Synthesis of Fuzzy Control for Inverter Pendulum Robot with H_∞ Performance Constraint

Wen-Jer Chang, Wei-Han Huang and Wei Chang

***Abstract*—The purpose of this paper is to give one set of stability and stabilization conditions for an inverted pendulum robot which simulating human stance under the framework of Discrete Perturbed Time-Delay Affine (DPTDA) Takagi–Sugeno (T–S) fuzzy models. In the beginning, the mathematical model of inverted pendulum robot system and the corresponding DPTDA T–S fuzzy model are introduced. Next, some sufficient conditions are derived on robust H_∞ disturbance attenuation, in which the robust stability and prescribed performance are achieved. In order to find suitable fuzzy controllers, the Iterative Linear Matrix Inequality (ILMI) algorithm is employed to solve these sufficient conditions. Finally, a numerical simulation for the nonlinear inverted pendulum robot system is given to show the applications of the presented controller design approach.**

***Index Terms*—Inverted Pendulum Robot, Takagi-Sugeno Fuzzy Model, Time-delay, *S*-procedure, Iterative Linear Matrix Inequality.**

I. Introduction

Human being stance has been investigated in detail for a long time [1]. In recent years, the researchers wish to simulate human stance on the machine. In this paper, the model is constructed based on purely inverted pendulum dynamics and on a movable supportive base. This work was based on the assumption that the act of maintaining an erect posture could be viewed. However, the problems often are a complicated nonlinear system. In general, the methods of linear control and those of local linearization and moving linearization are not well suited for the control problem of inverted pendulums. This is due to the fact that inverted pendulums constantly move among widely separated regions of their workspace such that no linearization valid for all regions can be found. In fact, in many practical systems, the system plants contain severe nonlinear properties. Therefore, many researchers have studied to solve the difficulties of nonlinear control methods. One of them is the fuzzy logic control [2-4]. It is a successful control approach to many complex nonlinear systems or even non-analytic systems [5-6]. Some remarkable studies on the stabilizing controller design for fuzzy systems can be referred to [7-10], in which the so-called T–S fuzzy model [11] has mainly been used to represent fuzzy systems.

Generally, the T–S fuzzy models can be separated into the homogeneous fuzzy models [7-11] and the affine fuzzy models [12-15]. The homogeneous one can be referred to the T–S fuzzy model of which consequent part is linear without constant bias term. In contrast, the affine one means the T–S fuzzy model of which consequent part is affined by a constant bias term. In general, the affine T–S fuzzy model can preserve diverse nonlinear systems more than the homogeneous one. However, the analysis and synthesis of affine T–S fuzzy model are more difficult than the homogeneous one.

Due to there are few studies dealing with the H_∞ constraints and time delay effects for the DPTDA T–S fuzzy models. Hence, one major target presented in this paper is the robust stability for the affine T-S fuzzy models. In this paper, the issue of robust stability is proposed in the presence of norm-bounded uncertainty. The class of perturbed affine T–S fuzzy models is defined by a state-space model and time-varying norm-bounded parameter uncertainties. Moreover, the H_∞ control scheme [16] is used in this paper to attack the problem of robust performance design problems for the perturbed affine T–S fuzzy models. Besides, the presence of time delays in control loops usually degrades system performance and is even a source of instability [17].

The other target presented in this paper which we have to pay attention is the fuzzy controller design problem under the framework of LMI method [18]. The fuzzy controller design of the DPTDA T–S fuzzy models is a challenging problem for the designers because the closed-loop stability conditions are not LMI formulations but Bilinear Matrix Inequalities (BMI) ones. The BMI conditions can not be easily solved via a convex optimization algorithm. For this reason, an ILMI algorithm [12, 14-15] has been presented to solve the BMI problem. In this paper, an ILMI algorithm is developed to find feasible solutions for the synthesis problem of fuzzy controller design for the DPTDA T–S fuzzy models. Finally, in order to illustrate the applications of proposed fuzzy controller design approach for the inverted pendulum robot system, a simulation is provided in this paper.

II. System Descriptions and Problem Formulations

A. Inverted pendulum robot system

In this section, the mathematical model of the simple inverted pendulum robot system is introduced. Referring to Fig. 1, a simplified dynamic model for describing inverted pendulum robot system to simulate human stance is proposed as follows [4].

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$$x_1(\mathrm{k}+1) = x_1(\mathrm{k}) + T\,x_2(\mathrm{k}) + \mathrm{e}\,v(\mathrm{k}) \tag{1a}$$

$$\begin{aligned} x_2(\mathrm{k}+1) = x_2(\mathrm{k}) &+ \frac{T}{M + m\left(\sin x_2(\mathrm{k})\right)^2} \\ &\times\Big(u(\mathrm{k}) + ml\,x_4^2(\mathrm{k})\sin x_3(\mathrm{k}) - \mathrm{b}\,x_2 \\ &- mg\cos x_3(\mathrm{k})\sin x_3(\mathrm{k})\Big) \end{aligned} \tag{1b}$$

$$x_3(\mathrm{k}+1) = x_3(\mathrm{k}) + T\,x_4(\mathrm{k}) \tag{1c}$$

$$\begin{aligned} x_4(\mathrm{k}+1) = x_4(\mathrm{k}) &+ \frac{T}{l\Big(M + m\left(\sin x_3(\mathrm{k})\right)^2\Big)} \\ &\times\Big((M+m)g\sin x_3(\mathrm{k}) - u(\mathrm{k})\cos x_3(\mathrm{k}) \\ &+ b\,x_2(\mathrm{k})\cos x_3(\mathrm{k}) \\ &- ml\,x_4^{\,2}(\mathrm{k})\sin x_3(\mathrm{k})\cos x_3(\mathrm{k})\Big) \end{aligned} \tag{1d}$$

where

m is the mass of the black on the pendulum.

l is length of the pendulum.

g is acceleration due to gravity.

b is coefficient of viscous friction for motion of the cart.

u is applied force.

$v(\mathrm{t})$ is the denotes the disturbances.

The four state variables stand for $x_1 = x$, $x_2 = \dot{x}$, $x_2 = \theta$, $x_4 = \dot{\theta}$ with the position of the cart, and θ the angle the pendulum makes with vertical. This model is obtained by discretizing the continuous time model via Euler's method with T is 0.1s, $b = 12.98\,\mathrm{kg}/\mathrm{s}$, $M = 1.378$ kg, $l = 0.325$ m, $g = 9.8\,\mathrm{m}/\mathrm{s}^2$, $m = 0.051$ kg.

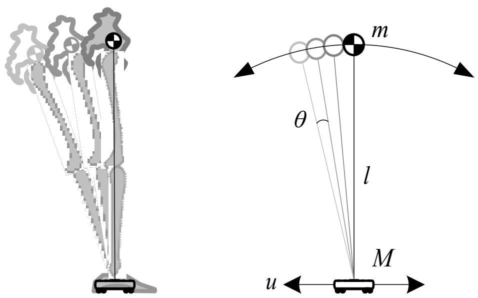

Fig. 1 Inverted pendulum robot system to simulate human stance

Considering premise nominal parameter uncertainties, the modified dynamic model for the inverted pendulum robot system can be described as follows:

$$x_1(\mathrm{k}+1) = \left(0.1\cos(\mathrm{t})+1\right)\sigma(\mathrm{k}) + 0.1x_2(\mathrm{k}) + 0.2\,v(\mathrm{k}) \tag{2a}$$

$$\begin{aligned} x_2(\mathrm{k}+1) = x_2(\mathrm{k}) &+ \frac{0.1}{1.378 + 0.051\left(\sin x_3(\mathrm{k})\right)^2} \\ &\times\Big(u(\mathrm{k}) - 12.98\,x_2 + 0.0166\,x_4^{\,2}(\mathrm{k})\sin x_3(\mathrm{k}) \\ &- 0.4998\cos x_3(\mathrm{k})\sin x_3(\mathrm{k})\Big) \end{aligned} \tag{2b}$$

$$x_3(\mathrm{k}+1) = x_3(\mathrm{k}) + 0.1x_4(\mathrm{k}) \tag{2c}$$

$$\begin{aligned} x_4(\mathrm{k}+1) = x_4(\mathrm{k}) &+ \frac{0.1}{0.4478\Big(0.0166\left(\sin x_3(\mathrm{k})\right)^2\Big)} \\ &\times\Big(13.4162\sin x_3(\mathrm{k}) - u(\mathrm{k})\cos x_3(\mathrm{k}) \\ &+ 12.98\,x_2(\mathrm{k})\cos x_3(\mathrm{k}) \\ &- 0.0166\,x_4^{\,2}(\mathrm{k})\sin x_3(\mathrm{k})\cos x_3(\mathrm{k})\Big) \end{aligned} \tag{2d}$$

where $\sigma(\mathrm{k}) = \rho x_1(\mathrm{k}) + (1-\rho)x_1(\mathrm{k}-\tau)$ and $\sigma(\mathrm{k})$ is a time-delay function.

B. DPTDA T–S fuzzy model

Based on the nonlinear inverted pendulum robot system (2) presented above, the stability analysis and fuzzy controller design problems for the nonlinear system (2) via T-S fuzzy model are introduced. Consider the DPTDA T–S fuzzy model described by the following IF-THEN rules.

Rule i: IF $z_1(\mathrm{k})$ is $\mathrm{M}_{\mathrm{i}1}$ and $z_2(\mathrm{k})$ is $\mathrm{M}_{\mathrm{i}2}$ and $\cdots$ and $z_{\mathrm{p}}(\mathrm{k})$ is M_{ip} THEN

$$\begin{aligned} x(\mathrm{k}+1) &= \left(\mathbf{A}_{\mathrm{i}} + \Delta\mathbf{A}_{\mathrm{i}}\right)x(\mathrm{k}) + \left(\mathbf{A}_{\mathrm{id}} + \Delta\mathbf{A}_{\mathrm{id}}\right)x(\mathrm{k}-\tau) \\ &\quad + \left(\mathbf{B}_{\mathrm{i}} + \Delta\mathbf{B}_{\mathrm{i}}\right)u(\mathrm{k}) + \left(\mathbf{a}_{\mathrm{i}} + \Delta\mathbf{a}_{\mathrm{i}}\right) + \mathbf{E}\,v(\mathrm{k}), \\ &\quad \mathrm{i} = 1,2,\ldots,\mathrm{r},\ x(\mathrm{k}) \in \mathbf{X}_{\mathrm{i}},\ \mathrm{i} \in \hat{\mathrm{I}} \\ x(\mathrm{k}) &= \psi(\mathrm{k}),\ \text{for } \mathrm{k} \in \left[-\bar{\tau},\ 0\right] \end{aligned} \tag{3}$$

where $\mathbf{A}_{\mathrm{i}} \in \Re^{\mathrm{n}\times\mathrm{n}}$, $\mathbf{A}_{\mathrm{id}} \in \Re^{\mathrm{n}\times\mathrm{n}}$, $\mathbf{B}_{\mathrm{i}} \in \Re^{\mathrm{n}\times\mathrm{m}}$, $\mathbf{a}_{\mathrm{i}} \in \Re^{\mathrm{n}}$ and $\mathbf{E} \in \Re^{\mathrm{n}}$ are constant matrices, $z_1(\mathrm{k}), \ldots, z_{\mathrm{p}}(\mathrm{k})$ are known premise variables that may be functions of the state variables, p is the premise variable number and r is the number of fuzzy model rules. $x(\mathrm{k}) \in \Re^{\mathrm{n}}$ is the state vector, $\mathrm{u}(\mathrm{k}) \in \Re^{\mathrm{m}}$ is the input vector, $v(\mathrm{k}) \in \Re$ denotes the disturbance which belongs to $L_2[0,\ \mathrm{k_f}]$, where $L_2[0,\ \mathrm{k_f}]$ denotes the Lebsegue space consists of square-integrable functions on the interval $[0,\ \mathrm{k_f}]$ and $\mathrm{k_f}$ is the terminal time of the control. M_{ip} is the fuzzy set, τ is the constant time delay in the state and $\tau > 0$. $\psi(\mathrm{k})$ is the initial condition of the state defined on $-\tau \le \mathrm{k} \le 0$. Besides, the region $\mathbf{X}_{\mathrm{i}} \subseteq \Re^{\mathrm{n}}$ is assumed to be a fuzzy subspace and $\mathbf{X}_{\mathrm{i}}$ is called as a cell. The set of cell indices is denoted as $\hat{\mathrm{I}}$ and the union of all cells $x(\mathrm{k}) = conv\left(\cup_{\mathrm{i}\in\hat{\mathrm{I}}}\mathbf{X}_{\mathrm{i}}\right)$ is referred to as the whole fuzzy space, where $conv(\bullet)$ refers to the convex combination. Let $\hat{\mathrm{I}}_0 \subseteq \hat{\mathrm{I}}$ be the set of indices for the fuzzy rules that contain the origin and $\hat{\mathrm{I}}_1 \subseteq \hat{\mathrm{I}}$ be the set of indices for the fuzzy rules that does not contain the origin. The origin is an equilibrium point of the DPTDA T–S fuzzy models and it is assumed that $\mathbf{a}_{\mathrm{i}} = 0$ for $\mathrm{i} \in \hat{\mathrm{I}}_0$. Besides, $\Delta\mathbf{A}_{\mathrm{i}}$, $\Delta\mathbf{A}_{\mathrm{id}}$, $\Delta\mathbf{B}_{\mathrm{i}}$ and $\Delta\mathbf{a}_{\mathrm{i}}$ are time-varying matrices with appropriate dimensions and they are structured in the following norm-bounded form:

$$\left[\Delta\mathbf{A}_{\mathrm{i}} \quad \Delta\mathbf{A}_{\mathrm{id}} \quad \Delta\mathbf{B}_{\mathrm{i}} \quad \Delta\mathbf{a}_{\mathrm{i}}\right]$$

$$=\mathbf{D}\mathbf{\Delta}(t)\left[\mathbf{Q}_{1i}\quad \mathbf{Q}_{2i}\quad \mathbf{Q}_{3i}\quad \mathbf{Q}_{4i}\right] \qquad (4)$$

where $\mathbf{D}$, $\mathbf{Q}_{1i}$, $\mathbf{Q}_{2i}$, $\mathbf{Q}_{3i}$ and $\mathbf{Q}_{4i}$ are known real constant matrices of appropriate dimensions, and $\mathbf{\Delta}_i(t)$ is an unknown matrix function with Lebesgue-measurable elements and satisfies $\mathbf{\Delta}_i^T(k)\mathbf{\Delta}_i(k)\le \mathbf{I}$.

Lemma 1 [19]:

Let $\mathbf{\Gamma}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{Q}}$ and $\mathbf{\Delta}_i(k)$ be real matrices of appropriate dimensions with $\mathbf{\Delta}_i(k)^T\mathbf{\Delta}_i(k)\le\mathbf{I}$. Then for $\mathbf{\Xi}>0$ any scalar $\varepsilon>0$ satisfying $\varepsilon\mathbf{I}-\tilde{\mathbf{D}}^T\mathbf{\Xi}\tilde{\mathbf{D}}>0$, one has

$$\left(\mathbf{\Gamma}+\tilde{\mathbf{D}}\mathbf{\Delta}_i(k)\tilde{\mathbf{Q}}\right)^T\mathbf{\Xi}\left(\mathbf{\Gamma}+\tilde{\mathbf{D}}\mathbf{\Delta}_i(k)\tilde{\mathbf{Q}}\right)$$
$$\le \mathbf{\Gamma}^T\mathbf{\Xi}\mathbf{\Gamma}+\mathbf{\Gamma}^T\mathbf{\Xi}\tilde{\mathbf{D}}\left(\varepsilon\mathbf{I}-\tilde{\mathbf{D}}^T\mathbf{\Xi}\tilde{\mathbf{D}}\right)^{-1}\tilde{\mathbf{D}}^T\mathbf{\Xi}\mathbf{\Gamma}+\varepsilon\tilde{\mathbf{Q}}^T\tilde{\mathbf{Q}} \qquad (5)$$

□

Given a pair of $(x(t),\ u(t))$, the final outputs of the DPTDA T–S fuzzy model (3) are inferred as follows:

$$x(k+1)=$$
$$\sum_{i=1}^{r}\omega_i(z(k))\{(\mathbf{A}_i+\Delta\mathbf{A}_i)x(k)+(\mathbf{A}_{id}+\Delta\mathbf{A}_{id})x(k-\tau)$$
$$+(\mathbf{B}_i+\mathbf{\Delta B}_i)u(k)+(\mathbf{a}_i+\mathbf{\Delta a}_i)\}\Big/\sum_{i=1}^{r}\omega_i(z(k))+\mathbf{E}\,v(k)$$
$$=\sum_{i=1}^{r}h_i(z(k))\{(\mathbf{A}_i+\mathbf{\Delta A}_i)x(k)+(\mathbf{A}_{id}+\mathbf{\Delta A}_{id})x(k-\tau)$$
$$+(\mathbf{B}_i+\mathbf{\Delta B}_i)u(k)+(\mathbf{a}_i+\mathbf{\Delta a}_i)\}+\mathbf{E}\,v(k) \qquad (6)$$

where

$$x(k)=\left[x_1(k),\ x_2(k),\ \ldots,\ x_n(k)\right]^T,$$
$$\omega_i(x(k))=\prod_{j=1}^{n}M_{ij}(x_j(k)),$$
$$h_i(x(k))=\frac{\omega_i(x(k))}{\sum_{i=1}^{r}\omega_i(x(k))},\ h_i(x(k))\ge 0$$
$$\text{and }\sum_{i=1}^{r}h_i(x(k))=1 \qquad (7)$$

The PDC [8] offers a scheme to design a fuzzy controller from the given T–S fuzzy model (6). The PDC type fuzzy controller has the following form:

Rule i: IF $z_1(k)$ is M_{i1} and $z_2(k)$ is M_{i2} and $\cdots$ and $z_p(k)$ is M_{ip} THEN

$$u(k)=-\mathbf{F}_i\,x(k),\ i=1,2,\ldots,r\quad \text{for } x(k)\in\mathbf{X}_i\ ,\ i\in\hat{I} \qquad (8)$$

where $\mathbf{F}_i\in\Re^{m\times n}$ are constant matrices. The output of the PDC type fuzzy controller is determined by the following summation:

$$u(k)=-\sum_{i=1}^{r}h_i(z(k))\{\mathbf{F}_i\,x(k)\} \qquad (9)$$

Substituting (9) into (6), one can obtain corresponding closed-loop system as follows:

$$x(k+1)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(k))h_j(z(k))$$
$$\times\left\{\left(\mathbf{H}_{1ij}+\mathbf{D}\Delta(k)\overline{\mathbf{H}}_{1ij}\right)x(k)+\left(\mathbf{H}_{2ij}+\mathbf{D}\Delta(k)\overline{\mathbf{H}}_{2ij}\right)x(t-\tau(k))\right.$$
$$\left.+\left(\mathbf{H}_{3ij}+\mathbf{D}\Delta(k)\overline{\mathbf{H}}_{3ij}\right)\right\}+\mathbf{E}\,v(k) \qquad (10)$$

where

$$\mathbf{H}_{ij}=\frac{\mathbf{G}_{ij}+\mathbf{G}_{ji}}{2},\ \mathbf{H}_{2ij}=\frac{\mathbf{A}_{id}+\mathbf{A}_{jd}}{2},\ \mathbf{H}_{3ij}=\frac{\mathbf{a}_i+\mathbf{a}_j}{2},$$
$$\overline{\mathbf{H}}_{1ij}=\frac{\overline{\mathbf{G}}_{ij}+\overline{\mathbf{G}}_{ji}}{2},\ \overline{\mathbf{H}}_{2ij}=\frac{\mathbf{Q}_{2i}+\mathbf{Q}_{2j}}{2},\ \overline{\mathbf{H}}_{3ij}=\frac{\mathbf{Q}_{4i}+\mathbf{Q}_{4j}}{2},$$
$$\mathbf{G}_{ij}=\mathbf{A}_i-\mathbf{B}_i\mathbf{F}_j \text{ and } \overline{\mathbf{G}}_{ij}=\mathbf{Q}_{1i}-\mathbf{Q}_{3i}\mathbf{F}_j \qquad (11)$$

For the more, in order to deal with the robust performance design problems, we have

Definition 1 (H_∞ Performance Constraint)

Given a positive real number γ, the model of the form (10) is said to have $L_2[0,\ k_f]$ gain less than γ if

$$\sum_{k=0}^{k_f}x^T(k)\mathbf{S}\,x(k)<\gamma^2\sum_{k=0}^{k_f}v^T(k)v(k),\ \forall v(k)\ne 0 \qquad (12)$$

with zero initial condition for all $v(k)\in L_2[0,\ k_f]$, where k_f is the terminal time of the control, γ is a prescribed value which denotes the worst case effect of $v(k)$ on $x(k)$. Besides, $\mathbf{S}=\mathbf{S}^T>0$ is a positive definite weighting matrix and $\mathbf{S}\in\Re^{n\times n}$.

□

The purpose of this paper is to find a fuzzy controller (9) such that the closed-loop system (10) is quadratically stable which satisfy the H_∞ constraint (12). In next section, we analyze the quadratically stable conditions for DPTDA T–S fuzzy model (10) firstly. According to these stability conditions, a fuzzy controller is developed via ILMI algorithm in section IV.

III. Stability Analysis for DPTDA T-S Fuzzy Model

Stability analysis for a closed-loop DPTDA T–S fuzzy model (10) is discussed in this section. It is shown that the stability analysis issue to closed-loop DPTDA T–S fuzzy models is considered based on Lyapunov stability criterion and Razumikhin theorem. The sufficient condition for guaranteeing the closed-loop stability is introduced in the following theorem.

Theorem 1

Given a H_∞ attenuation parameter $\gamma > 0$. The DPTDA T–S fuzzy model described in (10) is quadratically stable in the large and the H_∞ control performance (12) is guaranteed for an attenuation γ, if there exist positive definite matrices $\mathbf{P} > 0$, $\mathbf{S} > 0$, $\mathbf{P}_d > 0$, control gains $\mathbf{F}_i$ and scalars $\xi_{ijq} \geq 0$ such that

$$\Upsilon_{ij} < 0 \qquad \text{for } i \in \hat{I}_0 \tag{13}$$

and

$$\bar{\Upsilon}_{ij} - \sum_{q=1}^{n} \xi_{ijq} \mathbf{\Omega}_{ijq}(s) < 0 \qquad \text{for } i \in \hat{I}_1 \tag{14}$$

where

$$\Upsilon_{ij} \triangleq \left\{ \boldsymbol{\varphi} + \begin{bmatrix} \mathbf{H}_{ij}^{T} \\ \mathbf{H}_{2ij}^{T} \\ \mathbf{E}^{T} \end{bmatrix} \mathbf{PD}\left(\varepsilon I - \mathbf{D}^{T}\mathbf{PD}\right)^{-1} \mathbf{D}^{T}\mathbf{P}\begin{bmatrix} \mathbf{H}_{ij} & \mathbf{H}_{2ij} & \mathbf{E} \end{bmatrix} \right\} \tag{15a}$$

$$\boldsymbol{\varphi} = \begin{bmatrix} \mathbf{H}_{1ij}^{T}\mathbf{P}\mathbf{H}_{1ij} - \mathbf{P} + \mathbf{P}_d + \mathbf{S} + \bar{\mathbf{H}}_{1ij}^{T}\varepsilon\bar{\mathbf{H}}_{1ij} & * & * \\ \mathbf{H}_{2ij}^{T}\mathbf{P}\mathbf{H}_{1ij} + \bar{\mathbf{H}}_{2ij}^{T}\varepsilon\bar{\mathbf{H}}_{1ij} & \mathbf{H}_{2ij}^{T}\mathbf{P}\mathbf{H}_{2ij} - \mathbf{P}_d + \bar{\mathbf{H}}_{2ij}^{T}\varepsilon\bar{\mathbf{H}}_{2ij} & * \\ \mathbf{E}^{T}\mathbf{P}\mathbf{H}_{1ij} & \mathbf{E}^{T}\mathbf{P}\mathbf{H}_{2ij} & \mathbf{E}^{T}\mathbf{P}\mathbf{E} - \gamma^{2}\mathbf{I} \end{bmatrix} \tag{15b}$$

$$\bar{\Upsilon}_{ij} \triangleq \left\{ \bar{\boldsymbol{\varphi}} + \begin{bmatrix} \mathbf{H}_{ij}^{T} \\ \mathbf{H}_{2ij}^{T} \\ \mathbf{E}^{T} \\ \mathbf{H}_{3ij}^{T} \end{bmatrix} \mathbf{PD}\left(\varepsilon I - \mathbf{D}^{T}\mathbf{PD}\right)^{-1} \mathbf{D}^{T}\mathbf{P}\begin{bmatrix} \mathbf{H}_{ij} & \mathbf{H}_{2ij} & \mathbf{E} & \mathbf{H}_{3ij} \end{bmatrix} \right\} \tag{15c}$$

$$\bar{\boldsymbol{\varphi}}_{ij} \triangleq \begin{bmatrix} \boldsymbol{\varphi}_{ij} & * \\ \mathbf{H}_{3ij}^{T}\mathbf{P}\begin{bmatrix} \mathbf{H}_{1ij} & \mathbf{H}_{2ij} & \mathbf{E} \end{bmatrix} + \bar{\mathbf{H}}_{3ij}^{T}\varepsilon\begin{bmatrix} \bar{\mathbf{H}}_{1ij} & \bar{\mathbf{H}}_{2ij} & 0 \end{bmatrix} & \mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{H}_{3ij} + \bar{\mathbf{H}}_{3ij}^{T}\varepsilon\bar{\mathbf{H}}_{3ij} \end{bmatrix} \tag{15d}$$

$$\mathbf{\Omega}_{ijq}(s) = \begin{bmatrix} \mathbf{T}_{ijq} & 0 & \cdots & 0 & \mathbf{n}_{ijq} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \mathbf{n}_{ijq}^{T} & 0 & \cdots & 0 & v_{ijq} \end{bmatrix}_{s\times s} \tag{16}$$

Besides, the *S*-procedure [12, 18] weighting parameters $\mathbf{T}_{ijq} \in \Re^{n\times n}$, $\mathbf{n}_{ijq} \in \Re^{n\times 1}$, and $v_{ijq} \in \Re$ are defined such that

$$\sigma_{ijq}\left(x(t)\right) \triangleq x^{T}(t)\mathbf{T}_{ijq}\,x(t) + 2\mathbf{n}_{ijq}^{T}\,x(t) + v_{ijq} \leq 0, \quad q = 1\cdots p \ \text{ and } \ i = 1\cdots r \tag{17}$$

for all $x(t)$ which activates rule i (i.e., $h_i\left(x(t)\right) > 0$).

Proof:

Select a discrete-type Lyapunov function as

$$V\left(x(k)\right) = x^{T}(k)\mathbf{P}\,x(k) + \sum_{\varpi=k-\tau}^{k-1} x^{T}(\varpi)\mathbf{P}_d\,x(\varpi) \tag{18}$$

By evaluating the first-forward difference of the Lyapunov function $V\left(x(k)\right)$ along the trajectories of DPTDA T–S fuzzy model (10), one has

$$\begin{aligned} &\Delta V\left(x(k)\right) \\ &= V\left(x(k+1)\right) - V\left(x(k)\right) \\ &= x^{T}(k+1)\mathbf{P}\,x(k+1) - x^{T}(k)\mathbf{P}\,x(k) + x^{T}(k)\mathbf{P}_d\,x(k) \\ &\quad - x(k-\tau)^{T}\mathbf{P}_d\,x(k-\tau) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} h_i\left(z(k)\right)h_j\left(z(k)\right)\,h_k\left(z(k)\right)h_l\left(z(k)\right) \\ &\quad \times\tilde{\mathbf{x}}(k)^{T}\left\{\left(\mathbf{O}_{ij} + \mathbf{D}\Delta(k)\mathbf{J}_{ij}\right)^{T}\mathbf{P}\left(\mathbf{O}_{ij} + \mathbf{D}\Delta(k)\mathbf{J}_{ij}\right)\right\}\tilde{\mathbf{x}}(k) \\ &\quad - x^{T}(k)\mathbf{P}\,x(k) + x^{T}(k)\mathbf{P}_d\,x(k) - x^{T}(k-\tau)\mathbf{P}_d\,x(k-\tau) \end{aligned} \tag{19}$$

where

$$\mathbf{O}_{ij} = \begin{bmatrix} \mathbf{H}_{ij} & \mathbf{H}_{2ij} & \mathbf{E} & \mathbf{H}_{3ij} \end{bmatrix},\ \mathbf{J}_{ij} = \begin{bmatrix} \bar{\mathbf{H}}_{ij} & \bar{\mathbf{H}}_{2ij} & 0 & \bar{\mathbf{H}}_{3ij} \end{bmatrix},$$

and $\tilde{\mathbf{x}}(k) = \begin{bmatrix} x^{T}(k) & x^{T}(k-\tau) & v^{T}(k) & 1 \end{bmatrix}^{T}$ (20)

From Lemma 1, one can obtain

$$\begin{aligned} &\Delta V\left(x(k)\right) \\ &\leq \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} h_i\left(z(k)\right)h_j\left(z(k)\right)\,h_k\left(z(k)\right)h_l\left(z(k)\right) \\ &\quad \times\tilde{\mathbf{x}}^{T}(k)\left\{\mathbf{O}_{ij}^{T}\mathbf{P}\mathbf{O}_{ij} + \mathbf{O}_{ij}^{T}\mathbf{PD}\left(\varepsilon I - \mathbf{D}^{T}\mathbf{PD}\right)^{-1}\mathbf{D}^{T}\mathbf{P}\mathbf{O}_{ij}\right. \\ &\quad \left. + \varepsilon\mathbf{J}_{ij}^{T}\mathbf{J}_{ij}\right\}\tilde{\mathbf{x}}(k) - x^{T}(k)\mathbf{P}\,x(k) + x^{T}(k)\mathbf{P}_d\,x(k) \\ &\quad - x^{T}(k-\tau)\mathbf{P}_d\,x(k-\tau) \end{aligned} \tag{21}$$

Next, let us define the following performance index

$$J_\infty \triangleq \sum_{k=0}^{k_f}\left\{x^{T}(k)\mathbf{S}\,x(k) - \gamma^{2}\,v^{T}(k)v(k)\right\} \tag{22}$$

with zero initial condition for all $v(k) \in L_2[0,\ k_f]$. Hence, for any nonzero $v(k)$ one has

$$\begin{aligned} J_\infty &\leq \sum_{k=0}^{k_f}\left\{x^{T}(k)\mathbf{S}\,x(k) - \gamma^{2}\,v^{T}(k)v(k)\right\} + V\left(x(k)\right) \\ &= \sum_{k=0}^{k_f}\left\{x^{T}(k)\mathbf{S}\,x(k) - \gamma^{2}\,v^{T}(k)v(k)\right\} + \sum_{k=0}^{k_f}\Delta V\left(x(k)\right) \\ &= \sum_{k=0}^{k_f}\left\{x^{T}(k)\mathbf{S}\,x(k) - \gamma^{2}\,v^{T}(k)v(k) + \Delta V\left(x(k)\right)\right\} \end{aligned}$$

$$\triangleq \sum_{k=0}^{k_f}\left\{H(x,v,k)\right\} \tag{23}$$

where $H(x,v,k)$ is defined as follows according to (21).

$$\begin{aligned} &H(x,v,k)\\ &=\Delta V(x(k))+x^T(k)\mathbf{S}x(k)-\gamma^2 v^T(k)v(k)\\ &=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(x(k))h_j(x(k))\left\{\tilde{\mathbf{x}}(k)^T(k)\bar{\Upsilon}_{ij}\tilde{\mathbf{x}}(k)\right\} \end{aligned} \tag{24}$$

where $\bar{\Upsilon}_{ij}$ is defined in (15c). Converting (24) to an LMI by applying the *S*-procedure described in [12, 18], one has

$$\begin{aligned} H(x,v,k)\le &\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(x(k))h_j(x(k))\left\{\tilde{\mathbf{x}}(k)^T(k)\bar{\Upsilon}_{ij}\tilde{\mathbf{x}}(k)\right\}\\ &-\sum_{q=1}^{n}\xi_{ijq}\sigma_{ijq}(x(k)) \end{aligned} \tag{25}$$

where $\sigma_{ijq}(x(k))\in\Re$ is defined in (17). Since $\xi_{ijq}\ge 0$ and $\sigma_{ijq}(x(k))\le 0$, then (25) can be represented as

$$\begin{aligned} H(x,v,k)\le &\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(x(k))h_j(x(k))\\ &\times\left\{\tilde{\mathbf{x}}(k)^T\left(\bar{\Upsilon}_{ij}-\sum_{q=1}^{n}\xi_{ijq}\mathbf{\Omega}_{ijq}(s)\right)\tilde{\mathbf{x}}(k)\right\} \end{aligned} \tag{26}$$

Obviously, if (14) hold for all $x(k)\in\mathbf{X}_i$, $i\in\hat{I}_1$, then $\sum_{k=0}^{k_f}\left\{H(x,v,k)\right\}<0$. It means that

$$J_\infty<0 \quad\text{or}\quad \sum_{k=0}^{k_f}\left\{x^T(k)\mathbf{S}x(k)\right\}<\gamma^2\sum_{k=0}^{k_f}\left\{v^T(k)v(k)\right\} \tag{27}$$

Since (27) is equivalent to (12), it is easy to find that H_∞ performance constraint (12) is achieved with a prescribed γ. In the next step, we have to show that the DPTDA T–S fuzzy model in (10) is quadratically stable. From (26), if $\bar{\Upsilon}_{ij}-\sum_{q=1}^{n}\xi_{ijq}\mathbf{\Omega}_{ijq}<0$ hold, it implies that $H(x,v,k)<0$. Assume that the disturbance $v(t)$ is zero, and then one has

$$H(x,v,k)=\Delta V(x(k))+x^T(k)\mathbf{S}x(k)-\gamma^2 v^T(k)v(k)<0 \tag{28}$$

or

$$\Delta V(x(k))<-x^T(k)\mathbf{S}x(k) \tag{29}$$

Therefore, the DPTDA T–S fuzzy model described in (10) is quadratically stable in the large. Besides, for the case of $x(t)\in\mathbf{X}_i$, $i\in\hat{I}_0$, the stability condition (13) can be obtained by setting the state bias term $\mathbf{a}_i=0$ and ignoring the *S*-procedure from the similar proof procedure.

□

From Theorem 1, it can be noted that the matrix inequalities in $\mathbf{P}$, $\mathbf{S}$, $\mathbf{P}_d$, $\mathbf{F}_i$ and ξ_{ijq} belong to the class of BMIs and the controller synthesis can not be solved by the MATLAB LMI-toolbox. In the next section, Theorem 2 is provided to introduce modified stability conditions which can be solved by MATLAB LMI-toolbox through an ILMI algorithm [12, 14-15].

IV. Fuzzy Controller Design for DPTDA T-S Fuzzy Models via ILMI Algorithm

In this section, the ILMI algorithm [12, 14-15] is used to develop a fuzzy controller design procedure for the DPTDA T–S fuzzy model (10). The idea of the ILMI algorithm used in solving BMI problems is based on holding some matrix variables as constant values and then converting it into a LMI problem. One can thus use the MATLAB LMI-toolbox to solve the proposed fuzzy controller design problem.

A. Stabilization condition of DPTDA T–S fuzzy model

Theorem 2

Given a H_∞ attenuation parameter $\gamma>0$ and the auxiliary constant matrix $\mathbf{R}>0$. The conditions of Theorem 1 are satisfied if there exist $\alpha<1$, positive definite matrices $\mathbf{P}>0$, $\mathbf{S}>0$, $\mathbf{P}_d>0$, control gains $\mathbf{F}_i$ and scalars $\xi_{ijq}\ge 0$ such that

$$\begin{cases}\mathbf{\Theta}_{ij}<0\\ \mathbf{R}^T\mathbf{P}\mathbf{R}-\mathbf{R}\le 0\end{cases}\quad \text{for } i\in\hat{I}_0 \tag{30}$$

and

$$\begin{cases}\bar{\mathbf{\Theta}}_{ij}<0\\ \mathbf{R}^T\mathbf{P}\mathbf{R}-\mathbf{R}\le 0\end{cases}\quad \text{for } i\in\hat{I}_1 \tag{31}$$

where

$$\mathbf{\Theta}_{ij}\triangleq\begin{bmatrix} -\alpha\mathbf{P}+\mathbf{P}_d+\mathbf{S} & * & * & * & * & * \\ \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{1ij} & \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{2ij}-\mathbf{P}_d+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{2ij} & * & * & * & * \\ \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{1ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{2ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}-\gamma^2\mathbf{I} & * & * & * \\ \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{ij} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{2ij} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{E} & -\varepsilon I+\mathbf{D}^T\mathbf{R}^{-1}\mathbf{D} & * & * \\ \bar{\mathbf{H}}_{1ij} & 0 & 0 & 0 & -\varepsilon^{-1} & * \\ \mathbf{H}_{1ij} & 0 & 0 & 0 & 0 & -\mathbf{R} \end{bmatrix} \tag{32}$$

$$\bar{\boldsymbol{\Theta}}_{ij} \triangleq \begin{bmatrix} -\alpha\mathbf{P}+\mathbf{P}_d+\mathbf{S}-\xi_{ijq}\mathbf{T}_{ijq} & * & * & * & * & * & * \\ \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{1ij} & \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{2ij}-\mathbf{P}_d+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{2ij} & * & * & * & * & * \\ \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{1ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{2ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}-\gamma^2\mathbf{I} & * & * & * & * \\ \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{n}^T & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{2ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{2ij} & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{E} & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{3ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{3ij}-\xi_{ijq}\mathbf{v}_{ij} & * & * & * \\ \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{ij} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{2ij} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{E} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{3ij} & -\varepsilon\mathrm{I}+\mathbf{D}^T\mathbf{R}^{-1}\mathbf{D} & * & * \\ \bar{\mathbf{H}}_{1ij} & 0 & 0 & 0 & 0 & -\varepsilon^{-1} & * \\ \mathbf{H}_{1ij} & 0 & 0 & 0 & 0 & 0 & -\mathbf{R} \end{bmatrix} \tag{33}$$

Proof:

Rewriting (31), one has

$$\begin{bmatrix} -\mathbf{P}+\mathbf{P}_d+\mathbf{S}-\xi_{ijq}\mathbf{T}_{ijq} & * & * & * & * & * & * \\ \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{1ij} & \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{2ij}-\mathbf{P}_d+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{2ij} & * & * & * & * & * \\ \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{1ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{2ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}-\gamma^2\mathbf{I} & * & * & * & * \\ \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{n}^T & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{2ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{2ij} & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{E} & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{3ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{3ij}-\xi_{ijq}\mathbf{v}_{ij} & * & * & * \\ \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{ij} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{2ij} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{E} & \mathbf{D}^T\mathbf{R}^{-1}\mathbf{H}_{3ij} & -\varepsilon\mathrm{I}+\mathbf{D}^T\mathbf{R}^{-1}\mathbf{D} & * & * \\ \bar{\mathbf{H}}_{1ij} & 0 & 0 & 0 & 0 & -\varepsilon^{-1} & * \\ \mathbf{H}_{1ij} & 0 & 0 & 0 & 0 & 0 & -\mathbf{R} \end{bmatrix} < \begin{bmatrix} (\alpha-1)\mathbf{P} & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

By using the Schur-complement [18], the inequality (34) becomes

$$\begin{bmatrix} \mathbf{H}_{1ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}-\mathbf{P}+\mathbf{P}_d+\mathbf{S}+\bar{\mathbf{H}}_{1ij}^T\varepsilon\bar{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{T}_{ij} & * & * & * \\ \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{1ij} & \mathbf{H}_{2ij}^T\mathbf{R}^{-1}\mathbf{H}_{2ij}-\mathbf{P}_d+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{2ij} & * & * \\ \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{1ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{H}_{2ij} & \mathbf{E}^T\mathbf{R}^{-1}\mathbf{E}-\gamma^2\mathbf{I} & * \\ \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{n}^T & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{2ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{2ij} & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{E} & \mathbf{H}_{3ij}^T\mathbf{R}^{-1}\mathbf{H}_{3ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{3ij}-\xi_{ijq}\mathbf{v}_{ij} \end{bmatrix} + \mathbf{Z} < \begin{bmatrix} (\alpha-1)\mathbf{P} & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{H}_{1ij}{}^T \\ \mathbf{H}_{2ij}{}^T \\ \mathbf{E}^T \\ \mathbf{H}_{3ij}{}^T \end{bmatrix} \mathbf{R}^{-1}\mathbf{D}\left(\varepsilon\mathrm{I}-\mathbf{D}^T\mathbf{P}\mathbf{D}\right)^{-1}\mathbf{D}^T\mathbf{R}^{-1}\begin{bmatrix} \mathbf{H}_{ij} & \mathbf{H}_{2ij} & \mathbf{E} & \mathbf{H}_{3ij} \end{bmatrix} \tag{36}$$

If the matrix $\mathbf{P}>0$ exist such that $\mathbf{R}^T\mathbf{P}\mathbf{R}-\mathbf{R}\leq 0$ is held, then the following inequality is obvious.

$$\left[\begin{matrix} \mathbf{H}_{1ij}^T\mathbf{P}\mathbf{H}_{1ij}-\mathbf{P}+\mathbf{P}_d+\mathbf{S}+\bar{\mathbf{H}}_{1ij}^T\varepsilon\bar{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{T}_{ij} \\ \mathbf{H}_{2ij}^T\mathbf{P}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{2ij}^T\varepsilon\bar{\mathbf{H}}_{1ij} \\ \mathbf{E}^T\mathbf{P}\mathbf{H}_{1ij} \\ \mathbf{H}_{3ij}^T\mathbf{P}\mathbf{H}_{1ij}+\bar{\mathbf{H}}_{3ij}^T\varepsilon\bar{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{n}^T \end{matrix}\right.$$

$$
\begin{array}{l}
* \\
\mathbf{H}_{2ij}^{T}\mathbf{P}\mathbf{H}_{2ij}-\mathbf{P}_{d}+\overline{\mathbf{H}}_{2ij}^{T}\varepsilon\overline{\mathbf{H}}_{2ij} \\
\mathbf{E}^{T}\mathbf{P}\mathbf{H}_{2ij} \\
\mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{H}_{2ij}+\overline{\mathbf{H}}_{3ij}^{T}\varepsilon\overline{\mathbf{H}}_{2ij}
\end{array}
\left.\begin{array}{ll}
* & * \\
* & * \\
\mathbf{E}^{T}\mathbf{P}\mathbf{E}-\gamma^{2}\mathbf{I} & * \\
\mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{E} & \mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{H}_{3ij}+\overline{\mathbf{H}}_{3ij}^{T}\varepsilon\overline{\mathbf{H}}_{3ij}-\xi_{ijq}\mathbf{v}_{ij}
\end{array}\right]+\mathbf{Z}
$$

$$
<\begin{bmatrix} (\alpha-1)\mathbf{P} & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{37}
$$

where $\alpha < 1$. Thus, one has

$$
\left[\begin{array}{l}
\mathbf{H}_{1ij}^{T}\mathbf{P}\mathbf{H}_{1ij}-\mathbf{P}+\mathbf{P}_{d}+\mathbf{S}+\overline{\mathbf{H}}_{1ij}^{T}\varepsilon\overline{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{T}_{ij} \\
\mathbf{H}_{2ij}^{T}\mathbf{P}\mathbf{H}_{1ij}+\overline{\mathbf{H}}_{2ij}^{T}\varepsilon\overline{\mathbf{H}}_{1ij} \\
\mathbf{E}^{T}\mathbf{P}\mathbf{H}_{1ij} \\
\mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{H}_{1ij}+\overline{\mathbf{H}}_{3ij}^{T}\varepsilon\overline{\mathbf{H}}_{1ij}-\xi_{ijq}\mathbf{n}^{T}
\end{array}\right.
$$

$$
\begin{array}{l}
* \\
\mathbf{H}_{2ij}^{T}\mathbf{P}\mathbf{H}_{2ij}-\mathbf{P}_{d}+\overline{\mathbf{H}}_{2ij}^{T}\varepsilon\overline{\mathbf{H}}_{2ij} \\
\mathbf{E}^{T}\mathbf{P}\mathbf{H}_{2ij} \\
\mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{H}_{2ij}+\overline{\mathbf{H}}_{3ij}^{T}\varepsilon\overline{\mathbf{H}}_{2ij}
\end{array}
\left.\begin{array}{ll}
* & * \\
* & * \\
\mathbf{E}^{T}\mathbf{P}\mathbf{E}-\gamma^{2}\mathbf{I} & * \\
\mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{E} & \mathbf{H}_{3ij}^{T}\mathbf{P}\mathbf{H}_{3ij}+\overline{\mathbf{H}}_{3ij}^{T}\varepsilon\overline{\mathbf{H}}_{3ij}-\xi_{ijq}\mathbf{v}_{ij}
\end{array}\right]+\mathbf{Z}<0 \tag{38}
$$

The inequality (38) is equivalent to (14). Thus, the proof is completed. Besides, for the case of $x(t)\in\mathbf{X}_i$, $i\in\hat{I}_0$, the stability condition (30) can be obtained by setting the state bias term $\mathbf{a}_i=0$ and ignoring the *S*-procedure from the similar proof procedure.

□

B. ILMI Algorithm

According to Theorem 2, an ILMI algorithm [12, 14-15] is developed to find the feasible solutions for the stability conditions (30-31). The purpose of this algorithm is to iteratively search for $\mathbf{P}$, $\mathbf{S}$, $\mathbf{P}_d$, $\mathbf{F}_i$, ξ_{ijq}, α and to update the auxiliary constant matrix $\mathbf{R}$ until $\alpha<1$. The detail of the proposed fuzzy controller design procedure is concluded as follows.

<ILMI Algorithm>

Step 1 Define the iterative auxiliary variables as $\mathbf{R}(\kappa)=\mathbf{P}^{-1}(\kappa-1)$, where κ denotes an iteration index. When $\kappa=1$, the initial conditions of $\mathbf{R}(1)$ can be obtained as follows:

$$
\mathbf{R}(1)=\mathbf{P}^{-1}(0) \tag{39}
$$

For the given initial $\mathbf{P}(0)$, one can solve it from the following discrete Riccati equation.

$$
\begin{array}{l}
\hat{\mathbf{A}}^{T}\mathbf{P}(0)\hat{\mathbf{A}}-\mathbf{P}(0) \\
-\left(\hat{\mathbf{A}}^{T}\mathbf{P}(0)\hat{\mathbf{B}}\right)\left(1+\hat{\mathbf{B}}^{T}\mathbf{P}(0)\hat{\mathbf{B}}\right)^{-1}\left(\hat{\mathbf{B}}^{T}\mathbf{P}(0)\hat{\mathbf{A}}\right)+\mathbf{Q}=0
\end{array} \tag{40}
$$

where $\hat{\mathbf{A}}=\frac{1}{r}\sum_{i=1}^{r}\mathbf{A}_i$, $\hat{\mathbf{B}}=\frac{1}{r}\sum_{i=1}^{r}\mathbf{B}_i$ and $\mathbf{Q}>0$. The matrix $\mathbf{Q}$ is assigned by the designers.

Step 2 Set $\kappa=1$ and start the algorithm.

Step 3 Given the auxiliary variables $\mathbf{R}(\kappa)$ to solve the optimization problem for

Minimize $\alpha(\kappa)$

Subject to $\mathbf{P}(\kappa)>0$, $\mathbf{S}(\kappa)>0$, $\mathbf{P}_d(\kappa)>0$, $\mathbf{F}_i(\kappa)$ and $\xi_{ijq}(\kappa)\ge 0$, (30) for $i\in\hat{I}_0$, and (31) for $i\in\hat{I}_1$ (41)

If $\alpha(\kappa)<1$, then $\mathbf{P}(\kappa)$, $\mathbf{S}(\kappa)$, $\mathbf{P}_d(\kappa)$, $\mathbf{F}_i(\kappa)$, and $\xi_{ijq}(\kappa)$ obtained in (41) are feasible solutions for the Theorem 2 and stop the iterative manner. Otherwise, if $\alpha(\kappa)\ge 1$ then go to Step 4.

Step 4 Given $\alpha(\kappa)$ obtained in Step 3 and the same auxiliary variables $\mathbf{R}(\kappa)$ used in Step 3. Resolve the optimization problem for $\mathbf{P}(\kappa)$, $\mathbf{S}(\kappa)$, $\mathbf{P}_d(\kappa)$, $\mathbf{F}_i(\kappa)$ and $\xi_{ijq}(\kappa)$ such that

Minimize $trace(\mathbf{P}(\kappa))$

Subject to $\mathbf{P}(\kappa)>0$, $\mathbf{S}(\kappa)>0$, $\mathbf{P}_d(\kappa)>0$, $\mathbf{F}_i(\kappa)$ and $\xi_{ijq}(\kappa)\ge 0$, (30) for $i\in\hat{I}_0$, and (31) for $i\in\hat{I}_1$ (42)

If the condition $\left\|\mathbf{R}^{-1}(\kappa)-\mathbf{P}(\kappa)\right\|<\upsilon$ is satisfied for a predetermined small value υ. Then the Theorem 2 may not be feasible and stop the iterative manner. Otherwise, go to Step 5.

Step 5 Update the auxiliary variables $\mathbf{R}(\kappa+1)$ using $\mathbf{P}(\kappa)$, where $\mathbf{P}(\kappa)$ is determined in (42). Set $\kappa=\kappa+1$ and go back to Step 3.

□

In next section, a numerical example is presented to show the usefulness of the above fuzzy controller design procedure for the inverted pendulum robot system under the framework of DPTDA T–S fuzzy model.

V. A Numerical Example

According to the results developing in previous sections, this section provides a numerical simulation for the inverted pendulum robot system presented in section Ⅱ. Considering the inverted pendulum robot system (2), one can choose three operating points to obtain the linearized models for the system (2). Let us choose three operating points as follows:

$$
\begin{aligned}
&\left(x^{+},\ x_{\mathrm{d}}^{+},\ u^{+}\right)_{\mathrm{oper1}} = \left(\begin{array}{cccc|cccc|c} 0 & 0 & 88^{\circ} & 0 & 0 & 0 & 0^{\circ} & 0 & 0 \end{array}\right),\\
&\left(x,\ x_{\mathrm{d}},\ u\right)_{\mathrm{oper2}} = \left(\begin{array}{cccc|cccc|c} 0 & 0 & 0^{\circ} & 0 & 0 & 0 & 0^{\circ} & 0 & 0 \end{array}\right),\\
&\left(x^{-},\ x_{\mathrm{d}}^{-},\ u^{-}\right)_{\mathrm{oper3}} = \left(\begin{array}{cccc|cccc|c} 0 & 0 & -88^{\circ} & 0 & 0 & 0 & 0^{\circ} & 0 & 0 \end{array}\right)
\end{aligned}
\tag{43}
$$

Then, three linear subsystems can be constructed by these operating points. In which, $\left(x,\ x_{\mathrm{d}},\ u\right)_{\mathrm{oper2}}$ is the maintain equilibrium point and the others are the off-equilibrium points. Through the above three linear subsystems and membership functions defining in Fig. 2, one can obtain the DPTDA T–S fuzzy model for the inverted pendulum robot system (2), which is composed by three fuzzy rules as follows:

Rule i: IF $x_3(\mathrm{k})$ is about $\mathrm{M}_{\mathrm{i}1}$ THEN

$$
\begin{aligned}
x(\mathrm{k}+1) &= \left(\mathbf{A}_{\mathrm{i}} + \Delta\mathbf{A}_{\mathrm{i}}\right)x(\mathrm{k}) + \left(\mathbf{A}_{\mathrm{id}} + \Delta\mathbf{A}_{\mathrm{id}}\right)x(\mathrm{k}-\tau)\\
&\quad + \mathbf{B}_{\mathrm{i}}\,u(\mathrm{k}) + \mathbf{a}_{\mathrm{i}} + \mathbf{E}\,v(\mathrm{k}),\quad \mathrm{i} = 1\cdots 3
\end{aligned}
\tag{44}
$$

where

$$
\mathbf{A}_1 = \begin{bmatrix} 0.998 & 0.1 & 0 & 0\\ 0 & 0.0917 & 0.035 & 0\\ 0 & 0 & 1 & 0.1\\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
\mathbf{A}_2 = \begin{bmatrix} 0.998 & 0.1 & 0 & 0\\ 0 & 0.0581 & -0.0363 & 0\\ 0 & 0 & 1 & 0.1\\ 0 & 2.8983 & 3.127 & 1 \end{bmatrix},
$$

$$
\mathbf{A}_3 = \begin{bmatrix} 0.998 & 0.1 & 0 & 0\\ 0 & 0.0917 & 0.035 & 0.0464\\ 0 & 0 & 1 & 0.1\\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
\Delta\mathbf{A}_1 = \Delta\mathbf{A}_2 = \Delta\mathbf{A}_3 = \begin{bmatrix} 0.001\mathit{sin}(\mathrm{t}) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0.003\mathit{cos}(\mathrm{t}) & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\mathbf{A}_{1\mathrm{d}} = \mathbf{A}_{2\mathrm{d}} = \mathbf{A}_{3\mathrm{d}} = \begin{bmatrix} 0.002 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\mathbf{a}_1 = \begin{bmatrix} 0\\ 0.0549\\ 0\\ -3.0154 \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} 0\\ 0\\ 0\\ 0 \end{bmatrix},\ \mathbf{a}_3 = \begin{bmatrix} 0\\ -0.0549\\ 0\\ 3.0154 \end{bmatrix},
$$

$$
\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 = \begin{bmatrix} 0\\ 0.0726\\ 0\\ -0.2233 \end{bmatrix},\ \text{and}\ \mathbf{E} = \begin{bmatrix} 0\\ 0\\ 0.2\\ 0 \end{bmatrix}.
\tag{45}
$$

and the corresponding matrices of *S*-procedure [12, 18] are presented as follows:

For ***Rule***$_{\mathrm{s}}$ 11, i.e., $90^{\circ} \le x_3(\mathrm{t}) \le 80^{\circ}$

$$
\mathbf{T}_{111} = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \mathbf{n}_{111} = \begin{bmatrix} 0\\ 0\\ -\frac{1}{2}\left[(80+90)\pi/180\right]\\ 0 \end{bmatrix}
$$

$$
\text{and}\ \mathrm{v}_{111} = (90\pi/180)\times(80\pi/180)
\tag{46}
$$

For ***Rule***$_{\mathrm{s}}$ 33, i.e., $-90^{\circ} \le x_3(\mathrm{t}) \le -80^{\circ}$, the matrices of *S*-procedure are given as follows:

$$
\mathbf{T}_{331} = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \mathbf{n}_{331} = \begin{bmatrix} 0\\ 0\\ -\frac{1}{2}\left[(-80-90)\pi/180\right]\\ 0 \end{bmatrix}
$$

$$
\text{and}\ \mathrm{v}_{331} = (-90\pi/180)\times(-80\pi/180)
\tag{47}
$$

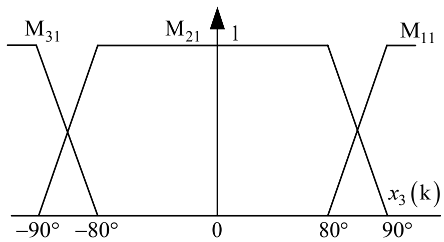

Fig. 2 Membership functions of $x_3(\mathrm{k})$

For the DPTDA T–S fuzzy model (44), the fuzzy controller can be designed by applying Theorem 2 and the ILMI algorithm [12, 14-15]. In this example, it is assumed that the H_∞ control performance is guaranteed for an attenuation $\gamma^2 = 0.32$. Then, we can get a feasible solution after four iterations of the ILMI algorithm. The final decay rate α is 0.9999 and the feasible solutions are obtained as follows:

$$\mathbf{P}=\begin{bmatrix} 4.4530 & 2.0824 & 7.3099 & 1.4670 \\ 2.0824 & 3.3681 & 9.5516 & 1.8320 \\ 7.3099 & 9.5516 & 41.5488 & 7.8826 \\ 1.4670 & 1.8320 & 7.8826 & 1.5450 \end{bmatrix},$$

$$\mathbf{P}_{d}=\begin{bmatrix} 0.0327 & 0.0186 & 0.0430 & 0.0118 \\ 0.0186 & 0.0238 & 0.0366 & 0.0134 \\ 0.0430 & 0.0366 & 0.1136 & 0.0225 \\ 0.0118 & 0.0314 & 0.0225 & 0.0081 \end{bmatrix},$$

$$\mathbf{S}=\begin{bmatrix} 0.0076 & 0.0046 & 0.0107 & 0.0030 \\ 0.0046 & 0.0059 & 0.0092 & 0.0034 \\ 0.0107 & 0.0092 & 0.0284 & 0.0056 \\ 0.0030 & 0.0034 & 0.0056 & 0.0020 \end{bmatrix},$$

$$\mathbf{R}=\begin{bmatrix} 0.3419 & -0.1082 & 0.0601 & -0.5031 \\ -0.1082 & 0.8916 & -0.1470 & -0.2054 \\ 0.0601 & -0.1470 & 0.7670 & -3.7935 \\ -0.5031 & -0.2054 & -3.7935 & 20.7108 \end{bmatrix},$$

$$\xi_{111}=186.4773,\ \xi_{331}=186.4773 \tag{48}$$

and the fuzzy controller has the following form:

Rule 1: IF $x_3(\mathrm{k})$ is about M_{11} THEN

$u(\mathrm{k})=-[-5.2530 \quad -1.0054 \quad -43.6819 \quad -9.2813]x(\mathrm{k})$

Rule 2: IF $x_3(\mathrm{k})$ is about M_{21} THEN

$u(\mathrm{k})=-[-5.1953 \quad -18.5910 \quad -49.8976 \quad -9.2234]x(\mathrm{k})$

Rule 3: IF $x_3(\mathrm{k})$ is about M_{31} THEN

$u(\mathrm{k})=-[-5.2530 \quad -1.0054 \quad -43.6819 \quad -9.2813]x(\mathrm{k})$

(49)

The output of the PDC type fuzzy controller (49) is determined by the following summation

$$u(\mathrm{k})=-\sum_{i=1}^{3}\mathrm{h}_{i}\left(x_{1}(\mathrm{k})\right)\left\{\mathbf{F}_{i}\,x(\mathrm{k})\right\} \tag{50}$$

The disturbance input noise $v(\mathrm{k})$ is given with variance one. The simulation results are shown in Fig. 3 to Fig. 6. From the simulated results, one can find that the controlled nonlinear perturbed time-delay inverted pendulum robot system (2) is quadratically stable under the fuzzy controller (50).

VI. Conclusion

A robust fuzzy controller design procedure has been developed for the nonlinear inverted pendulum robot system which can achieve the H_∞ performance constraints and cope with the worst case effect of disturbances. Firstly, the Lyapunov criterion was applied to analyze the stability conditions for the nonlinear inverted pendulum robot system. Secondly, an ILMI algorithm was developed to solve the stabilization conditions of synthesis problems for the nonlinear inverted pendulum robot system. Finally, in order to illustrate the applicability of the present fuzzy controller design procedure, a numerical simulation for the inverted pendulum robot system has been shown.

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References

[1] I. D. Loram, P. J. Gawthrop and M. Lakie, “The Frequency of Human, Manual Adjustments in Balancing an Inverted Pendulum is Constrained by Intrinsic Physiological Factors”, *Physiology*, Vol. 577, No. 1, pp. 417-432, 2006.

[2] H. J. Zimmermann, *Fuzzy Set Theory and Its Application*, MA: Kluwer, Boston, 1991.

[3] S. Dutta, “Fuzzy Logic Applications: Technological and Strategic Issues,” *IEEE Transactions on Engineering Management*, Vol. 40, No. 3, pp. 237–254, 1993.

[4] R. Gurumoorthy and S. R. Sanders, “Controlling Nonminimum Phase Nonlinear Systems- the Inverted Pendulum on a Cart Example”, *Control Conference on American*, pp. 680-685, 1994.

[5] B. Chen and X. Liu, “Fuzzy Approximate Disturbance Decoupling of MIMO Nonlinear Systems by Back Stepping and Application to Chemical Processes,” *IEEE Transactions on Fuzzy Systems*, Vol. 13, No. 6, pp. 832–847, 2005.

[6] S. R. Munasinghe, M. S. Kim, and J. J. Lee, “Adaptive Neuro Fuzzy Controller to Regulate UTSG Water Level in Nuclear Power Plants,” *IEEE Transactions on Nucleus Science*, Vol. 52, No. 1, pp. 421–429, 2005.

[7] K. Tanaka and M. Sugeno, “Stability Analysis and Design of Fuzzy Control System,” *Fuzzy Sets and Systems*, Vol. 45, No. 2, pp. 135–156, 1992.

[8] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis-A Linear Matrix Inequality Approach*, John Wiley & Sons, Inc., New York, 2001.

[9] W. J. Chang, “Fuzzy Controller Design via the Inverse Solution of Lyapunov Equations”, *ASME, J. Dynamic Systems, Measurement and Control*, Vol. 125, No. 1, pp. 42-47, 2003.

[10] W. J. Chang, C. C. Sun and H. Y. Chung, “Fuzzy Controller Design for Discrete Controllability Canonical Takagi-Sugeno Fuzzy Systems”, *IEE Proceeding, Part D, Control Theory and Applications*, Vol. 151, No. 3, pp. 319-328, 2004.

[11] T. Takagi and M. Sugeno, “Fuzzy Identification of Systems and Its Applications to Modeling and Control,” *IEEE Transactions on Systems, Man, Cybernetics*, Vol. 15, No. 1, pp. 116–132, 1985.

[12] E. Kim and D. Kim, “Stability Analysis and Synthesis for an Affine Fuzzy System via LMI and ILMI: Discrete Case,” *IEEE Trans. Systems, Man, and Cybernetics, Part B*, Vol. 31, No. 1, pp. 132-140, 2001.

[13] C. C. Hsiao, S. F. Su, T. T. Lee and C. C. Chuang, “Hybrid Compensation Control for Affine TSK Fuzzy Control Systems,” *IEEE Transactions on Systems Man and Cybernetics, Part B -Cybernetics*, Vol. 34, No. 4, pp. 1865-1873, 2004.

[14] W. J. Chang and W. Chang, “Synthesis of Nonlinear Discrete Control Systems via Time-Delay Affine Takagi-Sugeno Fuzzy Models”, *ISA Transactions*, Vol. 44, No. 2, pp. 243-257, 2005.

[15] W. J. Chang and W. Chang, “Discrete Fuzzy Control of Time-Delay Affine Takagi-Sugeno Fuzzy Models with H_∞ Constraint”, *IEE Proceeding, Part D, Control Theory and Applications*, Vol. 153, No. 6, pp.745-752, 2006.

[16] I. R. Petersen, V. A. Ugrinovskii and A. V. Savkin, *Robust Control Design Using H_∞ Methods*, Springer, 2000

[17] Y. Y. Cao and P. M. Frank, “Analysis and Synthesis of Nonlinear Time-Delay Systems via Fuzzy Control Approach,” *IEEE Transactions on Fuzzy Systems,* Vol. 8, No. 2, pp. 200-211, 2000.

[18] S. P. Boyd, *Linear Matrix Inequalities in System and Control Theory,* Society for Industrial and Applied Mathematics, 1994.

[19] Y. Wang, L. Xie and C. E. de Souna, “Robust Control of A Class of Uncertain Nonlinear Systems,” *Systems and Control Letters,* Vol. 19, pp. 139-149, 1992.

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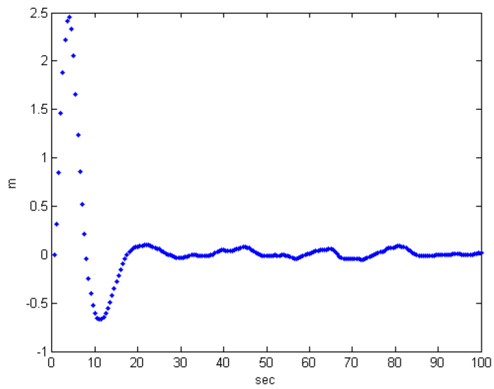

Fig. 3 Responses of $x_1(k)$

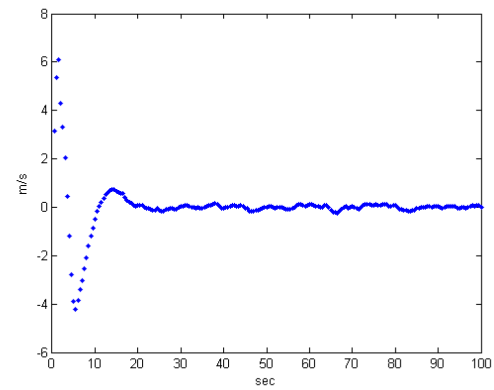

Fig. 4 Responses of $x_2(k)$

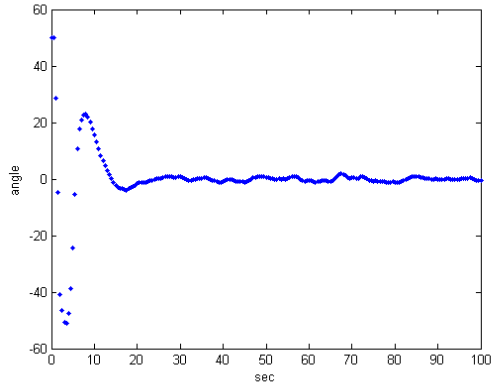

Fig. 5 Responses of $x_3(k)$

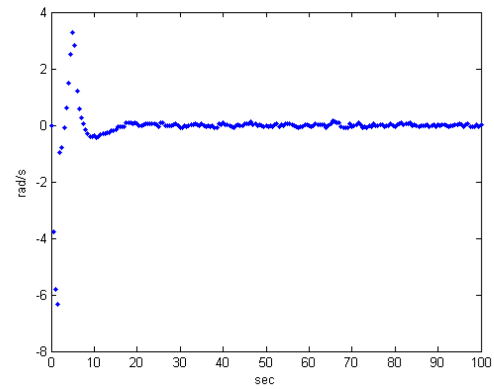

Fig. 6 Responses of $x_4(k)$

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