Minimum Dominating Quotient Ceil Energy of Graph

Prema M., Ruby Salestina M and Purushothama S.

***Abstract*— In this paper, we introduce the concept of minimum dominating quotient ceil energy of graph, denoted by $\overline{Q_D}E(G)$ and compute this energy for several families of graphs. Furthermore, we establish bounds for the minimum dominating quotient ceil energy.**

***Index Terms*: Minimum Dominating Set, Quotient Ceil Energy, Quotient Ceil Matrix, Dominating Quotient Ceil Matrix**

I. INTRODUCTION

Let $G = (V, E)$ be a graph with n nodes and m edges. The degree of v_i written by $d(v_i)$ is the number of edges incident with v_i. The maximum node of degree is denoted by $\Delta(G)$ and the minimum node of degree is denoted by $\delta(G)$. The adjacency matrix $A_D(G)$ of G is defined by its entries as $a_{ij} = 1$ if $v_i v_j \in E(G)\ or\ v_i \in D\ if\ (i = j)$ where D is a dominating set of G and 0 otherwise. The eigenvalues of graph G are the eigenvalues of its adjacency matrix$A_D(G)$, denoted by$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A graph G is considered singular if it has at least one eigenvalue equal to zero. In the case of singular graphs, it is clear that $det(A) = 0$. A graph is considered singular if it has at least one eigenvalue equal to zero. A graph G is referred to be k-regular if every node in G has degree k.

The energy of a graph G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This concept was introduced by I. Gutman in 1978 [3]. Initially, the concept of graph energy went largely unnoticed by mathematicians. However, once its importance was recognized, it became a subject of global mathematical research. Today, similar energy-related quantities are also being studied for other types of matrices.

In this paper, we are defining a matrix, called the minimum dominating quotient ceil matrix, denoted by $\overline{Q_D}E(G)$ and we study its eigenvalues and the energy. Further, we study the mathematical aspects of the minimum dominating quotient ceil energy of a graph. The graphs we are considering are assumed to be finite, simple, undirected having no isolated vertices, and of order at least two.

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For more details in quotient ceil energy of a graph refer [8].

Theorem 1.1 [9] Let a_i and b_i, $1 \leq i \leq p$ be positive real numbers, then

$$\sum_{i=1}^{p} a_i^2 \sum_{i=1}^{p} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{i=1}^{p} a_i b_i\right)^2$$

Where $M_1 = \max_{1\leq i\leq p}(a_i)$, $M_2 = \max_{1\leq i\leq p}(b_i)$, $m_1 = \min_{1\leq i\leq p}(a_i)$ and $m_2 = \min_{1\leq i\leq p}(b_i)$

Theorem 1.2 [10] Let a_i and b_i, $1 \leq i \leq p$ are positive real numbers then

$$\sum_{i=1}^{p} a_i^2 \sum_{i=1}^{p} b_i^2 - \left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \frac{p^2}{4}(M_1 M_2 - m_1 m_2)^2$$

Where, $M_1 = \max_{1\leq i\leq p}(a_i)$, $M_2 = \max_{1\leq i\leq p}(b_i)$, $m_1 = \min_{1\leq i\leq p}(a_i)$ and $m_2 = \min_{1\leq i\leq p}(b_i)$

Theorem 1.3 [10] Let a_i and b_i, $1 \leq i \leq p$ are positive real numbers then

$$\sum_{i=1}^{p} b_i^2 + tT \sum_{i=1}^{p} a_i^2 \leq (t + T)\left(\sum_{i-1}^{p} a_i b_i\right)$$

To prove our results we make use of the following inequalities:

(i) Hardy's inequality:

If $\{x_p\}$ is a sequence of positive real numbers, then for every real number $s > 1$,

$$\sum_{i=1}^{n}\left(\frac{x_1 + x_2 + \cdots + x_p}{p}\right)^2 \leq \left(\frac{s}{s-1}\right)^s \left(\sum_{i=1}^{n} x_i^s\right)$$

(ii) Cauchy's - Schwarz inequality:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

I.I. QUOTIENT ENERGY OF GRAPHS

For a graph G, the quotient matrix $\bar{Q} = \bar{Q}(G) = q_{ij}$ is a $p \times p$ matrix defined as

$$\bar{q}_{ij} = \begin{cases} \left\lceil \frac{d(v_i)}{d(v_j)} \right\rceil, & for \quad v_i v_j \in E \\ 0, & otherwise \end{cases}$$

The characteristic polynomial of $\bar{Q}(G))$ is $f(G,\lambda) = det\ (\bar{Q} - \lambda\, I)$. The quotient spectrum of the graph G is the eigenvalues of the matrix $\bar{Q}$ and is denoted as $\bar{Q} - Spec(G)$.

I.II THE MINIMUM DOMINATING ENERGY OF A GRAPH

Given a simple graph $G = (V, E)$ of order n, where n refers to the number of vertices in the graph $V(G) = \{v_1, v_2, \ldots\ldots, v_n\}$ is the set of vertices. Let $D \subseteq V(G)$ be a subset that is said to dominate the graph G if for every v_i $(1 \le\ i \le\ n)$ in $V - D$ is connected to some v_i in D. Such a set D with minimum number of vertices is known as a minimum dominating set. Corresponding to this we obtain a matrix for a graph G of order $n \times\ n$ and is denoted by $A_D(G)$ whose entries are defined as follows: $a^d{}_{i,j}$

$$= \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1, & \text{if } v_i = v_j \text{ and } v_i \in D \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of the matrix $A_D(G)$, is denoted by $f_n(G, \lambda) = \det(\lambda I - A_D(G))$. The minimum dominating eigenvalues of the graph G are the eigenvalues of $A_D(G)$. Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and are labeled in non-increasing order $\lambda_1 \ge \lambda_2\ \ge \cdots \ge \lambda_n$. The minimum dominating energy [7] of G is defined as

$$E_D(G) = \sum_{i=1}^{n} |\lambda_i|$$

II. MINIMUM DOMINATING QUOTIENT CEIL ENERGY OF GRAPH

Let G be simple graph of order n with node set $V = \{v_1, v_2, \ldots, v_n\}$ edge set E. Let D be the minimum dominating set of a graph G. The minimum dominating quotient ceil matrix of G is the $n \times\ n$ matrix defined by $A_Q(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} \left\lceil \frac{d(v_i)}{d(v_j)} \right\rceil, & if\ v_i v_j \in E \\ 1, & if\ v_i = v_j\ and\ v_i \in D \\ 0, & if\ otherwise \end{cases}$$

The characteristic polynomial of $A_{\bar{Q}}^{D}(G)$ is indicated by $f(G,\lambda) = det\ (\lambda\ I - A_{\bar{Q}}^{D}(G)\ )$. The minimum dominating quotient ceil eigenvalues of the graph G are the eigenvalues of $A_{\bar{Q}}^{D}(G)$. Since $A_{\bar{Q}}^{D}(G)$ is real and symmetric, its eigenvalues are real numbers and are labeled in non-increasing order $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. The minimum dominating quotient ceil energy of G is defined as

$$\overline{Q_D}E(G) = \sum_{i=1}^{n} |\lambda_i|$$

III. MINIMUM DOMINATING QUOTIENT CEIL ENERGY OF SOME STANDARD GRAPHS

Theorem 3.1. If K_n is the complete graph with n vertices then $\overline{Q}_D E(G) = (n-2) + \sqrt{n^2 - 2n + 5}$.

Proof: Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The minimum dominating set $D = \{v_1\}$. Then the minimum dominating quotient ceil matrix of K_n is given by

$$A_{\bar{Q}}^{D}(K_n) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 1 & 1 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1 & 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}$$

Its characteristic polynomial is

$$[\lambda + 1]^{(n-2)}[\lambda^2 - (n-1)\lambda - 1].$$

The minimum dominating quotient Ceil eigenvalues are

$$Spec(K_n) = \begin{pmatrix} -1 & \frac{(n-2)+\sqrt{n^2-2n+5}}{2} & \frac{(n-2)-\sqrt{n^2-2n+5}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

The minimum dominating quotient ceil energy for a complete graph is

$$\bar{Q}_D E(K_n) = |-1|\ (n-2) + \left| \frac{(n-2) + \sqrt{n^2 - 2n + 5}}{2} \right| + \left| \frac{(n-2) - \sqrt{n^2 - 2n + 5}}{2} \right|$$

$$= (n-2) + \sqrt{n^2 - 2n + 5}$$

Theorem 3.2 If $K_{1,n}$ is the star graph with $n+1$ vertices then $\bar{Q}_D E(G) = \sqrt{1 + 4n^2}$

Proof: Let $K_{1,n}$ be the star graph with vertex set $V = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$. The minimum dominating set $D = \{v_1\}$. Then the minimum dominating quotient ceil matrix of K_n is given by

$$A_{\bar{Q}}^{D}(K_{1,n}) = \begin{pmatrix} 1 & n & n & \ldots & n & n \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

Its characteristic polynomial is

$$\lambda^{(n-1)}[\lambda^2-\lambda-n^2].$$

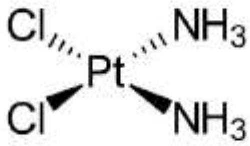
The minimum dominating quotient ceil eigenvalues are $Spec(K_n)$

$$=\begin{pmatrix} 0 & \frac{1+\sqrt{1+4n^2}}{2} & \frac{1-\sqrt{1+4n^2}}{2} \\ n-1 & 1 & 1 \end{pmatrix}$$

The minimum dominating quotient ceil energy for a star graph is

$$\bar{Q}_D E(G) = |\ 0\ |\ (n-2) + \left|\frac{1+\sqrt{1+4n^2}}{2}\right| + \left|\frac{1-\sqrt{1+4n^2}}{2}\right| = \sqrt{1+4n^2}$$

Structural formula: $P_t(NH_3)_2Cl_2$



Theorem 3.3 The minimum dominating quotient ceil energy of $P_t(NH_3)_2Cl_2$ is 8.0622

Proof: The minimum dominating quotient ceil matrix of $P_t(NH_3)_2Cl_2$ is given by

$$A_{\bar{Q}}^{D}(P_t(NH_3)_2Cl_2)=\begin{pmatrix} 0&1&0&0&0\\ 4&1&4&4&4\\ 0&1&0&0&0\\ 0&1&0&0&0\\ 0&1&0&0&0 \end{pmatrix}$$

The characteristic equation is $\lambda^5-\lambda^4-16\lambda^3=0$ and the minimum dominating quotient ceil eigenvalues are $\lambda_1 = 4.5311, \lambda_2 = 0, \lambda_3 = -3.5311, \lambda_4 = 0, \lambda_5 = 0$ Therefore, the minimum dominating ceil energy is $\bar{Q}_D E(P_t(NH_3)_2Cl_2) = 8.0622$.

IV. Bounds on Minimum Dominating Quotient Ceil Energy of Graphs

Proposition 4.1 The first two coefficients of $Q_D(G,\lambda)$ are given as follows:

(i) $a_0 = 1$.

(ii) $a_1 = |\ D\ |$.

Theorem 4.1 Let $G=(V,E)$ be any simple (p,q) graph. If $\lambda_1, \lambda_2, \dots, \lambda_p$ are the minimum dominating quotient ceil eigenvalues of the matrix, then the following conditions hold:

(i) $\sum_{i=1}^{p} \lambda_i = |D|$

(ii) $\sum_{i=1}^{p} {\lambda_i}^2 = |D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil$

Proof: (i) Since sum of the $A_{\bar{Q}}^{D}(G)$ is same as the trace of $A_{\bar{Q}}^{D}(G)$

$$\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} a_{ii} = |\ D\ |$$

(ii) Since the sum of squares of the spectrum of $A_{\bar{Q}}^{D}(G)$ is the trace of $\left(A_{\bar{Q}}^{D}(G)\right)^2$ we get

$$\sum_{i=1}^{p} {\lambda_i}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} \bar{q}_{ij}\bar{q}_{ji} = \sum_{i=1}^{p} (\bar{q}_{ij})^2 + \sum_{j=1}^{p} (\bar{q}_{ji}\bar{q}_{ij})^2$$

$$= 2\sum_{i<j}^{p} (\bar{q}_{ii})^2 + \sum_{i<j}^{p} \bar{q}_{ji}\bar{q}_{ij} + \sum_{i>j}^{p} \bar{q}_{ji}\bar{q}_{ij}$$

$$= |D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil$$

This implies

$$\sum_{i=1}^{p} {\lambda_i}^2 = |D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil.$$

Theorem 4.2 If $\{\lambda_i\}$, $1 \le i \le p$ is a sequence of absolute values of $\bar{Q}$ spectrum and $s=2$, then

$$\bar{Q}_D E(G) \le 2p\sqrt{\left(|D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil\right)\beta}$$

Where $\beta = \sum_{i=1}^{p-1} \left(\frac{\lambda_1+\lambda_2+\cdots+\lambda_i}{p}\right)^2$

Proof: Let $\lambda_i, 1 \le i \le p$ denote the sequence the of $\bar{Q}_D$ −spectrum of G. Then, using theorem 1.1, we get

$$\sum_{i=1}^{p} \left(\frac{\lambda_1+\lambda_2+\cdots+\lambda_i}{p}\right)^2 \le 4\sum_{i=1}^{p} {\lambda_i}^2$$

This implies

$$\sum_{i=1}^{p-1} \left(\frac{\lambda_1+\lambda_2+\cdots+\lambda_i}{p}\right)^2 + \left(\frac{\lambda_1+\lambda_2+\cdots+\lambda_i}{p}\right)^2 \le 4\left(|D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil\right)$$

Put

$\beta = \sum_{i=1}^{p-1} \left(\frac{\lambda_1+\lambda_2+\cdots+\lambda_i}{p}\right)^2$ Then

$$\beta + \left(\frac{\bar{Q}_D E(G)}{p}\right)^2 \le 4\left(|D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil\right)$$

$$\left(\frac{\bar{Q}_D E(G)}{p}\right)^2 \le 4\left(|D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil\right)\beta$$

Simplifying this, we get

$\bar{Q}_D E(G) \le 2p\sqrt{\left(|D| + 2\sum_{i<j}^{p} \left\lceil \frac{d_i}{d_j} \right\rceil\right)\beta}$

Theorem 4.3 For a graph$G$, let $\Delta = |\ \bar{Q}(G)\ |$. Then

$$\sqrt{|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|+p(p-1)\Delta^{\frac{2}{p}}} \le \bar{Q}_D E(G)$$
$$\le \sqrt{p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)}$$

Proof: Put $a_i = 1, b_i = |\lambda_i|$ in Cauchy's - Schwarz inequality then,

$$\left(\sum_{i=1}^{p}|\lambda_i|\right)^2 \le \left(\sum_{i=1}^{p}1\right)\left(\sum_{i=1}^{p}|\lambda_i|^2\right)$$

This implies

$$\left(\bar{Q}_D E(G)\right)^2 \le p\sum_{i=1}^{p}|\lambda_i|^2 = p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)$$

Therefore,

$$\bar{Q}_D E(G) \le \sqrt{p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)} --(4.3)$$

Consider

$$\left(\bar{Q}_D E(G)\right)^2 = \sum_{i=1}^{p}|\lambda_i|^2$$
$$= \sum_{i=1}^{p}|\lambda_i|^2 + \sum_{i\neq j}|\lambda_i|\,|\lambda_j|$$
$$= \left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right) + \sum_{i\neq j}|\lambda_i|\,|\lambda_j| --- (4.3.1)$$

Where $\sum_{i=1}^{p}|\lambda_i|^2 = \left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)$

Since the geometric mean cannot exceed the arithmetic mean, we have

$$\frac{\sum_{i\neq j}|\lambda_i|\,|\lambda_j|}{p(p-1)} \ge \left[\prod_{i\neq j}|\lambda_i|\,|\lambda_j|\right]^{\frac{1}{p(p-1)}}$$

$$i.e., \left[\prod_{i=1}^{p}|\lambda_i|^{2(p-1)}\right]^{\frac{1}{p(p-1)}} = \left[\prod_{i=1}^{p}|\lambda_i|\right]^{\frac{2}{p}} = \left|\det\left(E_{\bar{Q}}^{D}(G)\right)^2\right|^{\frac{2}{p}}$$
$$= \Delta^{\frac{2}{p}}$$

Which implies

$$\sum_{i\neq j}|\lambda_i|\,|\lambda_j| \ge p(p-1)\Delta^{\frac{2}{p}} --(4.3.2)$$

Put (4.3.2) in (4.3.1) to get

$$\left(\bar{Q}_D E(G)\right)^2 \ge |D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + p(p-1)\Delta^{\frac{2}{p}} --(4.3.3)$$

$$\bar{Q}_D E(G) \ge \sqrt{|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|+p(p-1)\Delta^{\frac{2}{p}}}$$

From (4.3) and (4.3.3) we get

$$\sqrt{|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|+p(p-1)\Delta^{\frac{2}{p}}} \le \bar{Q}_D E(G)$$
$$\le \sqrt{p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)}$$

Theorem 4.4. For any (p,q) connected graph G

$$\sqrt{|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|} \le \bar{Q}_D E(G) \le \sqrt{p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)}$$

Proof: Put $a_i = 1, b_i = |\lambda_i|$ Cauchy's - Schwarz inequality then,

$$\left(\sum_{i=1}^{p}|\lambda_i|\right)^2 \le \left(\sum_{i=1}^{p}1\right)\left(\sum_{i=1}^{p}|\lambda_i|^2\right)$$

This implies

$$\left(\bar{Q}_D E(G)\right)^2 \le p\sum_{i=1}^{p}|\lambda_i|^2$$
$$= p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right) --(4.4.1)$$

Therefore,

$$\left(\bar{Q}_D E(G)\right)^2 \le p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)$$
$$\left(\bar{Q}_D E(G)\right) \le \sqrt{p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)}$$

Consider

$\left(\bar{Q}_D E(G)\right)^2 = \left(\sum_{i=1}^{p}|\lambda_i|\right)^2 \ge \sum_{i=1}^{p}\lambda_i^2 = \left(|D| + 2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right) --(4.4.2)$

Which implies

$$\bar{Q}_D E(G) \ge \sqrt{\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)}$$

From (4.4.1) and (4.4.2) we get

$$\sqrt{|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|} \le \bar{Q}_D E(G) \le \sqrt{p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right)}$$

Theorem 4.5 Let G be a (p,q) graph. Then

$$\bar{Q}_D E(G) \ge \sqrt{p\left(|D|+2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right) - \frac{p^2}{4}\left(\lambda_1-\lambda_p\right)^2}$$

where λ_1 and λ_p are the maximum and minimum values of $|\lambda_i|$'s

Proof: Let $\lambda_1, \lambda_2, \dots \lambda_p$ be the $\bar{Q}$ spectrum of G. Put $a_i = 1, b_i = |\lambda_i|$, in Theorem 1.2 we get

$$\sum_{i=1}^{p} 1^2 \sum_{i=1}^{p} \lambda_i^2 - \sum_{i=1}^{p} |\lambda_i|^2 \leq \frac{p^2}{4}\left(\lambda_1 - \lambda_p\right)^2$$

This implies

$$p\left(|D| + 2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right) - \left(\bar{Q}_D E(G)\right)^2 \leq \frac{p^2}{4}\left(\lambda_1 - \lambda_p\right)^2$$

i.e., $\bar{Q}_D E(G) \geq \sqrt{p\left(|D| + 2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right|\right) - \frac{p^2}{4}\left(\lambda_1 - \lambda_p\right)^2}$

Theorem 4.6 Let G be a (p,q) simple graph. Let Δ be the absolute value of the determinant of the quotient matrix $\bar{Q}$ of G. Then,

$$\sqrt{2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + |D| + p(p-1)\Delta^{\frac{2}{p}}} \leq \bar{Q}_D E(G)$$

$$\leq \sqrt{\left(2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + |D|\right)(p-1) + p\Delta^{\frac{2}{p}}}$$

Proof: Let $Z = p\left[\frac{1}{p}\sum_{i=1}^{p}\lambda_i^2 - \left(\prod_{i=1}^{p}\lambda_i^2\right)^{\frac{1}{p}}\right]$

$$= p\left[\frac{1}{p}\left(2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + |D|\right) - \left(\prod_{i=1}^{p}|\lambda_i|\right)^{\frac{2}{p}}\right]$$

$$= 2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + |D| - p\Delta^{\frac{2}{p}}$$

if $a_i = \lambda_i^2, i = 1,2,\dots,p$ then

$$Z \leq p\sum_{i=1}^{p}\lambda_i^2 - \left(\sum_{i=1}^{p}|\lambda_i|\right)^2 \leq (p-1)Z$$

Which implies

$$Z \leq p\left(2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + |D|\right) - \left(\bar{Q}_D E(G)\right)^2 \leq (p-1)Z$$

On simplifying, we get

$$\sqrt{2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + |D| + p(p-1)\Delta^{\frac{2}{p}}} \leq \bar{Q}_D E(G)$$

$$\leq \sqrt{\left(2\sum_{i<j}^{p}\left|\frac{d_i}{d_j}\right| + |D|\right)(p-1) + p\Delta^{\frac{2}{p}}}$$

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