multi-objective problem without constraints. Then, using the Alienor transformation, this multi-objective problem with several variables is reduced to a single-variable multi-objective problem. Subsequently, OPO is applied to this single-variable problem.

In the following, we will present the different steps of the proposed M-OPO method.

2) Steps of M-OPO: The Multiobjective Operator Preserving Optimum method proceeds in four steps:

Step I: For a multiobjective optimization problem with constraints, this step involves directly penalizing the problem to transform it into an unconstrained multiobjective problem. From equation (1), we get:

$$\begin{cases} \min\{L_i(x), i = 1, ..., m.\} \\ \qquad \eta_i \geq \dfrac{M_i - f_i(x)}{\sum\limits_{j=1}^{p} g_j(x)}, \overline{1, m}. \\ \qquad M_i = \max\limits_{x\in\mathbf{D}} f_i(x), i = \overline{1, m}. \\ \qquad x \in \mathbb{R}^n, \end{cases} \tag{5}$$

with $L_i(x) = f_i(x) + \eta_i\left[\sum\limits_{j=1}^{p}(g_j(x) + |g_j(x)|\right]$

Theorem 13. *For problem (5), any Pareto optimal solution $x \in \mathbf{D}$ is also a non-dominated solution (Pareto optimal solution) to the problem (1), and vice versa.*

Proof: Suppose x^* is a solution on the Pareto front of problem (5), then $\forall x \in \mathbf{D}$, $\max\limits_{i=1,...,m}\{L_i(x) - L_i(x^*)\} > 0$ thus $\max\limits_{i=1,...,m}\left\{f_i(x) + \eta_i\left[\sum\limits_{j=1}^{p}(g_j(x) + |g_j(x)|\right] - f_i(x^*) - \right.$

$\eta_i\left[\sum_{j=1}^{p}(g_j(x^*)+|g_j(x^*)|\right]\bigg\} > 0$. Since x and x^* belong to the feasible domain $\mathbf{D}$, we have $g_j(x) \leq 0 \Rightarrow g_j(x)+|g_j(x)| = 0$, $\forall x \in \mathbf{D}$, $j = \overline{1,p}$ and $g_j(x^*) \leq 0 \Rightarrow g_j(x^*)+|g_j(x^*)| = 0$, $\forall x^* \in \mathbf{D}$, $j = \overline{1,p}$; Thus, $\eta_i\left[\sum_{j=1}^{p}(g_j(x^*)+|g_j(x^*)|\right] = 0$, $\forall i = 1,\dots,m$; and $\eta_i\left[\sum_{j=1}^{p}(g_j(x)+|g_j(x)|\right] = 0$, $\forall i = 1,\dots,m$; hence $\forall x \in \mathbf{D}$, $\max_{i=1,\dots,m}\{f_i(x)-f_i(x^*)\} > 0$. Thus, x^* is the global minimum of problem (1).

Conversely, assume that $x^* \in \mathbf{D}$ is a global solution of problem (1), and we aim to prove that x^* is also a global solution of problem (5):

$$\forall x \in \mathbf{D},\ \max_{i=1,\dots,m}\{f_i(x)-f_i(x^*)\} > 0.$$

Since $x \in \mathbf{D}$ and $x^* \in \mathbf{D}$ with the last equations, we know $\max_{i=1,\dots,m}\bigg\{f_i(x)+\eta_i\left[\sum_{j=1}^{p}(g_j(x)+|g_j(x)|\right] - f_i(x^*)-\eta_i\left[\sum_{j=1}^{p}(g_j(x^*)+|g_j(x^*)|\right]\bigg\} > 0$. Thus $\forall x \in \mathbf{D}$, $\max_{i=1,\dots,m}\{L_i(x)-L_i(x^*)\} > 0$. Therefore, $x^* \in \mathbf{D}$ is a global solution of problem (5). ■

Step II: This step entails reformulating problem (5) by converting it from a multi-variable optimization framework into a single-variable optimization formulation. An Aliénor transformation allows expressing all variables in terms of a single variable ϱ, where $\varrho \in [0,\varrho_{\max}]$, $x_i = h_i(\varrho)$ thus,

$$\begin{cases} \min\{L_i(\varrho), i=1,...,m\}; \\ \quad \eta_i \geq \dfrac{M_i - f_i(h(\varrho))}{\sum_{j=1}^{p} g_j(h(\varrho))}, i=1,...,m.; \\ \quad M_i = \max_{h(\varrho)\in\mathbf{D}} f_i(h(\varrho)), i=1,...,m; \\ \quad \varrho \in [0,\varrho_{\max}]; \end{cases} \tag{6}$$

where $L_i(\varrho) = f_i(h(\varrho)) + \eta_i\left[\sum_{j=1}^{p}(g_j(h(\varrho)) + |g_j(h(\varrho))|\right]$

Theorem 14. *If $\varrho^* \in [0,\varrho_{\max}]$ is a Pareto optimal solution of problem (6), then all $x_i^* = h_i(\varrho^*) \in \mathbf{D}$ are a non-dominated solutions of problem (5), and vice versa.*

Proof: Suppose ϱ^* is an optimal solution of problem (6). Then, $\forall \varrho \in [0,\varrho_{\max}]$, $\max_{i=1,\dots,m}\{L_i(h(\varrho)) - L_i(h(\varrho^*))\} > 0$. Therefore, $\max_{i=1,\dots,m}\big\{f_i(h(\varrho)) + \eta_i\big[\sum_{j=1}^{p}(g_j(h(\varrho)) + |g_j(h(\varrho))|\big] - f_i(h(\varrho^*)) - \eta_i\big[\sum_{j=1}^{p}(g_j(h(\varrho^*)) + |g_j(h(\varrho^*))|\big]\big\} > 0$; As we have $g_j(h(\varrho^*)) < 0, j = 1,\cdots,p$, then $\sum_{j=1}^{n}(g_j(h(\varrho^*))+|g_j(h(\varrho^*))|) = 0$; At the same time, we have also $g_j(h(\varrho)) < 0 \Rightarrow \sum_{j=1}^{n}(g_j(h(\varrho))+|g_j(h(\varrho))|) = 0$. Thus, $\forall \varrho \in [0,\varrho_{\max}]$, $\max_{i=1,\dots,m}\{f_i(h(\varrho)) - f_i(h(\varrho^*))\} > 0$; where $h(\varrho) = x$ and $h(\varrho^*) = x^*$. Therefore, $\forall \varrho \in [0,\varrho_{\max}]$, $\max_{i=1,\dots,m}\{f_i(h(\varrho)) - f_i(h(\varrho^*))\} > 0$ This implies that $\forall x \in \mathbf{D}$, $\max_{i=1,\dots,m}\{f_i(x) - f_i(x^*)\} > 0$. Using of the last equations, we can write this inequality as $\max_{i=1,\dots,m}\bigg\{f_i(x) + \eta_i\left[\sum_{j=1}^{p}(g_j(x)+|g_j(x)|)\right] - f_i(x^*) - \eta_i\left[\sum_{j=1}^{p}(g_j(x^*)+|g_j(x^*)|)\right]\bigg\} > 0$; Then, we have $\Rightarrow \forall x \in \mathbf{D}$, $\max_{i=1,\dots,m}\{L_i(x) - L_i(x^*)\} > 0$; Therefore, $x^* \in \mathbf{D}$ is a Pareto optimal solution of problem (5).

Now, consider $x^* \in \mathbf{D}$ as a Pareto optimal solution of problem (5), i.e. $\forall x \in \mathbf{D}$, $\max_{i=1,\dots,m}\{L_i(x) - L_i(x^*)\} > 0$; which implies $\max_{i=1,\dots,m}\bigg\{f_i(x)+\eta_i\left[\sum_{j=1}^{p}(g_j(x)+|g_j(x)|)\right] - f_i(x^*)-\eta_i\left[\sum_{j=1}^{p}(g_j(x^*)+|g_j(x^*)|)\right]\bigg\} > 0$ also, using the last equations, we obtain $\forall x \in \mathbf{D}$, $\max_{i=1,\dots,m}\{f_i(x)-f_i(x^*)\} > 0$. Since $h(\varrho) = x$ and $h(\varrho^*) = x^*$, we have $\forall \varrho \in [0,\varrho_{\max}]$, $\max_{i=1,\dots,m}\{f_i(h(\varrho)) - f_i(h(\varrho^*))\} > 0$ and thus, $\max_{i=1,\dots,m}\big\{f_i(h(\varrho)) + \eta_i\big[\sum_{j=1}^{p}(g_j(h(\varrho)) + |g_j(h(\varrho))|)\big] - f_i(h(\varrho^*)) + \eta_i\big[\sum_{j=1}^{p}(g_j(h(\varrho^*)) + |g_j(h(\varrho^*))|)\big]\big\} > 0$; with these equations, we can write $\forall \varrho \in [0,\varrho_{\max}]$, $\max_{i=1,\dots,m}\{L_i(h(\varrho)) - L_i(h(\varrho^*))\} > 0$, thus $\varrho^* \in [0,\varrho_{\max}]$ is a Pareto optimal solution of problem (6). ■

Step III: This step involves solving the single-variable optimization problem after the preceding three steps. The principle of OPO* is to list the minima in a level set and then reduce this set to a single element through successive cuts after solving the equation $T_{L_i^*}(\varrho) = 0, i = 1,...,m$. For all fixed $\varrho_0 \in \big[0,\varrho_{max}\big]$, that is equivalent to: $[L_i^*(\varrho) - L_i^*(\varrho_0) + \mid L_i^*(\varrho) - L_i^*(\varrho_0) \mid] = 0, i = 1,\dots,m.$

Theorem 15. *Let ϱ_0 be an arbitrary point in $\big[0,\varrho_{max}\big]$. Let S be the set of solutions of the system $T_{L_i^*}(\varrho) = 0, \forall i = 1,\dots,m$. If S contains a solution, then this solution is Pareto optimal for problem (6).*

Proof: We can well define a set S as follows: $S = \big\{\varrho \in [0,\varrho_{\max}] \bigm| T_{L_i^*}(\varrho) = 0,\ \forall i = 1,...,m\big\}$. $T_{L_i^*}(\varrho) = 0, \forall i = 1,...,m$ is equivalent to $\frac{1}{2}\big[L_i^*(\varrho) - L_i^*(\varrho_0) + \mid L_m^*(\varrho) - L_i^*(\varrho_0) \mid\big] = 0, \forall i$. That can be rewritten as follows: $L_i^*(\varrho) - L_i^*(\varrho_0) = - \mid L_i^*(\varrho) - L_i^*(\varrho_0) \mid, \forall i$. That gives us $L_i^*(\varrho) - L_i^*(\varrho_0) \leq 0, \forall i = 1$. And then, we obtain $L_i^*(\varrho) \leq L_i^*(\varrho_0), \forall i = 1$. Therefore, $S = \{\varrho \in [0,\varrho_{\max}] \mid L_i^*(\varrho) \leq L_i^*(\varrho_0),\ \forall i = 1,...,m\}$. If ϱ^* is a Pareto optimal solution, then for all $\varrho_0 \in [0,\varrho_{\max}]$, $\max_{i=1,\dots,m}\{L_i(\varrho_0) - L_i(\varrho^*)\} > 0$. Moreover, for all $\varrho \in S \cap [0,\varrho_{\max}]$, we have $\max_{i=1,\dots,m}\{L_i(\varrho) - L_i(\varrho_0)\} > 0$. Therefore, for all $\varrho \in S \cap [0,\varrho_{\max}]$, $\max_{i=1,\dots,m}\{L_i(\varrho) - L_i(\varrho^*)\} > 0$.

Hence, the solution of the system is a Pareto optimal solution. ■

Step V: This step consists in computing the values of the decision variables in the original problem by applying the previously defined Aliénor transformation. Recall that this transformation is given by $x_i = h(\varrho^*), \quad i = 1, \ldots, m.$

3) Algorithm of M-OPO: Algorithm 1 is the pseudo-code of the Multiobjective Operator Preserving Optimum (MOpPO) method for solving multiobjective optimization problems.

Algorithm 1 .

Require: $\min_{x\in\mathbf{D}}\{f_i(x)\}, \quad i = 1, 2, \ldots, m$, Ub, Lb, η.

for $i = 1, \ldots, n$ **do**
  Set $\omega_i = 1500 + 0.005(i + 1)$
  Set $\varphi_i = 0.0005(i + 1)$
  Set $h_i(\varrho)$ using equation (2).
end for
Set $\varrho_{\max} = \left[(Ub - Lb)\dfrac{2\pi - \varphi_1}{\omega_1} + (Ub + Lb)\right]/2$
for $i = 1, \ldots, m$ **do**
  Set $L_i(x) = f_i(x) + \eta_i \left[\sum_{j=1}^{p}(g_j(x) + |g_j(x)|)\right]$
end for
for $i = 1, \ldots, m$ **do**
  Set $L_i(\varrho) = f_i(h(\varrho)) + \eta_i \left[\sum_{j=1}^{p}(g_j(h(\varrho)) + |g_j(h(\varrho))|)\right]$
end for
for $i = 1, \ldots, m$ **do**
  Set $T_{L_i^*}(\varrho) = \dfrac{1}{2}\left[L_i^*(\varrho) - L_i^*(\varrho_0) + \mid L_i^*(\varrho) - L_i^*(\varrho_0) \mid\right]$
end for
for $\varrho_0 \in [0, \varrho_{\max}]$ **do**
  Find ϱ^*, a solution of the system (**??**).
  for $i = 1, \ldots, n$ **do**
    Set $x_i = h_i(\varrho^*)$
  end for
end for

C. Computer Illustration

1) Algorithm parameters and test problems: We implemented our algorithm with the following parameters: $\eta = 10000$.

The parameters of the NSGA II method are: Population 100, Number of generations 100, Uniform mutation.

The characteristics of the computer used for the experimentation are as follows: ASUS Processor 11th Gen Intel(R) Core(TM) i3-1115G4 @ 3.00GHz; RAM Memory 8 GB; Operating System Windows 11 / 64 bits.

The set of test problems used in this work are represented in Table I.

TABLE I
LIST OF MULTIOBJECTIVE OPTIMIZATION PROBLEMS

Function	Sources	m	n	Parameters bounds
Minex	[7], [8], [10]	2	2	$x \in [0.1, 1] \times [0, 0.5]$
BNH1	[6], [7], [10]	2	2	$x \in [0, 3] \times [0, 5]$
BNH2	[7], [9], [10]	2	2	$x \in [0, 5]^2$
BNH3	[7], [10], [16]	2	2	$x \in [0, 1]^2$
Lamda1	[7], [10], [20]	2	2	$x \in [-2, 2]^2$
Lamda2	[7], [10], [20]	2	2	$x \in [-2, 2]^2$
Lamda3	[7], [10], [20]	2	2	$x \in [-2, 2]^2$
SSFYY1	[7], [10], [13]	2	2	$x \in [0, 1] \times [0, 2]$
JOS1	[7], [10], [13]	2	2	$x \in [0, 5]^2$
VU1	[7], [10], [21]	2	2	$x \in [-3, 3]^2$
POL	[6], [7], [10]	2	2	$x \in [-\pi, \pi]^2$

All for the test problems in the Table I are convex and continuous expect the last (function POL).

2) Metrics and performance profiles: Performance measures are used to study the efficiency of a new proposed multiobjective optimization method compared to other existing resolution methods. In our work, we utilize the purity, υ-Spread, and Γ-Spread performance profiles.

Purity Metric ([20], [6], [27])
The purity measure is utilized to evaluate the quality of Pareto fronts obtained from diverse multi-objective optimization algorithms. Let $\mathcal{S}$ represent the collection of optimization solvers and $\mathcal{P}$ denote the set of benchmark problems. For a given solver $s \in \mathcal{S}$ and a problem $p \in \mathcal{P}$, define $F_{p;s}$ as the approximate Pareto front generated by solver s for problem p, and let F_p signify an approximation of the reference Pareto front for problem p. To proceed, first construct the union $\bigcup_{s\in\mathcal{S}} F_{p;s}$, then remove all dominated solutions from this aggregated set.

The purity metric computes, for solver $s \in S$ and problem $p \in P$, the ratio $\bar{t}_{p,s} = \frac{C_{p;s}^{F_p}}{C_{p,s}}$, where $C_{p;s}^{F_p} = |F_{p,s} \cap F_p|$ and $C_{p;s} = |F_{p,s}|$. Consequently, the purity metric is defined as $t_{p,s} = \frac{1}{\bar{t}_{p,s}}$.

Spread Metrics ([6], [20])
Spread metrics aim to quantify the spread extent of a computed Pareto front. Two distinct formulas are adopted for spread evaluation:

The first formula evaluates the maximal size of gaps in an approximate Pareto front. Consider a solver $s \in S$ that generates, for a given problem $p \in P$, an approximate Pareto front consisting of N points, indexed from 1 to N. To ensure boundary completeness, this set is augmented with two extreme points, indexed as 0 and $N + 1$.

The Γ metric is defined as:

$$\tau_{p,s} = \max_{j\in(1,\ldots,m)} \left(\max_{i\in(1,\ldots,N)} \upsilon_{i,j} \right) \tag{7}$$

where $\upsilon_{i,j} = (f_{i+1,j} - f_{i,j})$ (assuming objective values are sorted in ascending order per objective j).

The second formula quantifies the distribution of an approximate front in higher-dimensional objective space:

$$\upsilon_{p,s} = \max_{j\in(1,\ldots,m)} \left(\frac{\upsilon_{0,j} + \upsilon_{N,j} + \sum_{i=1}^{N-1} |\upsilon_{i,j} - \overline{\upsilon_j}|}{\upsilon_{0,j} + \upsilon_{N,j} + (N-1)\overline{\upsilon_j}} \right) \tag{8}$$

Performance Profiles ([6], [20], [26])
Algorithm performance can be compared through performance profiles and associated data. These profiles are depicted via the cumulative distribution function $\rho(\tau)$, which

captures performance ratios across solvers. Using the previously defined sets, let $\zeta_{p,s}$ denote the performance of algorithm $s \in S$ on problem $p \in P$. A performance ratio $r_{p,s}$ is calculated as:

$$r_{p,s} = \frac{\zeta_{p,s}}{\min \{\zeta_{p,\bar{s}} \mid \bar{s} \in S\}}.$$

The performance profile of algorithm $s \in S$ corresponds to the proportion of problems where the performance ratio does not exceed α ($\alpha \geq 1$):

$$\rho(\tau) = \frac{|\{p \in P \mid r_{p,s} \leq \tau\}|}{|P|}. \quad (9)$$

For sufficiently large τ, $\rho(\tau)$ indicates the proportion of problems where algorithm s satisfies the convergence criterion.

3) Results Analysis: In this subsection, we begin the analysis of the results obtained by first presenting the Pareto fronts of the test problems listed in Table I, generated by our method.

TABLE II
PURITY AND DISPERSION VALUES

Methods	M-OPO			NSGA II		
Metrics	Purity	Γ	υ	Purity	Γ	υ
BNH1	0.43	**1.88**	**0.76**	**0.96**	7.19	0.83
BNH2	**1.00**	**0.68**	**0.70**	0.81	1.66	0.74
BNH3	**1.00**	**2.72**	0.86	0.69	8.14	**0.84**
JOS1	**1.00**	**0.03**	**0.80**	0.88	0.09	0.80
Lamda1	**1.00**	**0.48**	1.233	0.84	1.14	**0.84**
Lamda2	0.52	**20.11**	**0.99**	**0.90**	22.26	0.99
Lamda3	0.60	**2.98**	**0.25**	**0.91**	5.12	0.92
Minex	0.81	**5.00**	0.96	**0.99**	0.46	**0.87**
POL 9	0.54	**13.65**	1.50	**1.00**	17.20	**1.21**
SSFYY1	**1.00**	**0.10**	0.888	0.68	0.16	**0.82**
VU1	0.89	**1.08**	0.93	1.00	1.24	**0.92**

The Table II presents performance measures for the metrics applied to both methods, namely M-OPO and NSGA-II, on the test problems listed in Table I. The best results are highlighted in bold in the table. We observe that, for the purity and υ-Spread metrics, the M-OPO method outperforms the NSGA-II method on each metric in 45.45% of the problems. For the Γ-Spread metric, M-OPO dominates NSGA-II across all 11 problems.

Figure 12, Figure 13 and Figure 14 illustrates the performance profile across all test problems presented in Table I. Using the three metrics mentioned earlier—purity metric, Γ-Spread, and υ-Spread—we find that for the purity metric, for an interest factor τ greater than 2.6, both methods show no significant difference. Regarding the Γ-Spread metric, when the tolerance factor is less than 3.5, the M-OPO method outperforms NSGA-II. For a tolerance factor equal to or less than 4 in the υ-Spread metric, the M-OPO method is superior to NSGA-II, while for a tolerance factor greater than 4, they show no significant difference.

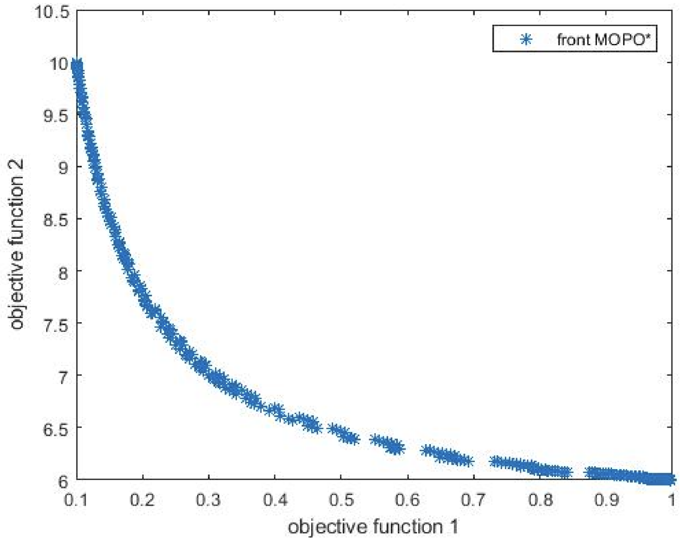

Fig. 1. Min-ex problem's Pareto front

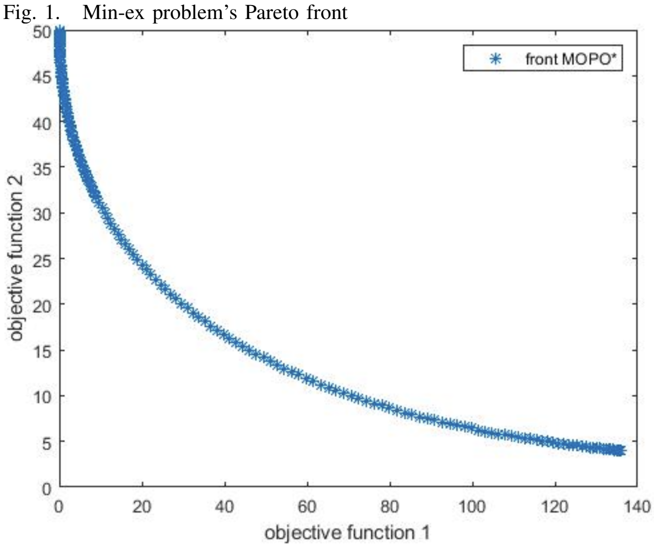

Fig. 2. BNH1 problem's Pareto front

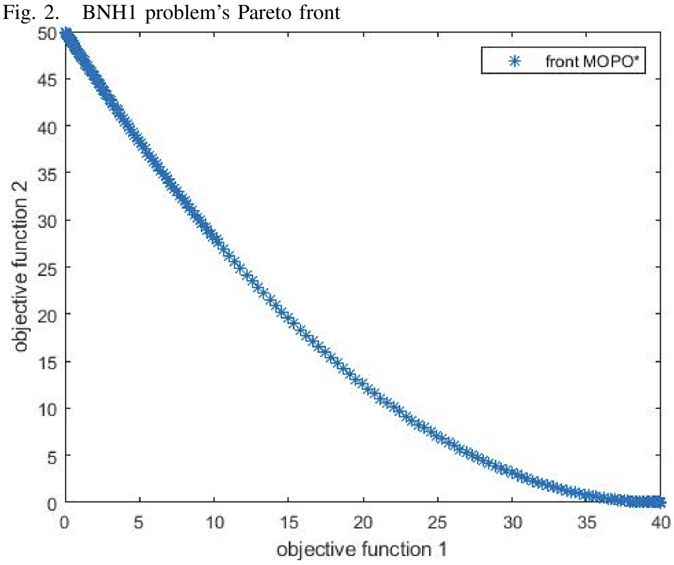

Fig. 3. BNH2 problem's Pareto front

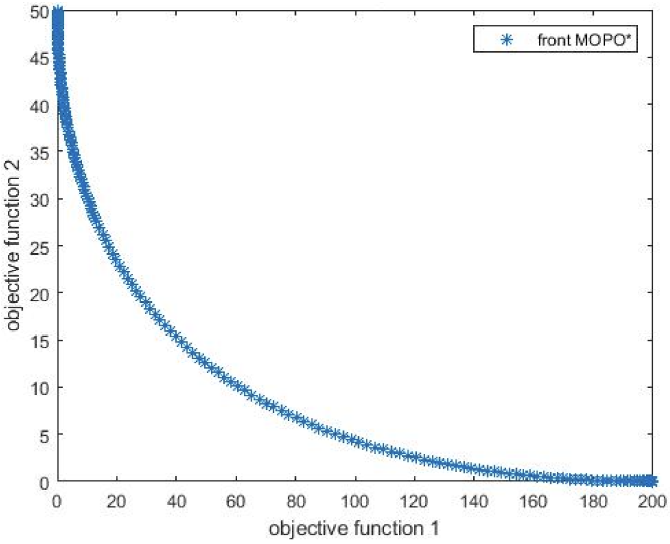

Fig. 4. BNH3 problem's Pareto front

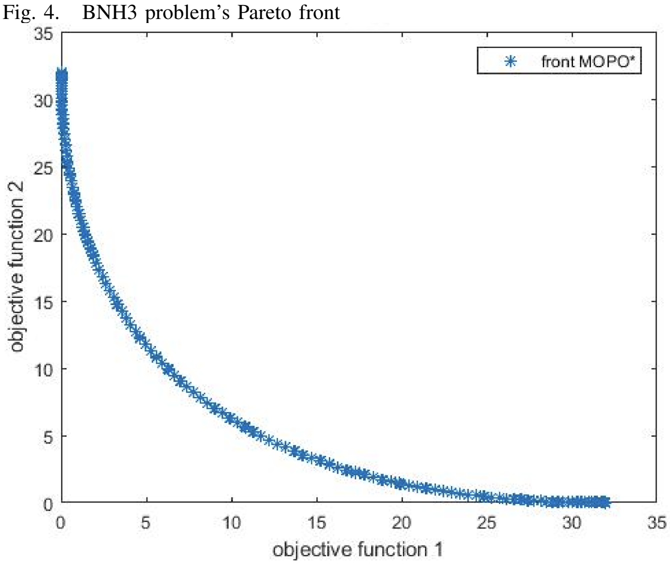

Fig. 5. Lambda1 problem's Pareto front

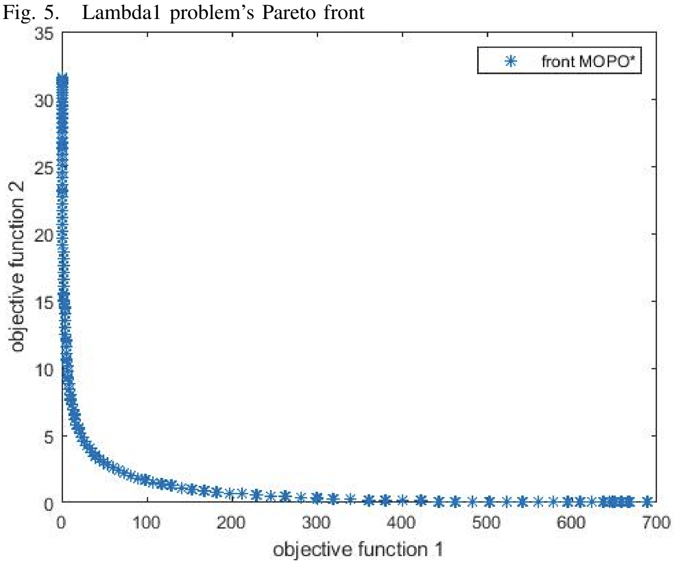

Fig. 6. Lambda2 problem's Pareto front

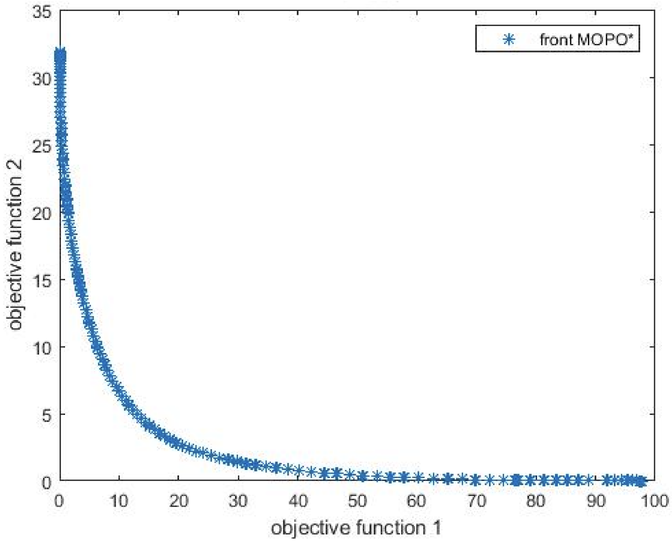

Fig. 7. Lambda3 problem's Pareto front

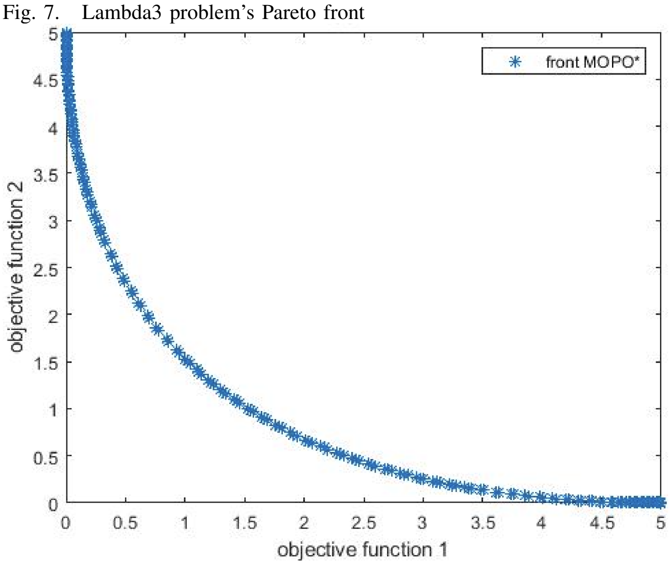

Fig. 8. SSFYY1 problem's Pareto front

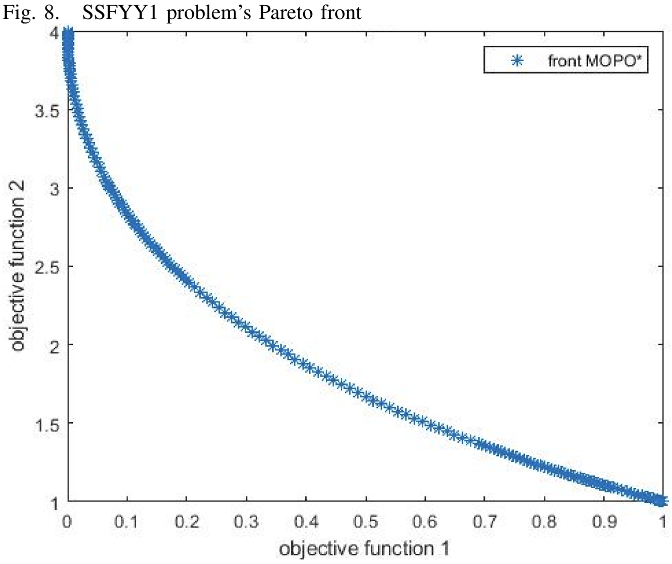

Fig. 9. JOS1 problem's Pareto front

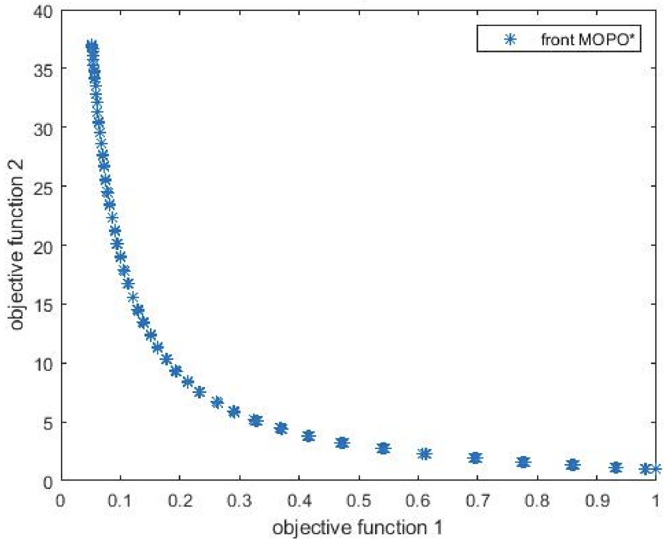

Fig. 10. VU1 problem's Pareto front

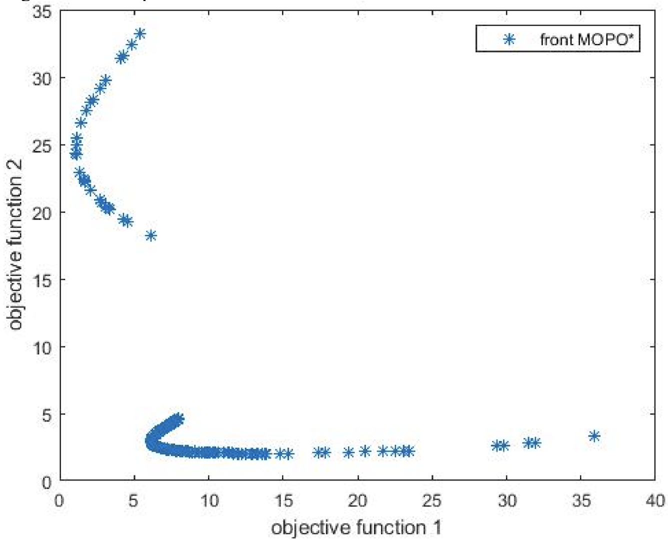

Fig. 11. POL problem's Pareto front

IV. Conclusion

In this paper, we proposed a new algorithm, named M-OPO, for solving multiobjective optimization problems. This algorithm directly transforms multiobjective optimization problems with constraints in multiple variables into single-variable unconstrained multi-objective optimization problems. These transformations rely on penalty functions, the Aliénor transformation, and the application of an OPO. To assess the performance of this novel algorithm, we applied it to 11 established benchmark problems drawn from existing literature. The results demonstrated that M-OPO achieves satisfactory performance in approximating the Pareto front for all the test problems used.

Furthermore, performance indicators such as purity metrics, υ-Spread, and Γ-Spread were applied to the problem results, proving the competitiveness of M-OPO compared to NSGA-II.

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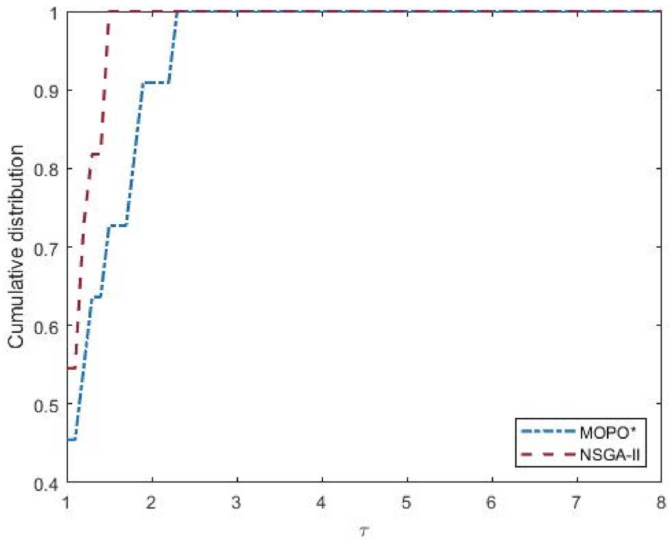

Fig. 12. Performance profil of Purity

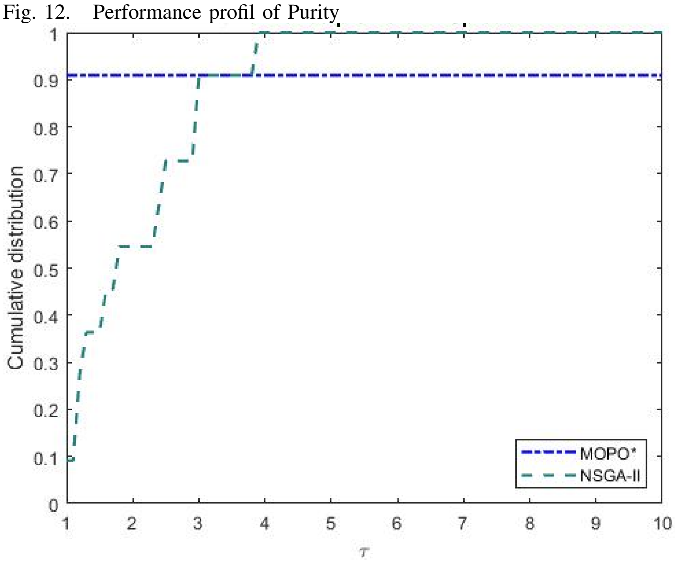

Fig. 13. Performance profil of γ-Spread

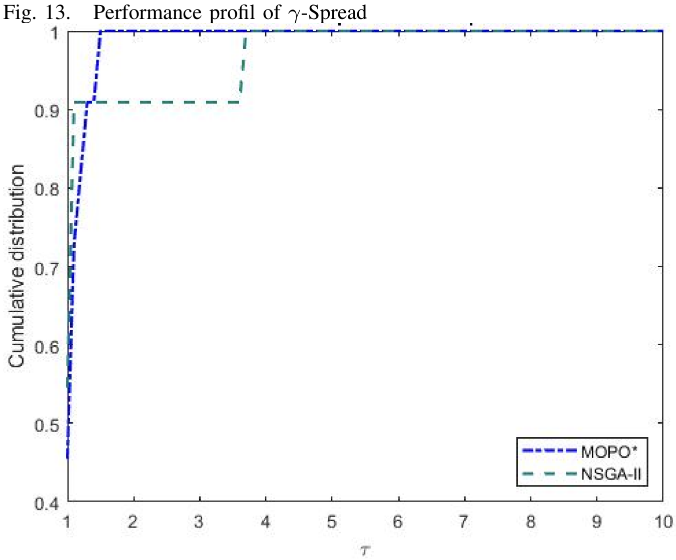

Fig. 14. Performance profil of υ-Spread

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