Quantum Search Algorithms in Analog and Digital Models

Hiromi Miyajima, Masataka Fujisaki, and Noritaka Shigei

***Abstract*—For searching any item in an unsorted database with N items, a classical computer takes $O(N)$ steps but Grover's quantum searching algorithm takes only $O(\sqrt{N})$ steps. However, it is also known that Grover's algorithm is effective only in the case where the initial amplitude distribution of dataset is uniform, but is not always effective in the non-uniform case. In this paper, we propose some quantum search algorithms. First, we propose an algorithm in analog time by solving the schrodinger equation and prove that the proposed algorithm shows best performance in optimum time. Next, we apply the result to Grover search algorithm. It is shown that the proposed algorithm gives better performance than the conventional one. Further, we propose the improved algorithm by generalizing the idea of the phase rotation. The algorithm shows best performance compared with the conventional ones.**

***Index Terms*—quantum search algorithm ; Grover search algorithm; initial amplitude distributions of dataset; observed probability**

I. Introduction

Many studies have been made with parallel computation. However, almost all models cannot perform plural solutions at a time. Therefore, quantum computer can be thought as one of best parallel models[1],[2]. However, it is not true that quantum computer can apply to all problems. Shor's prime factoring and Grover's database search algorithms are well known as typical applications[3],[4],[6]. Further, Ventura has proposed quantum associative memory by improving Grover's algorithm [5],[7]. Database search problem is to find any data effectively from unsorted dataset. For searching any item in an unsorted database with N items, a classical computer takes $O(N)$ steps, but Grover's algorithm takes only $O(\sqrt{N})$ steps. However, it is also known that Grover's algorithm is effective only in the case where the initial amplitude distribution of dataset is uniform, but is not always effective in the non-uniform case[5]. Ventura has proposed the quantum searching algorithm that it is effective in some cases for the initial amplitude distribution[8],[9],[14]. But, it is also not sufficient. Therefore, it is necessary to find effective algorithms even in the case where the initial amplitude distribution of dataset is not uniform. For example, associative memory needs non-uniform initial amplitude distribution[5],[13]. In the previous paper[15], we proposed some quantum search algorithms. First, we proposed an algorithm in analog time based on quantum walk by solving the schrodinger equation. The proposed algorithm showed best performance in optimum time. Next, we applied the result to the digital model. It was shown that the improved algorithm shows better performance than the conventional one. Further, we proposed the algorithm by generalizing the idea of the phase rotation[8]. However, we have not given the perfect proof and the meaning of the proposed algorithms in the previous paper yet. In this paper, we will verify the results from theory and numerical simulations. Further, we will propose the algorithm in the general case of digital model.

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H. Miyajima is with Graduate School of Science and Engineering, Kagoshima University, 1-21-40 Korimoto, Kagoshima 890-0065, Japan (corresponding author to provide phone/fax: +81-99-285-8422; email: miya@eee.kagoshima-u.ac.jp).

M. Fujisaki is with Graduate School of Science and Engineering, Kagoshima University, 1-21-40 Korimoto, Kagoshima 890-0065, Japan (email: k7605487@kadai.jp).

N. Shigei is with Graduate School of Science and Engineering, Kagoshima University, 1-21-40 Korimoto, Kagoshima 890-0065, Japan (email: shigei@eee.kagoshima-u.ac.jp).

II. Preliminary

The basic unit in quantum computation is a qubit $w_0|0\rangle + w_1|1\rangle$, which is a superposition of two independent states $|0\rangle$ and $|1\rangle$ corresponding to the states 0 and 1 in a classical computer, where w_0 and w_1 are complex numbers such that $|w_0|^2 + |w_1|^2 = 1$. We use the Dirac bracket notation, where the ket $|x\rangle$ is analogous to a column vector. Let n be a positive integer and $N = 2^n$. A system with n qubits is described using N independent state $|x\rangle (0 \le x \le N-1)$ as follows:

$$\sum_{x=0}^{N-1} w_x|x\rangle \tag{1}$$

where w_x is a complex number, $\sum_{x=0}^{N-1} |w_x|^2 = 1$ and $|w_x|^2$ is the probability of state $|x\rangle$.The direction of w_x on the complex plane is called the phase of state $|x\rangle$ and the absolute value $|w_x|$ is called the amplitude of state $|x\rangle$. In quantum system, starting from any quantum state, the desired state is formed by multiplying column vector of the quantum state by unitary matrix. Finally, we can obtain the desired state with high probability through observation[4]. The problem is how we can find unitary matrix. Grover has proposed the fast data search algorithm. Let us explain the Grover's algorithm shown in Fig.1. Grover has proposed an algorithm for finding one item in an unsorted database. In the conventional computation, if there are N items in the database, it would require $O(N)$ queries to the database. However, Grover has shown how to perform this using the quantum computation with only $O(\sqrt{N})$ queries[4]. Let $Z_N = \{0, 1, \cdots, N-1\}$. Let us define the following operators.

$$\boldsymbol{I}_a = \text{Identity matrix except for } \boldsymbol{I}(a+1, a+1) = -1, a \in Z_N \tag{2}$$

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1. Initial state $|\bar{0}\rangle$
2. $|\psi\rangle = \boldsymbol{W}|\bar{0}\rangle$
3. Repeat T_0 times
4. $|\psi\rangle = \boldsymbol{I}_\tau|\psi\rangle$
5. $|\psi\rangle = \boldsymbol{G}|\psi\rangle$
6. Observe the system

Fig. 1. Grover search algorithm

which inverts any state $|\psi\rangle$ and

$$\boldsymbol{W}(x,y) = \frac{1}{\sqrt{N}}(-1)^{x_0y_0+\cdots+x_{N-1}y_{N-1}}$$
$$\text{for } x = \sum_{i=0}^{N-1} x_i 2^i, y = \sum_{i=0}^{N-1} y_i 2^i, \quad (3)$$

which is called the Walsh or Hadamard transform and performs a special case of discrete Fourier transform. We begin with the $|\bar{0}\rangle$ state and apply $\boldsymbol{W}$ operator to it, where $|\bar{0}\rangle$ means that all states are 0 and the number of 0's for $\bar{0}$ is N. As a result, all the states have the same amplitude $1/\sqrt{N}$. Next, we apply the $\boldsymbol{I}_\tau$ operator, where $|\tau\rangle$ is the searching state. Further, we apply the operator

$$\boldsymbol{G} = -\boldsymbol{W}\boldsymbol{I}_0\boldsymbol{W} \quad (4)$$

, where $\boldsymbol{I}_0$ is as follows:

$$\boldsymbol{I}_0 = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix} \quad (5)$$

Followed by the I_τ operator $T_0 = \mathrm{CI}[(\pi/4)\sqrt{N}]$ times and observe the system[4], $\mathrm{CI}[x]$ means the rounding of x. The operator $\boldsymbol{G}$ has been described as inverting each of the state's amplitudes around the average one of all states.

Example 1 :

Let $N = 16$. Let searching data $|\tau\rangle = |8\rangle = |1000\rangle$ and the number of stored (memorized) data $k = 16$.

Let us apply Grover's algorithm shown in Fig.1, where $T_0 \simeq 3$.

At step2,

$$|\psi\rangle = \frac{1}{4}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)^{\mathrm{T}}, \quad (6)$$

where T shows the transposition.

Then $|\psi\rangle$ is updated as follows:

$$\begin{aligned}
\boldsymbol{I}_\tau|\psi\rangle &= \frac{1}{4}(1,1,1,1,1,1,1,1,-1,1,1,1,1,1,1,1)^{\mathrm{T}} \\
\boldsymbol{G}|\psi\rangle &= \frac{1}{16}(3,3,3,3,3,3,3,3,11,3,3,3,3,3,3,3)^{\mathrm{T}} \\
\boldsymbol{I}_\tau|\psi\rangle &= \frac{1}{16}(3,3,3,3,3,3,3,3,-11,3,3,3,3,3,3,3)^{\mathrm{T}} \\
\boldsymbol{G}|\psi\rangle &= \frac{1}{64}(5,5,5,5,5,5,5,5,61,5,5,5,5,5,5,5)^{\mathrm{T}} \\
\boldsymbol{I}_\tau|\psi\rangle &= \frac{1}{64}(5,5,5,5,5,5,5,5,-61,5,5,5,5,5,5,5)^{\mathrm{T}} \\
\boldsymbol{G}|\psi\rangle &= \frac{1}{256}(-13,-13,-13,-13,-13,-13,-13,-13 \\
&\quad 251,-13,-13,-13,-13,-13,-13,-13)^{\mathrm{T}}
\end{aligned}$$

Finally, the desired data 1000 is obtained with the probability 0.96 by using Grover's algorithm. We can get the searching data with high probability.

Next, assuming that stored data are $|0\rangle$, $|3\rangle$, $|8\rangle$, $|9\rangle$, $|12\rangle$, and $|15\rangle$, and searching data is $|8\rangle$, that is $k = 6$.

At step2,

$$|\psi\rangle = \frac{1}{\sqrt{6}}(1,0,0,1,0,0,0,0,1,1,0,0,1,0,0,1)^{\mathrm{T}} \quad (7)$$

Under this condition , let us apply Grover's algorithm. As a result, $|\psi\rangle$ is obtained as follows:

$$\begin{aligned}
|\psi\rangle = \frac{1}{8\sqrt{6}}&(5,-3,-3,5,-3,-3,-3,-3,13, \\
&5,-3,-3,5,-3,-3,5)^{\mathrm{T}}. \quad (8)
\end{aligned}$$

Then, the desired data 1000 is obtained with the probability 0.44. It shows that Grover's algorithm does not always give a good result in the case where $k \neq N$.

Therefore, it is needed to find effective algorithms in the case where the initial amplitude distribution of dataset is not uniform.

III. Quantum search in analog model based on quantum walk

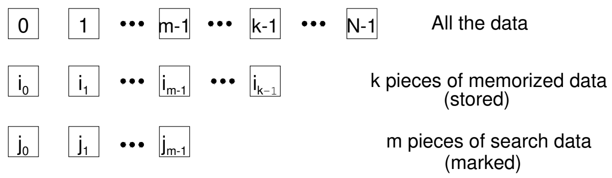

Fig. 2. Description of data search problem

In the following, we introduce a quantum search algorithm using analog model of quantum walk. In order to clarify the problem, we will explain the Fig.2. It is assumed that k pieces of data are memorized (stored) in the system of N pieces of data. Now we want to find any data in m pieces of data (marked) with high probability. Grover has shown the effective algorithm in the case of $k = N$, $m = 1$ (see Fig.1).

A. Schrodinger equation and quantum walk

In this chapter, we propose an algorithm based on quantum walk in analog model. As the state of system in quantum model is determined by the Schrodinger equation , we can obtain the result by solving the Schrodinger equation under the special condition[11],[12].

The Schrodinger equation for the state of system is represented as follows[11]:

$$i\hbar\frac{d|\psi\rangle}{dt} = \boldsymbol{H}|\psi\rangle, \quad (9)$$

where $\hbar$ is Plank constant and $\boldsymbol{H}$ is Hamiltonian which means all the energy of the system. Then $\boldsymbol{H} = \boldsymbol{H}^\dagger$ holds, where $\boldsymbol{H}^\dagger$ is the transposed matrix of complex conjugate for $\boldsymbol{H}$.

Then the solution for (9) is represented by

$$|\psi(t)\rangle = \boldsymbol{U}(t)|\psi(0)\rangle, \quad (10)$$

where

$$\boldsymbol{U}(t) = \exp(-it\boldsymbol{H}) \quad (11)$$

and $\boldsymbol{U}(t)$ is called time expansion operator of the state and $|\psi(0)\rangle$ is the initial state of system. Therefore, the state of system is determined by Hamiltonian $\boldsymbol{H}$. Let $P_w(t)$ be defined as the observed probability for data w of at time t as follows:

$$P_w(t) = |\langle w|\boldsymbol{U}(t)|\psi(0)\rangle|^2. \tag{12}$$

The state of system to search is called the marked one and the other is called the unmarked state. The number of marked states is m (see Fig.2). Hamiltonian H_ρ corresponding to the potential energy is represented by an identity matrix except for

$$\boldsymbol{H}_\rho(j_i, j_i) = -1, \tag{13}$$

where $i \in Z_m$.

Let $\boldsymbol{L}$ be the state assignment over the graph. Then the Hamiltonian $\boldsymbol{H}$ of the system is defined using the mobility r as follows[11]:

$$\boldsymbol{H} = -\gamma\boldsymbol{L} + \boldsymbol{H}_\rho. \tag{14}$$

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be the perfect graph to act for system, where $\boldsymbol{V}$ is the set of vertexes and $\boldsymbol{E}$ is the set of edges. Then $\boldsymbol{L}$ is represented as follows:

$$\begin{aligned} \boldsymbol{L} &= -N\boldsymbol{I} + \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \\ &= -N\boldsymbol{I} + \sum_x \sum_y |x\rangle\langle y| \\ &= -N\boldsymbol{I} + N|s\rangle\langle s|, \end{aligned} \tag{15}$$

where

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle. \tag{16}$$

Finally, Hamiltonian $\boldsymbol{H}$ is represented as follows:

$$\boldsymbol{H} = \gamma N\boldsymbol{I} - \gamma N|s\rangle\langle s| + \boldsymbol{H}_\rho. \tag{17}$$

It is known that any quantum arrives at any place (state) in $O(\sqrt{N})$ steps by using quantum walk. Assuming that the potential energy of the place to arrive is low, the high probability of the state is performed.

Then, let $\boldsymbol{U}$ be defined as follows:

$$\boldsymbol{U(t)} = \exp(-\frac{1}{2}it\gamma N\boldsymbol{I})\exp(it\boldsymbol{A}), \tag{18}$$

where

$$\boldsymbol{A} = \gamma N|s\rangle\langle s| - \boldsymbol{H}_\rho - \frac{1}{2}\gamma N\boldsymbol{I}. \tag{19}$$

B. Derivation of the time expansion operator $\boldsymbol{U}$

Let us compute (18).

Let $\boldsymbol{C}$ be any matrix. Then the following relation holds:

$$\exp(\boldsymbol{C}) = \sum_{r=0}^{\infty} \frac{1}{r!}\boldsymbol{C}^r. \tag{20}$$

In order to compute (18), $\boldsymbol{A}^r$ must be computed.

In the data search problem, the matrix $\boldsymbol{A}^r$ is represented as follows without loss of generality:

$$A^r(x,y) = \begin{cases} \epsilon(r) & 0 \le x \le l-1,\ y = x \\ \eta(r) & 0 \le x, y \le l-1,\ y \ne x \\ \beta(r) & 0 \le x \le l-1,\ l \le y \le N-1 \\ \zeta(r) & l \le x \le N-1,\ y = x \\ \delta(r) & l \le x \le N-1,\ 0 \le y \le l-1 \\ \alpha(r) & l \le x, y \le N-1,\ y \ne x, \end{cases} \tag{21}$$

where $x, y \in Z_N$, $r = 1, 2, \cdots$ and $l = k$ or m.

Then $\boldsymbol{A}^1$ as the initial condition is as follows:

$$\epsilon(1) = \gamma + \frac{1}{2} \tag{22}$$

$$\eta(1) = \gamma \tag{23}$$

$$\beta(1) = \gamma \tag{24}$$

$$\zeta(1) = \gamma - \frac{1}{2} \tag{25}$$

$$\delta(1) = \gamma \tag{26}$$

$$\alpha(1) = \gamma \tag{27}$$

From the relation that $\boldsymbol{A}^{r+1} = \boldsymbol{A}^r\boldsymbol{A}$ the following recursion formula are obtained:

$$\begin{aligned} \beta(r+1) &= \left\{(N-l)\gamma - \frac{1}{2}\right\}\beta(r) \\ &\quad + (l-1)\gamma\eta(r) + \gamma\epsilon(r) \end{aligned} \tag{28}$$

$$\begin{aligned} \eta(r+1) &= (N-l)\gamma\beta(r) + \left\{(l-1)\gamma + \frac{1}{2}\right\}\eta(r) \\ &\quad + \gamma\epsilon(r) \end{aligned} \tag{29}$$

$$\begin{aligned} \epsilon(r+1) &= (N-l)\gamma\beta(r) + (l-1)\gamma\eta(r) \\ &\quad + (\gamma + \frac{1}{2})\epsilon(r) \end{aligned} \tag{30}$$

$$\begin{aligned} \alpha(r+1) &= \left\{(N-l-1)\gamma - \frac{1}{2}\right\}\alpha(r) \\ &\quad + m\gamma\delta(r) + \gamma\zeta(r) \end{aligned} \tag{31}$$

$$\begin{aligned} \delta(r+1) &= (N-l-1)\gamma\alpha(r) + (l\gamma + \frac{1}{2})\delta(r) \\ &\quad + \gamma\zeta(r) \end{aligned} \tag{32}$$

$$\begin{aligned} \zeta(r+1) &= (N-l-1)\gamma\alpha(r) + l\gamma\delta(r) \\ &\quad + (\gamma - \frac{1}{2})\zeta(r). \end{aligned} \tag{33}$$

In order to from solve (28) to (33), let $\boldsymbol{v}_1(r), \boldsymbol{v}_2(r), \boldsymbol{K}_1 \text{and} \boldsymbol{K}_2$ be defined as follows:

$$\boldsymbol{v}_1(r) = \begin{pmatrix} \beta(r) \\ \eta(r) \\ \epsilon(r) \end{pmatrix} \tag{34}$$

$$\boldsymbol{v}_2(r) = \begin{pmatrix} \alpha(r) \\ \delta(r) \\ \zeta(r) \end{pmatrix} \tag{35}$$

$$\boldsymbol{K}_1 = \begin{pmatrix} \gamma(N-l) - \frac{1}{2} & \gamma(l-1) & \gamma \\ \gamma(N-l) & \gamma(l-1) + \frac{1}{2} & \gamma \\ \gamma(N-l) & \gamma(l-1) & \gamma + \frac{1}{2} \end{pmatrix} \tag{36}$$

$$\boldsymbol{K}_2 = \begin{pmatrix} \gamma(N-l-1) - \frac{1}{2} & \gamma l & \gamma \\ \gamma(N-l-1) & \gamma l + \frac{1}{2} & \gamma \\ \gamma(N-l-1) & \gamma l & \gamma - \frac{1}{2} \end{pmatrix}. \tag{37}$$

Then, the following relation hold:

$$\boldsymbol{v}_1(r+1) = \boldsymbol{K}_1\boldsymbol{v}_1(r) \quad (38)$$
$$\boldsymbol{v}_2(r+1) = \boldsymbol{K}_2\boldsymbol{v}_2(r). \quad (39)$$

By using the singular matrix $\boldsymbol{S}_1$, the matrix $\boldsymbol{K}_1$ is diagonalized as follows:

$$\boldsymbol{K}_1^D = \boldsymbol{S}_1^{-1}\boldsymbol{K}_1\boldsymbol{S}_1. \quad (40)$$

Let $\boldsymbol{w}_1(r) = \boldsymbol{S}_1^{-1}\boldsymbol{v}_1(r)$.

Then, the following holds:

$$\boldsymbol{w}_1(r+1) = \boldsymbol{K}_1^D\boldsymbol{w}_1(r) \quad (41)$$
$$\boldsymbol{w}_1(r+1) = (\boldsymbol{K}_1^D)^{r-1}\boldsymbol{w}_1(1). \quad (42)$$

Now, let us compute the matrix $\boldsymbol{S}_1$. From (36), the eigenvalues of $\boldsymbol{K}_1$ are obtained as follows:

$$\lambda_0 = \frac{1}{2} \quad (43)$$
$$\lambda_1 = \frac{1}{2} - \sqrt{\gamma l} \quad (44)$$
$$\lambda_2 = \frac{1}{2} + \sqrt{\gamma l}. \quad (45)$$

Then the matrix $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are obtained as follows:

$$\boldsymbol{S}_1 = \begin{pmatrix} 0 & P_1 & Q_1 \\ 1 & 1 & 1 \\ -(l-1) & 1 & 1 \end{pmatrix} \quad (46)$$
$$\boldsymbol{S}_1^{-1} = \frac{1}{l(Q_1-P_1)} \times \begin{pmatrix} 0 & Q_1-P_1 & -(Q_1-P_1) \\ -l & (l-1)Q_1 & Q_1 \\ l & -(l-1)P_1 & P_1 \end{pmatrix}, \quad (47)$$

where

$$P_1 = \frac{\sqrt{\gamma l}}{\sqrt{\gamma l}-1} \quad (48)$$
$$Q_1 = \frac{\sqrt{\gamma l}}{\sqrt{\gamma l}+1}. \quad (49)$$

From (40), the following holds:

$$\boldsymbol{K}_1^D = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \quad (50)$$
$$\boldsymbol{w}_1(1) = \boldsymbol{S}_1^{-1}\boldsymbol{v}_1(r) \quad (51)$$
$$= \begin{pmatrix} -\frac{1}{2l} \\ R_1 \\ -L_1 \end{pmatrix}, \quad (52)$$

where

$$R_1 = \frac{\gamma l Q_1 + \frac{1}{2}Q_1 - \gamma l}{l(Q_1-P_1)} \quad (53)$$
$$L_1 = \frac{\gamma l P_1 + \frac{1}{2}P_1 - \gamma l}{l(Q_1-P_1)}. \quad (54)$$

Therefore,

$$\boldsymbol{w}_1(r) = \begin{pmatrix} -\frac{1}{2l}\lambda_0^{r-1} \\ R_1\lambda_1^{r-1} \\ -L_1\lambda_2^{r-1} \end{pmatrix} \quad (55)$$
$$\boldsymbol{v}_1(r) = \begin{pmatrix} P_1R_1\lambda_1^{r-1} - Q_1L_1\lambda_2^{r-1} \\ -\frac{1}{2l}\lambda_0^{r-1} + R_1\lambda_1^{r-1} - L_1\lambda_2^{r-1} \\ \frac{l-1}{2l}\lambda_0^{r-1} + R_1\lambda_1^{r-1} - L_1\lambda_2^{r-1} \end{pmatrix}. \quad (56)$$

Likewise, (35) is determined as follows:

$$\boldsymbol{v}_2(r) = \begin{pmatrix} \frac{1}{2(N-l)}\lambda_0^{r-1} + R_2\lambda_1^{r-1} - L_2\lambda_2^{r-1} \\ P_2R_2\lambda_1^{r-1} - Q_2L_2\lambda_2^{r-1} \\ -\frac{1}{2}\frac{N-l-1}{N-l}\lambda_0^{r-1} + R_2\lambda_1^{r-1} - L_2\lambda_2^{r-1} \end{pmatrix}, \quad (57)$$

where

$$\begin{cases} P_2 = \frac{\sqrt{\gamma l}-1}{\sqrt{\gamma l}} \\ Q_2 = \frac{\sqrt{\gamma l}+1}{\sqrt{\gamma l}} \\ R_2 = \frac{\gamma(N-l)(1-Q_2)+\frac{1}{2}Q_2}{(N-l)(P_2-Q_2)} \\ L_2 = \frac{\gamma(N-l)(1-P_2)+\frac{1}{2}P_2}{(N-l)(P_2-Q_2)}. \end{cases} \quad (58)$$

From (18), (20) and $\boldsymbol{A}^r$, the operator $\boldsymbol{U}$ is obtained as follows:

$$U(t) = \begin{cases} \epsilon(t) & 0 \le x \le l-1,\ y = x \\ \eta(t) & 0 \le x, y \le l-1,\ y \ne x \\ \beta(t) & 0 \le x \le l-1,\ l \le y \le N-1 \\ \zeta(t) & l \le x \le N-1,\ y = x \\ \delta(t) & l \le x \le N-1,\ 0 \le y \le l-1 \\ \alpha(t) & l \le x, y \le N-1,\ y \ne x, \end{cases} \quad (59)$$

where

$$\beta(t) = \frac{P_1R_1}{\lambda_1}\exp(-i\sqrt{\gamma l}t) - \frac{Q_1L_1}{\lambda_2}\exp(i\sqrt{\gamma l}t) \quad (60)$$
$$\eta(t) = -\frac{1}{l} + \frac{R_1}{\lambda_1}\exp(-i\sqrt{\gamma l}t) - \frac{L_1}{\lambda_2}\exp(i\sqrt{\gamma l}t) \quad (61)$$
$$\epsilon(t) = \frac{l-1}{l} + \frac{R_1}{\lambda_1}\exp(-i\sqrt{\gamma l}t) - \frac{L_1}{\lambda_2}\exp(i\sqrt{\gamma l}t) \quad (62)$$
$$\alpha(t) = -\frac{1}{N-l}\exp(-it) + \frac{R_2}{\lambda_1}\exp(-i\sqrt{\gamma l}t) - \frac{L_2}{\lambda_2}\exp(i\sqrt{\gamma l}t) \quad (63)$$
$$\delta(t) = \frac{P_2R_2}{\lambda_1}\exp(-i\sqrt{\gamma l}t) - \frac{Q_2L_2}{\lambda_2}\exp(i\sqrt{\gamma l}t) \quad (64)$$
$$\zeta(t) = \frac{N-l-1}{N-l}\exp(-it) + \frac{R_2}{\lambda_1}\exp(-i\sqrt{\gamma l}t) - \frac{L_2}{\lambda_2}\exp(i\sqrt{\gamma l}t). \quad (65)$$

For example , $\beta(t)$ is obtained as follows:

$$\begin{aligned} \beta(t) &= \exp(-i\frac{1}{2}\gamma Nt)\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\beta(r) \\ &= \exp(-i\frac{1}{2}t) \times \left(\frac{P_1R_1}{\lambda_1}\sum_{r=0}^{\infty}\frac{(it\lambda_1)^r}{r!} - \frac{Q_1L_1}{\lambda_2}\sum_{r=0}^{\infty}\frac{(it\lambda_2)^r}{r!}\right) \\ &= \frac{P_1R_1}{\lambda_1}\exp(-i\sqrt{\gamma l}t) - \frac{Q_1L_1}{\lambda_2}\exp(i\sqrt{\gamma l}t). \end{aligned} \quad (66)$$

IV. Search algorithms based on quantum walk

A. Application to quantum search problem in analog time

The result of the chapter III is applied to quantum search problem. Remark that t is in analog time. The method based on (59) is called the method 1 (in analog time)[11]. The probability $P_w(t)$ is computed by (12). Then the initial amplitudes $\psi_a(0)$ and $\psi_b(0)$ of memorized and non-memorized data is as follows, respectively:

$$\psi_a(0) = \frac{1}{\sqrt{k}} \quad (67)$$

$$\psi_b(0) = 0. \quad (68)$$

From (10) and (59), the following result is obtained.

$$\begin{aligned}\psi_a(t) &= \frac{1}{\sqrt{k}}\left[(k-m)\alpha_m + (m-1)\beta_m + \delta_m\right] \\ &= \frac{1}{\sqrt{k}}\cos\sqrt{\gamma m}t + i\frac{\sqrt{l}}{\sqrt{mN}}\sin\sqrt{\gamma m}t \quad (69)\end{aligned}$$

As a result, the observed probability of search data $P_w(t)$ is obtained from (12) as follow:

$$P_w(t) = \frac{1}{k}\cos^2\sqrt{\frac{m}{N}}t + \frac{k}{mN}\sin^2\sqrt{\frac{m}{N}}t. \quad (70)$$

Let us consider the example of four cases, (case 1) $m = 1, k = N$, (case 2) $m = 2, k = N$, (case 3) $m = 1, k = N/2$, (case 4) $m = 2, k = N/2$ for $N = 1024$. The result shows $P_w((\pi/2)\sqrt{N/m}) = k/(mN)$, where $t = (\pi/2)\sqrt{N/m}$ is the optimum time. If $k = N$, then $P_w(t) = 1/m$ and $P_w(t) = 1$ for $m = 1$. Fig.3 shows the simulation result. The case1 and case2 for $k = N$ show high probability.

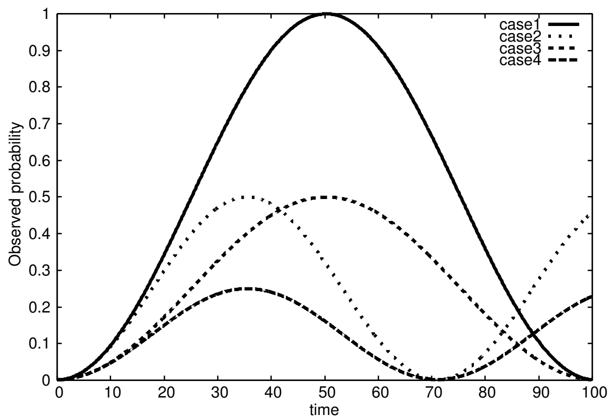

Fig. 3. The simulation result of the method 1 for four cases

Then, how is the case where $k \neq N$. In this case, the maximum probability is computed by solving $dP_w(t)/dt = 0$. Fig.4 shows the results which are $P_w(t) \neq 1$ except for $k = N$ and $P_w(t) \leq 0.5$ for $k < N/2$.

Example 2:

Let us show an example. Let $N = 16$. Let us search data $|\tau\rangle = |8\rangle$ and $k = 16$.

Then,

$$|\psi(0)\rangle = \frac{1}{4}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)^{\mathrm{T}} \quad (71)$$

and $t = (\pi/2)\sqrt{16} = 6.28$ in the optimum time. Therefore,

$$|\psi(6.28)\rangle = (0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0)^{\mathrm{T}} \quad (72)$$

The data $|8\rangle$ is obtained with the probability 1.

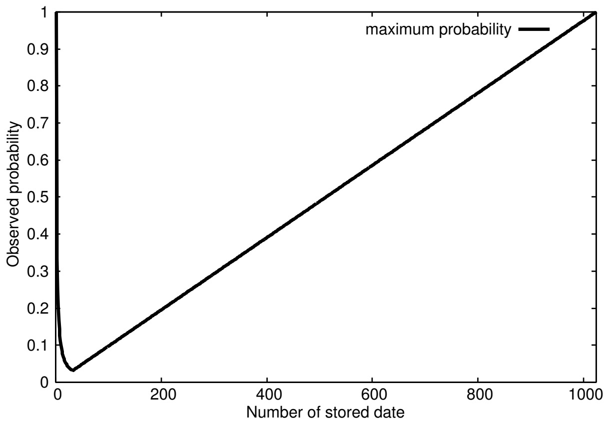

Fig. 4. Maximum probability for the number of memorized data

Next, let us consider the following case with $N = 16$ and $k = 6$.

$$|\psi(0)\rangle = \frac{1}{\sqrt{6}}(1,0,0,1,0,0,0,0,1,1,0,0,1,0,0,1)^{\mathrm{T}} \quad (73)$$

Then the optimum time is as follows:

$$t = (\pi/2)\sqrt{16} = 6.28 \quad (74)$$

Therefore,

$$\begin{aligned}|\psi(6.28)\rangle = (&-0.07 + i0.27, -0.07 + i0.14, \\ &-0.07 + i0.14, -0.07 + i0.27, \\ &-0.07 + i0.14 - 0.07 + i0.14, \\ &-0.07 + i0.14, -0.07 + i0.14, -0.61, \\ &-0.07 + i0.27, -0.07 + i0.14, \\ &-0.07 + i0.14, -0.07 + i0.27, \\ &-0.07 + i0.14, -0.07 + i0.14, \\ &-0.07 + i0.27)^{\mathrm{T}} \quad (75)\end{aligned}$$

Then, the desired data $|8\rangle$ is obtained with the probability 0.38.

The example shows that the desired data is not always obtained with the high probability in the case of $k \neq N$.

B. Improved search algorithm in analog time

The state $\psi_a(t_1)$ of memorized data after t_1 step and the state $\psi_b(t_1)$ of non-memorized data after t_1 step are represented as follows:

$$\psi_a(t_1) = \frac{1}{\sqrt{k}}\cos\sqrt{\frac{k}{N}}t_1 + i\frac{1}{\sqrt{N}}\sin\sqrt{\frac{k}{N}}t_1 \quad (76)$$

$$\psi_b(t_1) = i\frac{1}{\sqrt{N}}\sin\sqrt{\frac{k}{N}}t_1 \quad (77)$$

When $t_1 = (\pi/2)\sqrt{N/k}$, it holds $\psi_a(t_1) = \psi_b(t_1)$. Therefore, the states except for marked data at the step t_1 are identical, so the method 1 is possible to apply at the time $t = t_1$. The use of the method 1 for the time interval t_2 leads to the following probability:

$$\begin{aligned}\psi_w(t_2+t_1) &= \frac{i}{\sqrt{N}}\left[(N-m)\alpha_m + (m-1)\beta_m + \delta_m\right] \\ &= i\frac{1}{\sqrt{N}}\cos\sqrt{\frac{m}{N}}t_2 - \frac{1}{\sqrt{m}}\sin\sqrt{\frac{m}{N}}t_2 \quad (78)\end{aligned}$$

$$P_a(t_2+t_1)=\frac{1}{N}\cos^2\sqrt{\frac{m}{N}}t_2+\frac{1}{m}\sin^2\sqrt{\frac{m}{N}}t_2. \quad (79)$$

By taking $t_2=(\pi/2)\sqrt{N/m}$, it holds $P_a(t_1+t_2)=1/m$. The method is called the proposed method 2 in analog time. Fig.5 shows the numerical example the comparison between method 1 and proposed method 2 for $N=1024, m=1, k=512$.

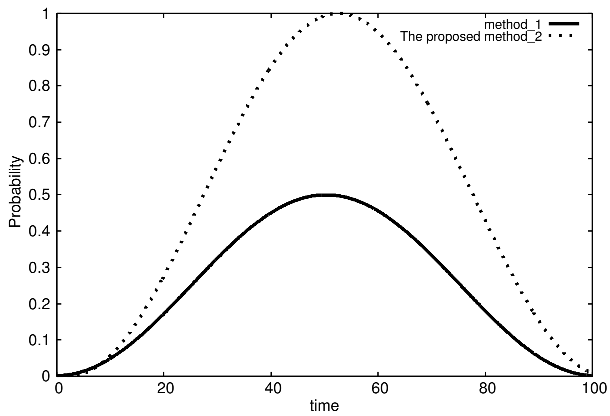

Fig. 5. The comparison between the method 1 and the proposed method 2

Example 3:
Let $N=16$. Let search data $|\tau\rangle=|8\rangle$ and $k=6$.

$$|\psi(0)\rangle=\frac{1}{\sqrt{6}}(1,0,0,1,0,0,0,0,1,1,0,0,1,0,0,1)^{\mathrm{T}} \quad (80)$$

Then, the optimum time is as follows: 5

$$t_1=(\pi/2)\sqrt{(16/6)}=2.57 \quad (81)$$

Therefore,

$$|\psi(2.57)\rangle=\frac{i}{4}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)^{\mathrm{T}}. \quad (82)$$

Further, it holds $t_2=(\pi/2)\sqrt{(16)}=6.28$ and $t_2+t_1=8.85$.
Then,

$$|\psi(8.85)\rangle=(0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0)^{\mathrm{T}}. \quad (83)$$

The data $|8\rangle$ is obtained with the probability 1.

C. The application to the digital time of the proposed methods

1. Initial state $|\psi\rangle$
2. Repeat T_1 times
3. $|\psi\rangle=\boldsymbol{I}_\rho|\psi\rangle$
4. $|\psi\rangle=\boldsymbol{G}|\psi\rangle$
5. Repeat T_g times
6. $|\psi\rangle=\boldsymbol{I}_\tau|\psi\rangle$
7. $|\psi\rangle=\boldsymbol{G}|\psi\rangle$
8. Observe the system

Fig. 6. The algorithm of the proposed method 3

In this section, let us apply the proposed method 2 obtained in the section B to the algorithm shown in Fig.1. Fig.6 shows the proposed method 3 (in digital time) corresponding to the proposed method 2. Let us compute the time T_1 as the same method used in the section B.

$$\psi_a(T_1) = \frac{1}{\sqrt{k}}\cos\omega_1 T_1 \quad (84)$$

$$\psi_b(T_1) = -\frac{1}{\sqrt{N-k}}\sin\omega_1 T_1, \quad (85)$$

where

$$\omega_1=\arccos\left(\frac{N-2k}{N}\right). \quad (86)$$

Therefore, we can find the time when $\psi_a(T_1)=\psi_b(T_1)$ as follows:

$$T_1=\mathrm{CI}\left(\frac{\pi-\arctan\left(\sqrt{\frac{N-k}{k}}\right)}{\arccos\left(\frac{N-2k}{N}\right)}\right), \quad (87)$$

where $\mathrm{CI}(x)$ means the rounding of x. Fig.7 shows the result

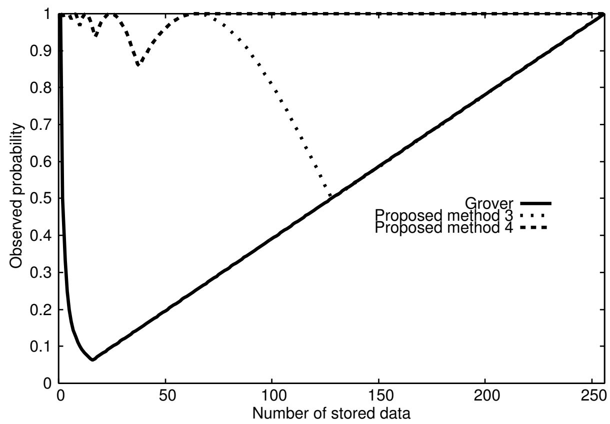

Fig. 7. The comparison among the proposed algorithms

of numerical simulation for $N=256$.

Example 4:
Let $N=16$. Let $|2\rangle,|6\rangle,|11\rangle,|12\rangle$ be stored data.

$$|\psi\rangle=\frac{1}{2}(0,0,1,0,0,0,1,0,0,0,0,1,1,0,0,0)^{\mathrm{T}} \quad (88)$$

The following result is obtained by applying $\boldsymbol{I}_\rho$ and $\boldsymbol{G}$ to the initial state of (88).

$$|\psi\rangle = \frac{1}{4}(-1,-1,1,-1,-1,-1,1,-1,-1,-1,-1,1, 1,-1,-1,-1)^{\mathrm{T}} \quad (89)$$

As $T_1=2$, $\boldsymbol{I}_\rho$ and $\boldsymbol{G}$ to (89) are iterated one more time. As a result, the following state is obtained:

$$|\psi\rangle = \frac{1}{4}(-1,-1,-1,-1,-1,-1,-1,-1,-1, -1,-1,-1,-1,-1,-1,-1)^{\mathrm{T}} \quad (90)$$

Further, iterating the steps 5 to 8 in Fig.6 to (90), the desired data is obtained with the high probability 0.96.

Next, supposing that the stored data are $|0\rangle,|2\rangle,|3\rangle,|6\rangle,|7\rangle,|10\rangle,|11\rangle,|12\rangle$ as follows:

$$|\psi\rangle=\frac{1}{2\sqrt{2}}(1,0,1,1,0,0,1,1,0,0,1,1,1,0,0,0)^{\mathrm{T}}. \quad (91)$$

When the steps 2 to 4 of the proposed method 3 are iterated for (91), the following state is obtained:

$$|\psi\rangle = -\frac{1}{2\sqrt{2}}(1,0,1,1,0,0,1,1,0,0,1,1,1,0,0,0)^{\mathrm{T}}. \quad (92)$$

Then, the desired data is obtained with the probability 0.5 after applying the steps 5 to 8. It shows that the proposed method 3 does not always give a good result.

As shown in Fig.7. we can not always get the maximum result, because the maximum value of T_1 is real number in analog model. Let us generalize $\boldsymbol{I}_0$ of $\boldsymbol{G} = -\boldsymbol{W}\boldsymbol{I}_0\boldsymbol{W}$ and $\boldsymbol{I}_\tau$ operators as follows:

$$\tilde{\boldsymbol{I}}_\tau = \begin{pmatrix} \mathrm{e}^{i\alpha_0} & & & 0 \\ & \mathrm{e}^{i\alpha_1} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{i\alpha_{N-1}} \end{pmatrix} \quad (93)$$

$$\tilde{\boldsymbol{I}}_0 = \begin{pmatrix} \mathrm{e}^{i\beta_0} & & & 0 \\ & \mathrm{e}^{i\beta_1} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{i\beta_{N-1}} \end{pmatrix}. \quad (94)$$

Now, let $\alpha_{i_0} = \cdots = \alpha_{i_{k-1}} = \alpha$ and $\alpha_j = 1$ for $j \notin \{i_0, \cdots, i_{k-1}\}$ and let $\beta_0 = 0$ and $\beta_1 = \cdots = \beta_{N-1} = \beta$, that is

$$\boldsymbol{U}(\alpha) = \begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & \mathrm{e}^{i\alpha} & & & & \\ & & & \ddots & & & \\ & & & & \mathrm{e}^{i\alpha} & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{pmatrix} \quad (95)$$

$$\boldsymbol{V}(\beta) = \begin{pmatrix} 1 & & & 0 \\ & \mathrm{e}^{i\beta} & & \\ & & \ddots & \\ 0 & & & \mathrm{e}^{i\beta} \end{pmatrix}. \quad (96)$$

Then, $\boldsymbol{G}$ and $\boldsymbol{I}_\tau$ are generalized as follows:

$$\boldsymbol{U}(\alpha) = \boldsymbol{I} - (1-\mathrm{e}^{i\alpha})\sum_{k=1}^{k}|\rho_k\rangle\langle\rho_k| \quad (97)$$

$$\boldsymbol{V}(\beta) = (1-\mathrm{e}^{i\beta})|s\rangle\langle s| + \mathrm{e}^{i\beta}\boldsymbol{I} \quad (98)$$

Let any state be defined by $|\psi\rangle = \sum_{x=0}^{N-1} w_x|x\rangle$.
Then $\boldsymbol{U}\boldsymbol{V}|\psi\rangle$ transits as follows:

$$\sum_{x=0,x\notin Y}^{N-1}(\bar{w} - (\bar{w} - w_x)\mathrm{e}^{i\beta})|x\rangle + \sum_{l=1}^{k}(\bar{w} - (\bar{w} - w_{\rho_l})\mathrm{e}^{i\beta})|\rho_l\rangle \quad (99)$$

, where $\bar{w} = (1/N)\sum_{x=0}^{N-1} w_x$, $Y = \{i_0, \cdots, i_{k-1}\}$ and $\rho_l \in Y$.

With (99), let $w_{\rho_l} = 1/\sqrt{k}$ for $l \in Y$ and $w_x = 0$ for $x \notin Y$

Then,

$$\begin{aligned} \boldsymbol{U}\boldsymbol{V}|\psi\rangle &= \sum_{x=0,x\notin Y}^{N-1}(\bar{w} - (\bar{w} - w_x)\mathrm{e}^{i\beta})|x\rangle \\ &\quad + \sum_{l=1}^{k}(\bar{w} - (\bar{w} - w_{\rho_l})\mathrm{e}^{i\beta})|\rho_l\rangle \\ &= \sum_{x=0,x\notin Y}^{N-1}\frac{\sqrt{k}}{N}\mathrm{e}^{i\alpha}(1-\mathrm{e}^{i\beta})|x\rangle \\ &\quad + \sum_{l=1}^{k}\{\frac{\sqrt{k}}{N}\mathrm{e}^{i\alpha}(1-\mathrm{e}^{i\beta}) + \frac{1}{\sqrt{k}}\mathrm{e}^{i(\alpha+\beta)}\}|\rho_l\rangle. \end{aligned} \quad (100)$$

As there does not exist the condition that the first and the second terms of the right-hand side of (100) agree, $(\boldsymbol{U}\boldsymbol{V})^2|\psi\rangle$ is computed.

As a result, the condition that the amplitudes of $|x\rangle$ and $|\rho_l\rangle$ for $(\boldsymbol{U}\boldsymbol{V})^2|\psi\rangle$ agree is as follows:

$$\begin{aligned} &\frac{\sqrt{k}}{N}(1-\mathrm{e}^{i\beta}) \\ &= \frac{\sqrt{k}}{N}\mathrm{e}^{i\alpha}(1-\mathrm{e}^{i\beta}) + \frac{1}{\sqrt{k}}\mathrm{e}^{i(\alpha+\beta)} \end{aligned} \quad (101)$$

Therefore, the following relation is needed in order that (101) holds:

$$\begin{cases} k(\cos\alpha + \cos\beta) - k + (N-k)\cos(\alpha+\beta) = 0 \\ (N-k)\sin(\alpha+\beta) + k\sin\beta + k\sin\alpha = 0. \end{cases} \quad (102)$$

Assume that $\alpha = -\beta$, Then,

$$\cos\beta = \frac{2k-N}{2k}. \quad (103)$$

As $|\cos\beta| \leq 1$, the following condition is necessary.

$$\frac{N}{4} \leq k \quad (104)$$

(103) agrees with Liu's result[10].

The algorithm is shown as Fig.8.

1. Initial state $|\psi\rangle$
2. Repeat 2 times
3. $|\psi\rangle = \boldsymbol{U}(-\beta)|\psi\rangle$
4. $|\psi\rangle = \boldsymbol{V}(\beta)|\psi\rangle$
5. Repeat T_g times
6. $|\psi\rangle = \boldsymbol{U}(\pi)|\psi\rangle$
7. $|\psi\rangle = \boldsymbol{V}(-\pi)|\psi\rangle$
8. Observe the system

Fig. 8. The algorithm for the proposed method 4

Example 5:

Let $N = 16$ and $k = 8$. Let the initial state be defined as (91). In the proposed method 4, α is set to $\pi/2$. As $T_2 = 2$, the steps 3 to 4 of the proposed method 4 are iterated two times. Then, the following state is obtained:

$$\begin{aligned} |\psi\rangle &= \frac{1}{4\sqrt{2}}(-1+i, -1+i, -1+i, -1+i, \\ &\quad -1+i, -1+i, -1+i, -1+i-1+i, \\ &\quad -1+i, -1+i, -1+i, -1+i, -1+i, \\ &\quad -1+i, -1+i)^{\mathrm{T}}. \end{aligned} \quad (105)$$

TABLE I
A SUMMARY OF THE PROPOSED METHODS.

	Analog model	Digital model
Schrodinger equation[11]	Method 1	Grover algorithm
	Proposed method 2	Proposed method 3
		Proposed method 4, [11]

Grover algorithm shows good performance only in the case of $N = k$. The proposed method 3 shows better performance compared with Grover algorithm, but does not always show good performance in the case of $N \neq k$.

The proposed algorithm 4 shows best performance of three algorithms in digital model.

V. CONCLUSIONS AND FUTURE WORK

The result in this paper is summarized in Table I. The method 1 in analog time is obtained by solving the schrodinger equation. As the method 1 does not always show good performance in the case of $k \neq N$, we propose the method 2 by improving the method 1. The proposed method 2 gave the optimum result in analog time. The method 1 and the proposed method 2 in analog time lead to Grover algorithm and the proposed method 3 in digital time, respectively. The proposed method 2 in analog time gives the optimum solution, but the proposed method 3 in digital method 3 does not give the optimum solution, because the optimum time is real number. Therefore, we propose the generalized model, and gave an algorithm as a special case of it using the phase rotation and showed that it gives better solution. The result agree with one of Liu[10]. The verification of the algorithms in shown in the theory and numerical simulations. As the future work, we will consider the relation between the proposed method 2 and 4, and find the optimum solution in digital model.

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