Morita Theory for The Category of Fuzzy S-acts

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***Abstract*—In this paper, we give the properties of tensor product and study the relationship between Hom functors and left (right) exact sequences in FS-Act. Also, we get some necessary conditions for equivalence of two fuzzy S-acts category. Moreover, we prove that two monoids S and T are Morita equivalent if and only if FS-Act and FT-Act are equivalent.**

***Index Terms*—fuzzy set, fuzzy semigroup, fuzzy S-act, Morita equivalent, category.**

I. Introduction

ZADEH [27] introduced the fuzzy set theory. In [20], Negoita and Ralescu applied fuzzy set theory to modules theory. Muganda [19] studied free fuzzy modules. Also, López-Permouth and Malik [13] gave some properties of fuzzy module category. Now, many authors have applied fuzzy set theory to semigroups, see [1], [9], [10], [11], [14], [24]. Ahsan et al. [1] studied fuzzy subacts of an S-act. Ali, Davvaz and Shabir [2] defined soft S-acts and characterized (α, β)-fuzzy subacts using soft S-acts.

Morita theory characterizes the relationship of two rings which preserves many ring properties. Morita theory plays an important role in ring theory and algebra theory. There are also many papers on Mortia theory for semigroups, see [3], [7], [8], [25]. In these papers, authors have also got equivalences between subcategories of S-acts. López-Permouth [12] characterized Morita theory of two rings R and S using fuzzy module categories. While, there is few paper on Morita theory for the category of fuzzy S-acts. The aim of this paper is to generalize the Morita theory in the paper [12], [25] to the category of fuzzy S-acts. These theory will be useful to the study of fuzzy semigroups and fuzzy S-acts.

The content of the paper is constructed as follows. In Section 2, we recall some definitions on semigroups and fuzzy sets. In Section 3, we study the properties of Hom sets and tensor products in the category of fuzzy S-acts. Using these properties, we get some necessary conditions for equivalences of FS-Act in Section 4. In Section 5, we get that two monoids S and T have equivalent categories of FS-Act and FT-Act if and only if S and T are Morita equivalent.

II. Preliminaries

Let S be a semigroup. A set A is called a (left) S-act, if there is a scalar multiplication $S \times A \longrightarrow A$, denoted by $(s, a) \longrightarrow sa$ and for all $s, t \in S, a \in A$, we have

$$(st)a = s(ta).$$

If A is a left S-act, we write ${}_SA$. If S is a monoid with 1 and A is a left S-act, for all $a \in A$, we have $1a = a$, then A is said to be unital. For ${}_SA$ and ${}_SB$ and a map $f : A \longrightarrow B$, if for all $s \in S, a \in A$, we have

$$f(sa) = sf(a),$$

then f is a (left) S-morphism. Let $\mathrm{Hom}_S(A, B)$ be the set of S-morphism from ${}_SA$ to ${}_SB$. We denote the category of left S-acts by S-Act.

Similarly, we can define right S-acts. We denoted by Act-S the category of right S-acts.

Suppose that S is a commutative semigroup and M is a left S-act, then it is obvious that M is also a right S-act.

If M is a left S-act and a right T-act, and for all $s \in S, t \in T, x \in M$, we have $(sx)t = s(xt)$, then M is called an S-T-biact. We write (S, T)-Act for the category of S-T-biacts.

Let A be a left S-act. If a map $\alpha : A \to [0, 1]$ satisfies $\alpha(sa) \geq \alpha(a)$, $\forall s \in S, a \in A$, then α_A is called a fuzzy (left) S-act ([1]).

Similarly, the fuzzy right S-acts can be defined.

Let A be an S-T-biact. If a map $\alpha : A \to [0, 1]$ satisfies $\alpha(sa) \geq \alpha(a)$ and $\alpha(at) \geq \alpha(a)$, $\forall s \in S, t \in T, a \in A$, then α_A is called a fuzzy S-T-biact.

For two fuzzy S-acts α_A and β_B, if a map $\tilde{f} : A \to B$ is an S-morphism and satisfies $\beta(\tilde{f}(a)) \geq \alpha(a)$, $\forall a \in A$, then $\tilde{f}$ is called a fuzzy S-morphism.

Let S be a semigroup. We denote the set of all fuzzy S-morphisms from α_A to β_B by $\mathrm{Hom}_S(\alpha_A, \beta_B)$. Let FS-Act be the category of fuzzy left S-acts.

Let $\tilde{f} : \alpha_A \to \beta_B$ be a fuzzy S-morphism. 1) If $\tilde{f}$ is an epimorphism (a monomorphism), then $\tilde{f}$ is a fuzzy epimorphism (monomorphism). 2) If $\beta(\tilde{f}(a)) = 1$, then $\alpha(a) = 1$, $\forall a \in A$, $\tilde{f}$ is a fuzzy quasi-monomorphism. 3) If $\tilde{f} : A \to B$ is an S-isomorphism, then $\tilde{f}$ is a fuzzy quasi-isomorphism. 4) If $\tilde{f} : A \to B$ is an S-isomorphism and for all $a \in A$, we have $\beta(\tilde{f}(a)) = \alpha(a)$, then $\tilde{f}$ is a fuzzy isomorphism. 5) The kernel of $\tilde{f}$ is $KER(\tilde{f}) = \{a \in A | \beta(\tilde{f}(a)) = 1\}$.

If $M \in S$-Act, 0_M represents the fuzzy S-act $0 : M \longrightarrow [0, 1]$ such that $0(m) = 0$, for all $m \in M$. 1_M represents the fuzzy S-act $1 : M \longrightarrow [0, 1]$ such that $1(m) = 1$, for all $m \in M$.

II

Proposition II.1 Let S be a semigroup. The category FS-Act has coproduct.

Proof Suppose that $\{\alpha_{i A_i} | i \in I\}$ are a family of fuzzy S-acts. Then the coproduct $\coprod_{i \in I} A_i$ in S-Act is the disjoint union $\bigcup_{i \in I} A_i$. Define a map $\alpha : \coprod_{i \in I} A_i \longrightarrow [0, 1]$ by putting $\alpha(a_i) = \alpha_i(a_i)$, if $a_i \in A_i$. Then α is the coproduct of $\{\alpha_{i A_i} | i \in I\}$ in FS-Act.

Definition II.2 For $\alpha_A \in FS$-Act, if for all $\eta_C \in FS$-Act, there is an epimorphism $\alpha_A^{(I)} \longrightarrow \eta_C$, where I is index set, then α_A is called a generator.

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III. Hom sets and tensor products in FS-Act

In this section, we shall study the properties of Hom sets and tensor products in FS-Act.

The following lemma is useful in Proposition III.2.

Lemma III.1 [14] Let S be a commutative semigroup. For two fuzzy S-acts α_A and β_B, we can make $\mathrm{Hom}_S(\alpha_A, \beta_B)$ to be a fuzzy S-act by the function $\tilde{\gamma} : \mathrm{Hom}_S(\alpha_A, \beta_B) \longrightarrow [0, 1]$, which is defined by $\tilde{\gamma}(\tilde{f}) = \bigwedge\{\beta(\tilde{f}(a))|a \in A\}$.

Proposition III.2 Let S be a commutative semigroup and $\alpha_A \in FS$-Act. Then $\tilde{\gamma}_{\mathrm{End}_{S(\alpha_A,\alpha_A)}}$ is a fuzzy monoid.

Proof Obviously, $\tilde{\gamma}_{\mathrm{End}_{S(\alpha_A,\alpha_A)}}$ is a monoid. Assume $\tilde{f}_1, \tilde{f}_2 \in \tilde{\gamma}_{\mathrm{End}_{S(\alpha_A,\alpha_A)}}$ and $a \in A$. Since

$$\alpha(\tilde{f}_1(\tilde{f}_2(a))) \geq \alpha(\tilde{f}_2(a))$$

and the definition of $\tilde{\gamma}$ in Lemma III.1, we get

$$\tilde{\gamma}(\tilde{f}_1\tilde{f}_2) \geq \tilde{\gamma}(\tilde{f}_2) \geq \bigwedge\{\tilde{\gamma}(\tilde{f}_1), \tilde{\gamma}(\tilde{f}_2)\}.$$

Hence, we get that $\tilde{\gamma}_{\mathrm{End}_{S(\alpha_A,\alpha_A)}}$ is a fuzzy monoid.

Let $A \in$ Act-S, $B \in S$-Act and C be a set. If a map $\varphi : A \times B \longrightarrow C$ satisfies $\varphi(as, b) = \varphi(a, sb)$, for all $a \in A, s \in S$ and $b \in B$, then φ is called a bimap ([4]); If for all set H and all bimap $f : A \times B \longrightarrow H$, there exists a unique map $g : C \to H$ such that $g\varphi = f$, then the pair (C, φ) is called a tensor product of A and B ([4]). We write $A \otimes_S B$ for the tensor product of A and B. By Proposition 8.1.8 of [4], for $a \otimes b$, $c \otimes d \in A \otimes B$, then $a \otimes b = c \otimes d \Longleftrightarrow$ either $(a, b) = (c, d)$, or there exist $g_1, \cdots, g_{n-1} \in A, h_1, \cdots, h_{n-1} \in B, r_1, \cdots, r_n, s_1 \cdots, s_{n-1} \in S$ such that

$$\begin{array}{ll} a = g_1r_1, & r_1b = s_1h_1, \\ g_1s_1 = g_2r_2, & r_2h_1 = s_2h_2, \\ g_is_i = g_{i+1}r_{i+1}, & r_{i+1}h_i = s_{i+1}h_{i+1}, \\ i = 2, \cdots, n-2, & \\ g_{n-1}s_{n-1} = cr_n, & r_nh_{n-1} = d \text{ ([4])}. \end{array}$$

Let α_A and β_B be two fuzzy sets. Define $\alpha_A \times \beta_B(a, b) = \bigwedge\{\alpha(a), \beta(b)\}$, for all $(a, b) \in A \times B$, then $\alpha_A \times \beta_B$ is a fuzzy set.

Let $\alpha_A \in$ Act-FS, $\beta_B \in FS$-Act and η_C be a fuzzy set. Let $\tilde{f} : A \times B \longrightarrow C$ be a map. If $\tilde{f}$ is a bimap and for all $(a, b) \in A \times B$, we have

$$\eta\tilde{f}(a, b) \geq (\alpha_A \times \beta_B)(a, b),$$

then $\tilde{f}$ is called a fuzzy bimap ([14]).

Theorem III.3 [14] Let $\alpha_A \in$ Act-FS and $\beta_B \in FS$-Act. The tensor product of α_A and β_B exists, where the function $\alpha \otimes \beta : A \otimes B \longrightarrow [0, 1]$ is defined by

$$\alpha \otimes \beta(a \otimes b) = \bigvee\{\bigwedge\{\alpha(a'), \beta(b')\}|a' \otimes b' = a \otimes b\}.$$

Moreover, it is unique up to isomorphism.

We write $\alpha_A \otimes \beta_B$ for the tensor product of α_A and β_B.

Denote by (FS, FT)-Act the category of fuzzy S-T-biacts. In the following, we shall study the associativity of the tensor product in fuzzy S-acts category.

Proposition III.4 Let $1_A \in$ Act-FS, $\beta_B \in (FS, FT)$-Act and $\eta_C \in FT$-Act. There is a quasi-isomorphism $1_A \otimes (\beta_B \otimes \eta_C) \cong (1_A \otimes \beta_B) \otimes \eta_C$.

Proof In S-Act, there is an isomorphism $f : 1_A \otimes (\beta_B \otimes \eta_C) \to (1_A \otimes \beta_B) \otimes \eta_C$ by $f(a \otimes (b \otimes c)) = (a \otimes b) \otimes c$.

For every $a \otimes (b \otimes c) \in 1_A \otimes (\beta_B \otimes \eta_C)$, we have

$$\begin{aligned} & 1 \otimes (\beta \otimes \eta)(a \otimes (b \otimes c)) \\ = & \bigvee\{\bigwedge\{1(a'), \beta \otimes \eta(b' \otimes c')\}|a' \otimes (b' \otimes c') = \\ & a \otimes (b \otimes c)\} \\ = & \bigvee\{\beta \otimes \eta(b' \otimes c')|a' \otimes (b' \otimes c') = a \otimes (b \otimes c)\} \\ = & \bigvee\{\bigvee\{\bigwedge\{\beta(b'), \eta(c')\}\}|a' \otimes (b' \otimes c') = \\ & a \otimes (b \otimes c)\} \\ = & \bigvee\{\bigwedge\{\beta(b'), \eta(c')\}|a' \otimes (b' \otimes c') = a \otimes (b \otimes c)\}. \end{aligned}$$

On the other hand, we can prove that

$$\begin{aligned} & (1 \otimes \beta) \otimes \eta((a \otimes b) \otimes c) \\ = & \bigvee\{\bigwedge\{(1 \otimes \beta)(a' \otimes b'), \eta(c')\}|(a' \otimes b') \otimes c' = \\ & (a \otimes b) \otimes c\} \\ = & \bigvee\{\bigwedge\{\bigvee\{\bigwedge\{(1(a'), \beta(b')\}\}, \eta(c')\}|(a' \otimes b') \otimes c' \\ & = (a \otimes b) \otimes c\} \\ = & \bigvee\{\bigwedge\{\bigvee\{\beta(b')\}, \eta(c')\}|(a' \otimes b') \otimes c' = \\ & (a \otimes b) \otimes c\}. \end{aligned}$$

Then $((1 \otimes \beta) \otimes \eta)f(a \otimes (b \otimes c)) \geq 1 \otimes (\beta \otimes \eta)((a \otimes b) \otimes c)$. We get the desired result.

The following statement gives the relationship between the tensor product and the coproduct in FS-Act.

Proposition III.5 Let $\xi_M \in FS$-Act and $\coprod_{i \in I} \alpha_{iA_i} \in$ Act-FS. Then there is an isomorphism

$$(\coprod_{i \in I} \alpha_{iA_i}) \otimes \xi_M \cong \coprod_{i \in I} (\alpha_{iA_i} \otimes \xi_M).$$

Similarly, let $\xi_M \in$ Act-FS and $\coprod_{i \in I} \alpha_{iA_i} \in FS$-Act. Then there is an isomorphism

$$\xi_M \otimes (\coprod_{i \in I} \alpha_{iA_i}) \cong \coprod_{i \in I} (\xi_M \otimes \alpha_{iA_i}).$$

Proof It is obvious by the definition of coproduct in Proposition II.1 and the related results in the category of S-acts.

Let $\tilde{\varphi} \in \mathrm{Hom}_S(\mu_L, \nu_M)$. If there is an element $\tilde{\psi} \in \mathrm{Hom}_S(\nu_M, \mu_L)$ satisfying $\tilde{\psi}\tilde{\varphi} = 1$, then $\tilde{\varphi}$ is split. Write $IMA(\tilde{\varphi}) = \{\tilde{\varphi}(l)|l \in L\}$.

Definition III.6 A sequence

$$\mu_L \xrightarrow{\tilde{\varphi}} \nu_M \xrightarrow{\tilde{\psi}} \eta_N$$

is called left exact, if $\tilde{\varphi}$ is quasi-monic and $IMA(\tilde{\varphi}) = KER(\tilde{\psi})$. Similarly, if $\tilde{\psi}$ is epic and $IMA(\tilde{\varphi}) = KER(\tilde{\psi})$, then the sequence is called right exact.

Similar to Theorem 2 in [21], we can prove the following theorems.

Theorem III.7 Let S be a commutative semigroup. Suppose that

$$\mu_L \xrightarrow{\tilde{\varphi}} \nu_M \xrightarrow{\tilde{\psi}} \eta_N$$

is left exact. If $\tilde{\varphi}$ is split, then $\mathrm{Hom}(\theta_P, -)$ preserves the sequence.

Proof Let $H = \mathrm{Hom}(\theta_P, -)$. We prove

$$\alpha_{\mathrm{Hom}(\theta_P,\mu_L)} \xrightarrow{H\tilde{\varphi}} \beta_{\mathrm{Hom}(\theta_P,\nu_M)} \xrightarrow{H\tilde{\psi}} \gamma_{\mathrm{Hom}(\theta_P,\eta_N)}$$

is also left exact.

1) Suppose $\beta(H\tilde{\varphi}(\tilde{f}))=1$, i.e. $\beta(\tilde{\varphi}\tilde{f})=1$, where $\tilde{f}\in$ Hom(θ_P,μ_L). By Lemma III.1, we have $\bigwedge\{\nu(\tilde{\varphi}\tilde{f}(p))\mid p\in P\}=1$. That is, for all $p\in P$, we have $\nu(\tilde{\varphi}\tilde{f}(p))=1$. Since $\tilde{\varphi}$ is quasi-monic, we have $\mu(\tilde{f}(p))=1,\forall p\in P$. This concludes that $\alpha(\tilde{f})=1$. Hence, $H\tilde{\varphi}$ is quasi-monic.

2) $IMA(H\tilde{\varphi})\subseteq KER(H\tilde{\psi})$. For $\tilde{f}\in IMA(H\tilde{\varphi})$, suppose $\tilde{g}\in$ Hom(θ_P,μ_L) and $\tilde{f}=H\tilde{\varphi}(\tilde{g})=\tilde{\varphi}\tilde{g}$. Note that $IMA(\tilde{\varphi})=KER(\tilde{\psi})$. We have $\eta(\tilde{\psi}\tilde{\varphi}\tilde{g}(p))=1,\forall p\in P$. Hence, $\gamma(H\tilde{\psi}(\tilde{g}))=1$. Then $\tilde{g}\in KER(F\tilde{\psi})$.

3) $KER(H\tilde{\psi})\subseteq IMA(H\tilde{\varphi})$. Suppose $\tilde{f}\in KER(H\tilde{\psi})$ and $\gamma(H\tilde{\psi}(\tilde{f}))=1$. Then $\eta(\tilde{\psi}\tilde{f}(p))=1$, $\forall p\in P$. Hence, $\tilde{f}(p)\in KER(\tilde{\psi})=IMA(\tilde{\varphi})$. Choose an element $l_p\in L$ such that

$$\mu(l_p)=\bigvee\{\mu(l)\mid\tilde{\varphi}(l)=\tilde{f}(p),l\in L\}.$$

Suppose $\tilde{k}\in$ Hom(ν_M,μ_L) and $\tilde{k}\tilde{\varphi}=1$. Then

$$\mu(l_p)=\mu(\tilde{k}\tilde{\varphi}(l_p))\geq\nu(\tilde{\varphi}(l_p))\geq\mu(l_p).$$

Then $\mu(l_p)=\nu\tilde{\varphi}(l_p)$. Define $\tilde{g}:\theta_P\longrightarrow\mu_L$ by $\tilde{g}(p)=l_p$. Then $H\tilde{\varphi}(\tilde{g})=\tilde{\varphi}\tilde{g}=\tilde{f}$. Since

$$\mu(\tilde{g}(p))=\mu(l_p)=\nu(\tilde{\varphi}(l_p))=\nu(\tilde{f}(p))\geq\theta(p),$$

this proves that $\tilde{g}$ is fuzzy.

Theorem III.8 Let S be a commutative semigroup. Suppose that

$$\mu_L\xrightarrow{\tilde{\varphi}}\nu_M\xrightarrow{\tilde{\psi}}\eta_N$$

is right exact. If $\tilde{\varphi}$ is split, then the sequence

$$\alpha_{\text{Hom}(\mu_L,\theta_P)}\xrightarrow{\text{Hom}(\tilde{\varphi},\theta_P)}\beta_{\text{Hom}(\nu_M,\theta_P)}\xrightarrow{\text{Hom}(\tilde{\psi},\theta_P)}\gamma_{\text{Hom}(\eta_N,\theta_P)}$$

is also right exact.

IV. Properties of equivalent functors

Notation In this section, we shall assume that S is a commutative semigroup.

For a fuzzy left S-act α_A, we have that $\tilde{\gamma}_{\text{Hom}_S(0_S,\alpha_A)}$ is a fuzzy left S-act by Lemma III.1. For $a\in A$, we define a map

$$\tilde{\rho_a}:0_S\longrightarrow\alpha_A$$

by putting

$$\tilde{\rho_a}(s)=sa.$$

Since $\alpha\rho_a(s)=\alpha(sa)\geq 0=0(s)$, we have $\rho_a\in\tilde{\gamma}_{\text{Hom}_S(0_S,\alpha_A)}$. On the other hand, for all $\tilde{f}\in\tilde{\gamma}_{\text{Hom}_S(0_S,\alpha_A)}$ and $s,q\in S$, we have

$$(s\cdot\tilde{f})(q)=\tilde{f}(qs)=q\tilde{f}(s)=\rho_{\tilde{f}(s)}(q).$$

Hence, $s\cdot\tilde{f}=\rho_{\tilde{f}(s)}$.

We now define a map as follows:

$$\Delta_{\alpha_A}:0_S\otimes\tilde{\gamma}_{\text{Hom}_S(0_S,\alpha_A)}\longrightarrow\alpha_A,$$
$$s\otimes\tilde{f}\longmapsto\tilde{f}(s).$$

Lemma IV.1 Let S be a monoid and $\alpha_A\in FS$-Act. Then Δ_{α_A} is a fuzzy S-epimorphism.

Proof The proof is similar to Lemma 4.1 in [25]. Suppose $a\otimes\tilde{f}=c\otimes\tilde{g}$, where $a,c\in S,\tilde{f},\tilde{g}\in\tilde{\gamma}_{\text{Hom}_S(0_S,\alpha_A)}$. By Proposition 8.1.8 of [4], we have $(a,\tilde{f})=(c,\tilde{g})$ or there exist $a_1,\cdots,a_{n-1}\in S,\tilde{f_1},\cdots,\widetilde{f_{n-1}}\in$ Hom$_S(0_S,\alpha_A),r_1,\cdots,r_n,s_1,\cdots,s_{n-1}\in S$ such that

$$\begin{array}{ll} a=a_1r_1, & s_1\cdot\tilde{f}=s_1\cdot\tilde{f_1},\\ a_1s_1=a_2r_2, & s_2\cdot\tilde{f_1}=s_2\cdot\tilde{f_2},\\ a_is_i=a_{i+1}r_{i+1}, & r_{i+1}\cdot\tilde{f_i}=t_{i+1}\cdot\widetilde{f_{i+1}},\\ i=2,\cdots,n-2, & \\ a_{n-1}s_{n-1}=cr_n, & s_n\cdot\widetilde{f_{n-1}}=\tilde{g}.\end{array}$$

Hence we get

$$\tilde{f}(a)=\tilde{\gamma}(a_1r_1)=r_1\cdot\tilde{f}(a_1)=s_1\cdot\tilde{f_1}(a_1)$$
$$=\tilde{f_1}(a_1s_1)=\cdots=\tilde{g}(c).$$

This concludes that $\tilde{f}(a)=\tilde{g}(c)$ and so Δ_{α_A} is well-defined.

It is clear that Δ_{α_A} is an S-morphism. Since $\alpha\Delta_{\alpha_A}(r\otimes\tilde{f})=\alpha(\tilde{f}(r))\geq 0=0\otimes\tilde{\gamma}(r\otimes\tilde{f})$, we get that Δ_{α_A} is a fuzzy morphism.

Suppose $a\in A$. Then $\tilde{\rho_a}\in\tilde{\gamma}_{\text{Hom}(0_S,\alpha_A)}$ and

$$\Delta_{\alpha_A}(1\otimes\tilde{\rho_a})=\tilde{\rho_a}(1)=1\cdot a=a.$$

It follows that Δ_{α_A} is surjective and we get the desired result.

Lemma IV.2 Let S be a monoid and $0_S\in S$-Act. We have $0_S\otimes\tilde{\gamma}_{\text{Hom}_S(0_S,0_S)}\cong 0_S$.

Proof By Lemma IV.1, we know that Δ_{0_S} is a fuzzy epimorphism. We have to prove that Δ_{0_S} is injective. Let $s,t\in S,\tilde{f},\tilde{g}\in\tilde{\gamma}_{\text{Hom}_S(0_S,0_S)}$ and $\Delta_{0_S}(s\otimes\tilde{f})=\Delta_{0_S}(t\otimes\tilde{g})$. By the definition of Δ_{0_S}, we have $\tilde{f}(s)=\tilde{g}(t)$. In $0_S\otimes\tilde{\gamma}_{\text{Hom}_S(0_S,0_S)}$, we have $1\otimes\widetilde{\rho_{\tilde{f}(s)}}=1\otimes\widetilde{\rho_{\tilde{g}(t)}}$. Hence, we get

$$s\otimes\tilde{f}=1\cdot s\otimes\tilde{f}=1\otimes s\cdot\tilde{f}=1\otimes\rho_{\tilde{f}(s)}=1\otimes\rho_{\tilde{g}(t)}$$
$$=1\otimes t\cdot\tilde{g}=1\cdot t\otimes\tilde{g}=t\otimes\tilde{g}.$$

This concludes that Δ_{0_S} is injective and so is an isomorphism.

Remark IV.3 Denote by F_0S-Act=$\{0_M|M\in S\text{-Act}\}$. Let $L_S:F_0S$-Act $\longrightarrow S$-Act be a functor such that $L_S(0_A)=A$ and for every fuzzy morphism $\tilde{f}:\alpha_A\longrightarrow\beta_B$, $L_S(\tilde{f})=\tilde{f}$. Then F_0S-Act and S-Act are equivalent and L_S is invertible.

The following statement gives a characterization of objects in F_0S-Act.

Lemma IV.4 Let S be a semigroup. $\alpha_A\in F_0S$-Act if and only if for any fuzzy morphism $\tilde{f}:\beta_B\longrightarrow\alpha_A$, where $\beta_B\in FS$-Act, if $\tilde{f}$ is both monomorphism and epimorphism, then $\tilde{f}$ is invertible.

Analogous to Lemma 5.3 in [12], we get the following proposition.

Proposition IV.5 Let S and T be two semigroups. If $G:FS$-Act $\longrightarrow FT$-Act induces an equivalence between FS-Act and FT-Act, then G induces an equivalence $G|_{F_0S-\text{Act}}:F_0S$-Act $\longrightarrow F_0T$-Act.

Proof If $0_A\in F_0S$-Act, by the Lemma IV.4, we have $G(0_A)\in F_0T$-Act. Then we can get the desired result.

Let us denote by QFS-Act $=\{A\in FS\text{-Act}|\ \Delta_{\alpha_A}$ is a quasi-isomorphism$\}$.

We now use the properties of equivalence to get various necessary conditions. The following statement is similar to Theorem 6.1 in [25].

Theorem IV.6 Let S and T be two commutative monoids. If there exists equivalence: $\Phi:QFS$-Act $\rightleftharpoons QFT$-Act $:\Psi$. Write $0_U=\Psi(0_T)$ and $0_V=\Phi(0_S)$. Then 0_U is a fuzzy unitary S-T-biact and 0_V is a fuzzy unitary T-S-biact. Moreover, the following conditions are satisfied:

(1) $\Phi \cong 0_T \otimes \tilde{\gamma}_{\mathrm{Hom}_S(0_U,-)}$, $\Psi \cong 0_S \otimes \tilde{\beta}_{\mathrm{Hom}_T(0_V,-)}$;

(2) $0_U \cong 0_S \otimes \tilde{\gamma}_{\mathrm{Hom}_T(0_V,0_T)}, 0_V \cong 0_T \otimes \tilde{\beta}_{\mathrm{Hom}_S(0_U,0_S)}$ as fuzzy biacts.

Proof We obviously have that 0_V is naturally a fuzzy right $\mathrm{End}_T V$-act. Since $\mathrm{End}_S S \cong \mathrm{End}_T V$, 0_V is a fuzzy right $\mathrm{End}_S S$-act. Defining $v \cdot s = G(\rho_s)(v)$, then 0_V is a fuzzy T-S-biact. Similarly, we can check that 0_U is a fuzzy S-T-biact.

(1) If $\alpha_M \in QFS$-Act, then $\Phi(\alpha_M) \in QFT$-Act. That is,

$$\Phi(\alpha_M) \cong 0_T \otimes \tilde{\Delta}_{\mathrm{Hom}_T(0_T,\Phi(\alpha_M))}.$$

On the other hand, as Φ and Ψ are eqivalent functors, we have

$$\tilde{\Delta}_{\mathrm{Hom}_T(0_T,\Phi(\alpha_M))} \cong \tilde{\gamma}_{\mathrm{Hom}_S(\Psi(0_T),\alpha_M)} = \tilde{\gamma}_{\mathrm{Hom}_S(0_U,\alpha_M)}.$$

Hence, we get $\Phi(\alpha_M) \cong 0_T \otimes \tilde{\gamma}_{\mathrm{Hom}_S(0_U,\alpha_M)}$. Then $\Phi \cong 0_T \otimes \tilde{\gamma}_{\mathrm{Hom}_S(0_U,-)}$. Similarly, $\Psi \cong 0_S \otimes \tilde{\beta}_{\mathrm{Hom}_T(0_V,-)}$.

(2) Let $\tilde{\xi} : \Phi \longrightarrow 0_T \otimes \tilde{\gamma}_{\mathrm{Hom}_S(0_U,-)}$ be the natural isomorphism. Then we have the following commutative diagram

$$\begin{array}{ccc} \Phi(0_S)(=0_V) & \overset{G(\tilde{\rho_s})}{\rightarrow} & \Phi(0_S)(=0_V) \\ \tilde{\xi}_S \downarrow & & \downarrow \tilde{\xi}_S \\ 0_T \otimes \tilde{\gamma}_{\mathrm{Hom}_S(0_U,0_S)} & \overset{1_T \otimes \tilde{\rho_s}^*}{\rightarrow} & 0_T \otimes \tilde{\gamma}_{\mathrm{Hom}_S(0_U,0_S)}, \end{array}$$

where $1_T \otimes \tilde{\rho_s}^*(t \otimes f) = t \otimes \tilde{\rho_s}(f) = t \otimes f \cdot s = (t \otimes f)s$. That is, for all $v \in 0_V$, we have

$$1_T \otimes \tilde{\rho_s}^* \tilde{\xi_S}(v)) = \tilde{\xi_S}(v) \cdot s.$$

Hence, we get

$$\tilde{\xi_S}(vs) = \tilde{\xi_S}(G(\tilde{\rho_s})(v)) = \tilde{\xi_S}(1_S \otimes \rho_s)(v) = (\tilde{\xi_S}(v))s.$$

This shows that $\tilde{\xi_S}$ is a fuzzy right S-morphism. Therefore, we have $0_V \cong 0_T \otimes \mathrm{Hom}_S(0_U, 0_S)$ as fuzzy T-S-biact. Similarly, we can get $0_U \cong 0_S \otimes_S \mathrm{Hom}_T(0_V, 0_T)$.

V. Equivalence of fuzzy S-act categories

In this section, we will prove that two monoids are Morita equivalent if and only if their fuzzy S-acts categories are equivalent.

Definition V.1 [6] Two monoids S and T are Morita equivalent if the two categories S-Act and T-Act are equivalent by the functors $\Phi = U \otimes_S -$ and $\Psi = V \otimes_T -$, where ${}_T U_S$ and ${}_S V_T$ are biacts.

Lemma V.2 Let S and T be two monoids and α_A be a fuzzy S-T-biact. Then $\alpha_A \otimes -$ takes values in FS-Act.

Proof Let $\beta_B \in FS$-Act. It is well-known that $A \otimes B$ is a left S-act. Since

$$\begin{array}{rl} & (\alpha \otimes \beta)(s(a \otimes b)) \\ = & \bigvee\{\bigwedge\{\alpha(a'), \beta(b')\} | a' \otimes b' = sa \otimes b\} \\ \geq & \bigvee\{\bigwedge\{\alpha(sa''), \beta(b'')\} | sa'' \otimes b'' = sa \otimes b\} \\ \geq & \bigvee\{\bigwedge\{\alpha(a'''), \beta(b''')\} | a''' \otimes b''' = a \otimes b\}. \end{array}$$

It follows that $(\alpha \otimes \beta)(s(a \otimes b)) \geq (\alpha \otimes \beta)(a \otimes b)$. Hence, $\alpha_A \otimes \beta_B$ is a fuzzy S-act.

We only need to show that if $\tilde{f} : \beta_B \longrightarrow \eta_C$ is a fuzzy S-morphism, then $1 \otimes \tilde{f}$ is a fuzzy S-morphism. Since

$$\begin{array}{rl} & (\alpha \otimes \eta)(1 \otimes \tilde{f})(a \otimes b) \\ = & (\alpha \otimes \eta)(a \otimes \tilde{f}(b)) \\ \geq & \bigvee\{\bigwedge\{\alpha(a'), \eta(\tilde{f}(b'))\} | a' \otimes b' = a \otimes b\} \\ \geq & \bigvee\{\bigwedge\{\alpha(a'), \beta(b')\} | a' \otimes b' = a \otimes b\}, \end{array}$$

we have that

$$(\alpha \otimes \eta)(1 \otimes \tilde{f})(a \otimes b) \geq (\alpha \otimes \beta)(a \otimes b).$$

It follows that $1 \otimes \tilde{f}$ is a fuzzy S-morphism and we get the desired result.

Analogous to Theorem 4.1 and 5.1 in [12], we get the following three theorems.

Theorem V.3 Let S and T be Morita equivalent monoids. Then FS-Act and FT-Act are equivalent.

Proof Let $\Phi = U \otimes_S - : S\text{-Act} \longrightarrow T\text{-Act}$ and $\Psi = V \otimes_T - : T\text{-Act} \longrightarrow S\text{-Act}$ be the inverse functors, where $U \otimes_S V \cong T$ and $V \otimes_T U \cong S$. Then Φ and Ψ induce two functors $\Phi_1 = 1_U \otimes_S - : FS\text{-Act} \longrightarrow FT\text{-Act}$ and $\Psi_1 = 1_V \otimes_T - : FT\text{-Act} \longrightarrow FS\text{-Act}$. We shall show that $\Phi_1\Psi_1 \approx 1_{FS\text{-Act}}$ and $\Psi_1\Phi_1 \approx 1_{FT\text{-Act}}$. First, we show that $\Psi_1\Phi_1 \approx 1_{FS\text{-Act}}$. Let $\alpha_A \in FS$-Act. There is an isomorphism $\tilde{\pi} : 1_V \otimes (1_U \otimes \alpha_A) \longrightarrow (1_V \otimes 1_U) \otimes \alpha_A$ such that $\tilde{\pi}(v \otimes (u \otimes a)) = (v \otimes u) \otimes a$, where $v \in V, u \in U, a \in A$. Let $\omega : V \otimes U \longrightarrow S$ be a biact isomorphism. Then $\tilde{\omega} : 1_V \otimes 1_U \longrightarrow 1_S$ is a fuzzy isomorphism and hence $\tilde{\omega} \otimes 1_A$ is also an isomorphism. Let $\tilde{\varphi} : 1_S \otimes \alpha_A \longrightarrow \alpha_A$ be a fuzzy isomorphism with $\tilde{\varphi}(r \otimes a) = ra$. We define a map $\tilde{\eta}_A = \tilde{\varphi} \circ (\tilde{\omega} \otimes 1) \circ \tilde{\pi} : 1_V \otimes (1_U \otimes \alpha_A) \longrightarrow \alpha_A$, then it is clearly a fuzzy isomorphism.

Because $\Psi\Phi \approx 1_{S\text{-Act}}$, for any $\tilde{f} : \alpha_A \longrightarrow \beta_B$, we have the following commutative diagram

$$\begin{array}{ccc} V \otimes (U \otimes A) & \overset{\Psi\Phi(\tilde{f})}{\rightarrow} & V \otimes (U \otimes B) \\ \tilde{\eta}_A \downarrow & & \downarrow \tilde{\eta}_B \\ A & \overset{\tilde{f}}{\rightarrow} & B. \end{array}$$

It follows that the following diagram

$$\begin{array}{ccc} 1_V \otimes (1_U \otimes \alpha_A) & \overset{\Psi_1\Phi_1(\tilde{f})}{\rightarrow} & 1_V \otimes (1_U \otimes \beta_B) \\ \tilde{\eta_A} \downarrow & & \downarrow \tilde{\eta_B} \\ \alpha_A & \overset{\tilde{f}}{\rightarrow} & \beta_B \end{array}$$

is commutative and $\Psi_1\Phi_1 \approx 1_{FS\text{-Act}}$. Similarly, we can prove that $\Phi_1\Psi_1 \approx 1_{FT\text{-Act}}$.

Theorem V.4 Let S and T be two monoids. If FS-Act and FT-Act are equivalent, then S and T are Morita equivalent.

Proof Let $\Phi : FS\text{-Act} \rightleftharpoons FT\text{-Act} : \Psi$ be two equivalent categories. Then the functor $\overline{\Phi} = L_T \circ \Phi \circ L_S^{-1} : S\text{-Act} \rightleftharpoons T\text{-Act} : \overline{\Psi} = L_S \circ \Psi \circ L_T^{-1}$ are two inverse functors. Using diagram chasing, we can easily prove that $\overline{\Phi} \circ \overline{\Psi} \approx 1_{T\text{-Act}}$ and $\overline{\Psi} \circ \overline{\Phi} \approx 1_{S\text{-Act}}$.

By Theorem V.3 and 5.4, we can ge the following main theorem.

Theorem V.5 Let S and T be two monoids. Then S and T are Morita equivalent if and only if FS-Act and FT-Act are equivalent.

VI. Conclusion

In this paper, we study the associativity of tensor product in FS-Act. We give some necessary conditions for equivalence of two fuzzy S-acts category. Moreover, we prove that two monoids S and T are Morita equivalent if and only if FS-Act and FT-Act are equivalent. The obtained results generalized the related theory in S-Act. The results will be helpful to study homological properties of FS-Act.

To conclude, there are still some questions on this topic. 1) Is adjoint the Hom functors and the tensor product functors in FS-Act? 2) Does the tensor product of any three fuzzy S-acts satisfy associativity in FS-Act?

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