The Hamiltonicity and Hamiltonian Connectivity of Some Shaped Supergrid Graphs*

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***Abstract*—A Hamiltonian path (cycle) of a graph is a simple path (cycle) in which each vertex of the graph is visited exactly once. The Hamiltonian path (cycle) problem is to determine whether a graph contains a Hamiltonian path (cycle). A graph is called Hamiltonian if it contains a Hamiltonian cycle, and it is said to be Hamiltonian connected if there exists a Hamiltonian path between any two distinct vertices. Supergrid graphs were first introduced by us and include grid graphs and triangular grid graphs as their subgraphs. These problems on supergrid graphs can be applied to compute the stitching traces of computerized sewing machines. In the past, we have proved the Hamiltonian path (cycle) problem on supergrid graphs to be NP-complete. Recently, we showed that rectangular supergrid graphs are Hamiltonian connected except one trivial forbidden condition. In this paper, we will verify the Hamiltonicity and Hamiltonian connectivity of some shaped supergrid graphs, including triangular, parallelogram, and trapezoid. The results can be used to solve the Hamiltonian problems on some special classes of supergrid graphs in the future.**

***Index Terms*—Hamiltonicity, Hamiltonian connectivity, supergrid graphs, triangular supergrid graphs, parallelogram supergrid graphs, trapezoid supergrid graphs, computer sewing machines.**

I. Introduction

A *Hamiltonian path* of a graph is a simple path in which each vertex of the graph appears exactly once. A *Hamiltonian cycle* in a graph is a simple cycle with the same property. The *Hamiltonian path* (resp., *cycle*) *problem* involves deciding whether or not a graph contains a Hamiltonian path (resp., cycle). A graph is called to be *Hamiltonian* if it contains a Hamiltonian cycle. A graph G is said to be *Hamiltonian connected* if for each pair of distinct vertices u and v of G, there exists a Hamiltonian path between u and v in G. If (u, v) is an edge of a Hamiltonian connected graph, then a Hamiltonian cycle containing (u, v) does exist. Thus, a Hamiltonian connected graph contains many Hamiltonian cycles, and, hence, the sufficient conditions of Hamiltonian connectivity are stronger than those of Hamiltonicity. It is well known that the Hamiltonian path and cycle problems are NP-complete for general graphs [10], [24]. The same holds true for bipartite graphs [25], split graphs [11], circle graphs [8], undirected path graphs [2], grid graphs [23], triangular grid graphs [12], and supergrid graphs [15]. In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks. Li *et al.* [26] proved the Hamiltonian connectivity of the recursive dual-net. The hypercomplete network [6] and the arrangement graph [29] were known to be Hamiltonian connected. The popular hypercubes are Hamiltonian but are not Hamiltonian connected. However, many variants of hypercubes, including augment hypercubes [14], generalized base-b hypercube [20], twisted cubes [22], crossed cubes [21], Möbius cubes [7], and enhanced hypercubes [28], have been shown to be Hamiltonian connected. For more related works and applications, we refer readers to [1], [4], [5], [9], [13], [17], [18], [27], [30], [31], [32], [33], [34].

The *two-dimensional integer grid* G^∞ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if the (Euclidean) distance between them is equal to 1. The *two-dimensional triangular grid* T^∞ is an infinite graph obtained from G^∞ by adding all edges on the lines traced from up-left to down-right. A *grid graph* is a finite, vertex-induced subgraph of G^∞. For a node v in the plane with integer coordinates, let v_x and v_y be the x and y *coordinates* of node v, respectively, denoted by $v = (v_x, v_y)$. If v is a vertex in a grid graph, then its possible neighbor vertices include (v_x, v_y+1), (v_x-1, v_y), (v_x+1, v_y), and (v_x, v_y-1). For example, Fig. 1(a) shows a grid graph. A *triangular grid graph* is a finite, vertex-induced subgraph of T^∞. If v is a vertex in a triangular grid graph, then its possible neighbor vertices include (v_x, v_y+1), (v_x-1, v_y), (v_x+1, v_y), (v_x, v_y-1), (v_x-1, v_y+1), and (v_x+1, v_y-1). For instance, Fig. 1(b) depicts a triangular grid graph. Thus, triangular grid graphs contain grid graphs as subgraphs. Note that triangular grid graphs defined above are isomorphic to the original triangular grid graphs studied in the literature [12] but these graphs are different when considered as geometric graphs. By the same construction of triangular grid graphs from grid graphs, we have proposed a new class of graphs, namely *supergrid graphs*, in [15]. The *two-dimensional supergrid* S^∞ is an infinite graph obtained from T^∞ by adding all edges on the lines traced from up-right to down-left. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph include (v_x, v_y+1), (v_x-1, v_y), (v_x+1, v_y), (v_x, v_y-1), (v_x-1, v_y+1), (v_x+1, v_y-1), (v_x+1, v_y+1), and (v_x-1, v_y-1). Then, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. For example, Fig. 1(c) shows a supergrid graph. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (points) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are

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bipartite [23] but triangular grid graphs and supergrid graphs are not bipartite.

The Hamiltonian problems on supergrid graphs can be applied to control the stitching trace of a computerized sewing machine as stated in [15]. We also proved that the Hamiltonian cycle and path problems are NP-complete for supergrid graphs [15]. Thus, an important line of investigation is to discover the complexities of the Hamiltonian related problems when the input is restricted to be in special subclasses of supergrid graphs. In [17], we showed that the Hamiltonian cycle problem for linear-convex supergrid graphs is linear solvable. Recently, we proved that rectangular supergrid graphs are always Hamiltonian connected except one trivial forbidden condition [18]. In this paper, we will show that some shaped supergrid graphs, including triangular, parallelogram, and trapezoid, are always Hamiltonian and Hamiltonian connected except few trivial forbidden conditions. The results can be applied to the Hamiltonian problems on some special subcalsses of supergrid graphs, such as solid and alphabet supergrid graphs.

The rest of the paper is organized as follows. Section II gives some notations and background results. In Section III, we propose constructive proofs to show that triangular and parallelogram supergrid graphs are Hamiltonian and Hamiltonian connected except two or three trivial conditions. Section IV verifies the Hamiltonicity and Hamiltonian connectivity of trapezoid supergrid graphs by using the Hamiltonicity and Hamiltonian connectivity of rectangular, triangular, and parallelogram supergrid graphs. Finally, we make some concluding remarks in Section V.

II. Notations and Background Results

In this section, we will introduce some notations. Some observations and previously established results for the Hamiltonian problems on rectangular supergrid graphs are also presented. For graph-theoretic terminology not defined in this paper, the reader is referred to [3].

A. Notations

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let S be a subset of vertices in G, and let u and v be two distinct vertices in G. We write $G[S]$ for the subgraph of G *induced* by S, $G-S$ for the subgraph $G[V-S]$, i.e., the subgraph induced by $V-S$. In general, we write $G-v$ instead of $G-\{v\}$. If (u, v) is an edge in G, we say that u is *adjacent* to v. A *neighbor* of v in G is any vertex that is adjacent to v. We use $N_G(v)$ to denote the set of neighbors of v in G. The subscript 'G' of $N_G(v)$ can be removed from the notation if it has no ambiguity. The *degree* of vertex v, denoted by $deg(v)$, is the number of vertices adjacent to vertex v. The notation $u \sim v$ (resp., $u \nsim v$) means that vertices u and v are adjacent (resp., non-adjacent). Two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are said to be *incident* if $u_1 \sim v_1$ and $u_2 \sim v_2$, denote this by $e_1 \approx e_2$. A path P of length $|P|$ in G, denoted by $v_1 \to v_2 \to \cdots \to v_{|P|-1} \to v_{|P|}$, is a sequence $(v_1, v_2, \cdots, v_{|P|-1}, v_{|P|})$ of vertices such that $(v_i, v_{i+1}) \in E$ for $1 \leqslant i < |P|$. The first and last vertices visited by P are denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote "P visits vertex v_i" and use $(v_i, v_{i+1}) \in P$ to denote "P visits edge (v_i, v_{i+1})". A path from v_1 to v_k is denoted by (v_1, v_k)-path. In addition, we use P to refer to the set of vertices visited by path P if it is understood without ambiguity. On the other hand, a path is called the *reversed path*, denoted by rev(P), of path P if it visits the vertices of P from $end(P)$ to $start(P)$ in proper sequence; that is, the reversed path rev(P) of path $P = v_1 \to v_2 \to \cdots \to v_{|P|-1} \to v_{|P|}$ is $v_{|P|} \to v_{|P|-1} \to \cdots \to v_2 \to v_1$. A path P is a cycle if $|V(P)| \geqslant 3$ and $end(P) \sim start(P)$. Two paths (or cycles) P_1 and P_2 of graph G are called vertex-disjoint if $V(P_1) \cap V(P_2) = \emptyset$. Two vertex-disjoint paths P_1 and P_2 can be concatenated into a path, denoted by $P_1 \Rightarrow P_2$, if $end(P_1) \sim start(P_2)$.

Let S^∞ be the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if the difference of their x or y coordinates is not larger than 1. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. For a vertex v in a supergrid graph, let v_x and v_y be respectively x and y coordinates of v. We color vertex v to be *white* if $v_x + v_y \equiv 0$ (mod 2); otherwise, v is colored to be *black*. Then there are eight possible neighbors of vertex v including four white vertices and four black vertices. Obviously, all grid graphs are bipartite [23] but supergrid graphs are not bipartite. The edge (u, v) in S^∞ is said to be *horizontal* (resp., *vertical*) if $u_y = v_y$ and $u_x \neq v_x$ (resp., $u_x = v_x$ and $u_y \neq v_y$), and is called *skewed* if it is neither a horizontal nor a vertical edge. In the figures, we assume that $(1, 1)$ are coordinates of the up-left vertex, i.e., the leftmost vertex of the first row, in a supergrid graph.

Rectangular supergrid graphs first appeared in [15], in which the Hamiltonian cycle problem was solved. Let $R(m, n)$ be the supergrid graph with vertex set $V(R(m, n)) = \{v = (v_x, v_y) \mid 1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n\}$. That is, $R(m, n)$ contains m columns and n rows of vertices in S^∞. A *rectangular supergrid graph* is a supergrid graph which is isomorphic to $R(m, n)$. Then m and n, the *dimensions*, specify a rectangular supergrid graph up to isomorphism. The size of $R(m, n)$ is defined to be mn, and $R(m, n)$ is called n-rectangle. Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. The vertex v is called the *up-left* (resp., *up-right*, *down-left*, *down-right*) *corner* of $R(m, n)$ if for any vertex $w = (w_x, w_y) \in R(m, n)$, $w_x \geqslant v_x$ and $w_y \geqslant v_y$ (resp., $w_x \leqslant v_x$ and $w_y \geqslant v_y$, $w_x \geqslant v_x$ and $w_y \leqslant v_y$, $w_x \leqslant v_x$ and $w_y \leqslant v_y$). There are four boundaries (borders) in a rectangular supergrid graph $R(m, n)$ with $m, n \geqslant 2$. The edge in the boundary of $R(m, n)$ is called *boundary edge*. For example, Fig. 2(a) shows a rectangular supergrid graph $R(10, 10)$ which is called 10-rectangle and contains $4 \times 9 = 36$ boundary edges. Fig. 2(a) also indicates the types of corners.

The triangular supergrid graphs are subgraphs of rectangular supergrid graphs and are defined as follows.

Definition 1. Let ℓ be a diagonal line of $R(n, n)$ with $n \geqslant 2$ from the up-left corner to the down-right corner. Let $\Delta(n, n)$ be the supergrid graph obtained from $R(n, n)$ by removing all vertices under ℓ. A *triangular supergrid graph* is a supergrid graph which is isomorphic to $\Delta(n, n)$.

For example, Fig. 2(b) shows a triangular supergrid graph $\Delta(10, 10)$. Each triangular supergrid graph contains three boundaries, namely *horizontal*, *vertical*, and *skewed*, and

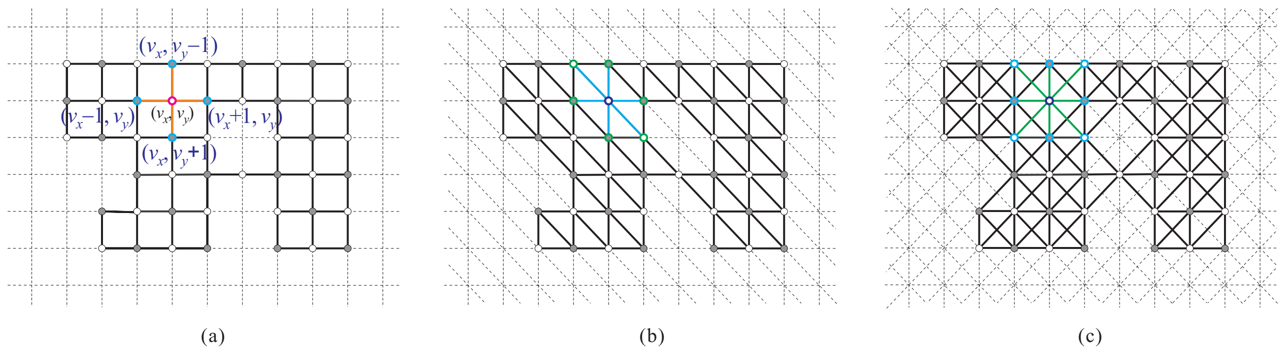

Fig. 1. (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, where circles represent the vertices and solid lines indicate the edges in the graphs.

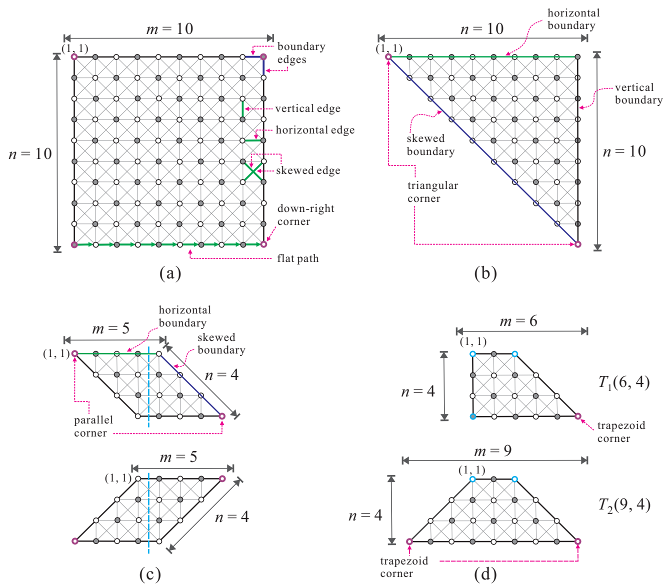

Fig. 2. (a) A rectangular supergrid graph $R(10, 10)$, (b) a triangular supergrid graph $\Delta(10, 10)$, (c) two types of parallelogram supergrid graph $P(5, 4)$, and (d) two types of trapezoid supergrid graphs $T_1(6, 4)$ and $T_2(9, 4)$, where solid arrow lines in (a) indicate a flat path on $R(10, 10)$ and dashed line in (c) indicates a vertical cut.

these boundaries form a triangle, as illustrated in Fig. 2(b). The triangular supergrid graph $\Delta(n, n)$ is called n-triangle, and the vertex v in $\Delta(n, n)$ is called *triangular corner* if $deg(v) = 2$ and it is the intersection of horizontal (or vertical) and skewed boundaries.

Parallelogram supergrid graphs are defined similar to rectangular supergrid graphs as follows.

Definition 2. Let $P(m, n)$ be the supergrid graph with $m \geqslant n$ and vertex set $V(P(m, n)) = \{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y + m - 1\}$ or $\{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $-v_y + 2 \leqslant v_x \leqslant m - (v_y - 1)\}$. A *parallelogram supergrid graph* is a supergrid graph which is isomorphic to $P(m, n)$.

In the above definition, there are two types of parallelogram supergrid graphs. We can see that they are isomorphic although they are different when considered as geometric graphs. In this paper, we can only consider the parallelogram supergrid graph $P(m, n)$ with $V(P(m, n)) = \{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y + m - 1\}$. Each parallelogram supergrid graph contains four boundaries, two *horizontal* boundaries and two *skewed* boundaries, and these boundaries form a parallelogram, as depicted in Fig. 2(c). The size of $P(m, n)$ is defined to be mn, and $P(m, n)$ is called n-parallelogram. The vertex w of $P(m, n)$ is called *parallel corner* if $deg(w) = 2$. We can see from definition that a parallelogram supergrid graph contains two parallel corners and it can be decomposed into two disjoint triangular supergrid subgraphs. For instance, Fig. 2(c) depicts a parallelogram supergrid graph $P(5, 4)$ which can be partitioned into two triangular supergrid graphs $\Delta(4, 4)$.

Next, we introduce trapezoid supergrid graphs. Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. A trapezoid supergrid graph $T_1(m, n)$ or $T_2(m, n)$ is obtained from $R(m, n)$ by removing one or two triangular supergrid graphs $\Delta(n-1, n-1)$. The trapezoid supergrid graphs $T_1(m, n)$ and $T_2(m, n)$ are defined as follows.

Definition 3. Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. A trapezoid supergrid graph $T_1(m, n)$ with $m \geqslant n + 1$ is obtained from $R(m, n)$ by removing a triangular supergrid graph $\Delta(n - 1, n - 1)$ from the corner of $R(m, n)$. A trapezoid supergrid graph $T_2(m, n)$ is constructed from $R(m, n)$ with $m \geqslant 2n$ by removing two triangular supergrid graphs $\Delta(n-1, n-1)$ from the up-left and up-right corners of $R(m, n)$. Fig. 2(d) shows these two types of trapezoid graphs.

In a trapezoid supergrid graph, a vertex v is called *trapezoid corner* if $deg(v) = 2$. We can see that $T_1(m, n)$ contains a trapezoid corner, $T_2(m, n)$ contains two trapezoid corners, $T_1(m, n)$ contains two horizontal, one vertical and one skewed boundaries, and $T_2(m, n)$ contains two horizontal and two skewed boundaries. By definition, each boundary of $T_1(m, n)$ and $T_2(m, n)$ contains at least two vertices. On the other hand, $T_1(m, n)$ and $T_2(m, n)$ are called n_{T_1}-trapezoid and n_{T_2}-trapezoid, respectively. For instance, Fig. 2(d) depicts $T_1(6, 4)$ and $T_2(9, 4)$.

Let G be a rectangular, triangular, parallelogram, or trapezoid supergrid graph. A path on one boundary of G is called *flat* if its vertices are in the boundary and it contains all boundary edges in the boundary. For example, the solid arrow lines in the down boundary of Fig. 2(a) indicate a flat path of $R(10, 10)$.

In proving our results, we need to partition a shaped supergrid graph into two disjoint parts. The decomposition is defined as follows.

Definition 4. Let $S(m,n)$ be a triangular, parallelogram, or trapezoid supergrid graph. A *cut* operation on $S(m,n)$ is a line partition through a set Z of edges so that the removal of Z from $S(m,n)$ results in two disjoint supergrid subgraphs S_1 and S_2. A cut is called *vertical* (resp., *horizontal*) if it is a vertical (resp., horizontal) line to separate $S(m,n)$ into S_1 and S_2 such that S_1 is to the left (resp., upper) of S_2, i.e., Z is a set of horizontal (resp., vertical) edges.

For instance, the bold dashed line in Fig. 2(c) depicts a vertical cut on $P(5,4)$ to partition it into two disjoint triangular supergrid subgraphs $\Delta(4,4)$.

In proving our result, we will construct a canonical Hamiltonian cycle and a canonical Hamiltonian path of a triangular, parallelogram, or trapezoid supergrid graph defined as follows.

Definition 5. Let $S(m,n)$ be a triangular, parallelogram, or trapezoid supergrid graph with κ boundaries, and let s and t be its two distinct vertices. A Hamiltonian cycle of $S(m,n)$ is called *canonical* if it contains $\kappa-1$ flat paths on $\kappa-1$ boundaries, and it contains at least one boundary edge in the other boundary. A Hamiltonian (s,t)-path of $S(m,n)$ is called *canonical* if it contains at least one boundary edge of each boundary in $S(m,n)$.

B. Background results

In [15], we have showed that rectangular supergrid graphs always contain Hamiltonian cycles except 1-rectangles. Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant n$, C be a cycle of $R(m,n)$, and let H be a boundary of $R(m,n)$, where H is a subgraph of $R(m,n)$. The restriction of C to H is denoted by $C_{|H}$. If $|C_{|H}|=1$, i.e. $C_{|H}$ is a boundary path on H, then $C_{|H}$ is called *flat face* on H. If $|C_{|H}|>1$ and $C_{|H}$ contains at least one boundary edge of H, then $C_{|H}$ is called *concave face* on H. A Hamiltonian cycle of $R(m,3)$ is called *canonical* if it contains three flat faces on two shorter boundaries and one longer boundary, and it contains one concave face on the other boundary, where the shorter boundary consists of three vertices. And, a Hamiltonian cycle of $R(m,n)$ with $n=2$ or $n \geqslant 4$ is said to be *canonical* if it contains three flat faces on three boundaries, and it contains one concave face on the other boundary. The following lemma states the result in [15] concerning the Hamiltonicity of rectangular supergrid graphs.

Lemma 1. *(See [15].) Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. Then, the following statements hold true:*
(1) *if $n=3$, then $R(m,3)$ contains a canonical Hamiltonian cycle;*
(2) *if $n=2$ or $n \geqslant 4$, then $R(m,n)$ contains four distinct canonical Hamiltonian cycles with concave faces being on different boundaries.*

Let (G,s,t) denote the supergrid graph G with two given distinct vertices s and t. Without loss of generality, we will assume that $s_x \leqslant t_x$, i.e., s is to the left of t, in the rest of the paper. The notation $L(G,s,t)$ indicates the length of longest path between s and t in G, where the length of a path is defined to be the number of vertices

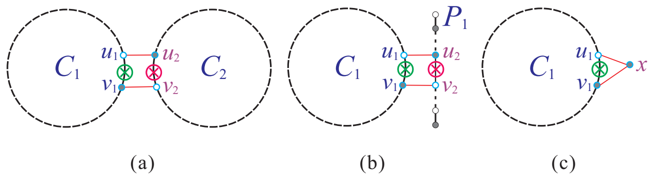

Fig. 3. A schematic diagram for (a) Proposition 3, (b) Proposition 4, and (c) Proposition 5, where bold dashed lines indicate the cycles (paths) and $\otimes$ represents the destruction of an edge while constructing a cycle or path.

visited by the path. We denote a Hamiltonian path between s and t in G by $HP(G,s,t)$. We say that $HP(G,s,t)$ exists if there exists a Hamiltonian (s,t)-path of G. By the definition, $L(G,s,t)=|V(G)|$ if $HP(G,s,t)$ does exist. The Hamiltonian cycle of $R(m,n)$ is called *canonical* if it satisfies Lemma 1. From Lemma 1, we know that $HP(R(m,n),s,t)$ does exist when $m,n \geqslant 2$ and (s,t) is an edge in the canonical Hamiltonian cycle of $R(m,n)$. In [18], we have proved that $HP(R(m,n),s,t)$ always exists for $m,n \geqslant 3$. For $(R(m,n),\ s,\ t)$ with $m \geqslant n \geqslant 3$, a Hamiltonian (s,t)-path of $R(m,n)$ is called *canonical* if it contains at least one boundary edge of each side (boundary) in $R(m,n)$. The following lemma is to show the Hamiltonian connectivity of rectangular supergrid graphs.

Lemma 2. *(See [18].) For $(R(m,n),\ s,\ t)$ with $m \geqslant n \geqslant 3$, $R(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(R(m,n),s,t)$ does exist.*

For the 1-rectangle, $HP(R(m,1),s,t)$ does not exist if s or t is not a corner. On the other hand, $HP(R(m,2),s,t)$ does not exist if (s,t) is a vertical and nonboundary edge of $R(m,2)$. For $n=1$ or 2, $HP(R(m,n),s,t)$ does exist except the above one trivial forbidden condition [18].

We next give some observations on the relations among cycle, path, and vertex. These propositions are presented in [18] and will be used in proving our results. Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be combined into a cycle of G. Then the following proposition holds true.

Proposition 3. *(See [18].) Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be combined into a cycle of G. (see Fig.* 3(a)*)*

Let C_1 be a cycle and let P_1 be a path in a graph G such that $V(C_1) \cap V(P_1) = \emptyset$. If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path P of G with $start(P)=start(P_1)$ and $end(P)=end(P_1)$. Fig. 3(b) depicts such a construction, and, hence, the following proposition holds true.

Proposition 4. *(See [18].) Let C_1 and P_1 be a cycle and a path, respectively, of a graph G such that $V(C_1) \cap V(P_1) = \emptyset$. If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path of G. (see Fig.* 3(b)*)*

The above observation can be extended to a vertex x, where $P_1 = x$, as shown in Fig. 3(c), and we then have the following proposition.

Proposition 5. *(See [17]) Let C_1 be a cycle (path) of a graph G and let x be a vertex in $G - V(C_1)$. If there exists an edge (u_1, v_1) in C_1 such that $u_1 \sim x$ and $v_1 \sim x$, then C_1 and x can be combined into a cycle (path) of G. (see Fig.* 3(c)*)*

III. THE HAMILTONICITY AND HAMILTONIAN CONNECTIVITY OF TRIANGULAR AND PARALLELOGRAM SUPERGRID GRAPHS

A. The Hamiltonicity and Hamiltonian connectivity of triangular supergrid graphs

In this subsection, we will verify the Hamiltonicity and Hamiltonian connectivity (except two trivial conditions) of triangular supergrid graphs. For a triangular supergrid graph, we will construct a canonical Hamiltonian cycle and a canonical Hamiltonian path. Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geqslant 2$, and let $s,t \in \Delta(n,n)$. Recall that a Hamiltonian cycle of $\Delta(n,n)$ is called *canonical* if it contains two flat faces on vertical and horizontal boundaries, and it contains at least one boundary edge in skewed boundary. A Hamiltonian (s,t)-path of $\Delta(n,n)$ is called *canonical* if it contains at least one boundary edge in each boundary. The following lemma proves the Hamiltonicity of triangular supergrid graphs.

Lemma 6. *Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geqslant 2$. Then, $\Delta(n,n)$ contains a canonical Hamiltonian cycle.*

Proof: We prove this lemma by induction on n. Initially, let $n = 2$ or 3. By inspection, $\Delta(2,2)$ and $\Delta(3,3)$ contain Hamiltonian cycles which contain all boundary edges of each boundary. Thus, the lemma holds true for $n = 2$ and 3. Assume that lemma holds true when $n = k \geqslant 3$. Then, $\Delta(k-1,k-1)$ and $\Delta(k,k)$ contain canonical Hamiltonian cycles. Now, assume that $n = k+1$. We first make a vertical cut on $\Delta(k+1,k+1)$ to obtain two disjoint subgraphs $\Delta(k-1,k-1)$ and T', where T' is a 2-rectangle attached by a 2-triangle, and the vertical boundary of $\Delta(k-1,k-1)$ is faced to one boundary of T'. By induction hypothesis, $\Delta(k-1,k-1)$ contains a canonical Hamiltonian cycle HC_{k-1} which contains two flat faces of vertical and horizontal boundaries. By visiting all boundary edges of T', we can construct a Hamiltonian cycle HC' of T'. Then, there exist two edges $e_1 \in HC_{k-1}$ and $e_2 \in HC'$ such that e_1 is a vertical boundary edge and contains the nontriangular corner of $\Delta(k-1,k-1)$, e_2 is a vertical boundary edge of T', and $e_1 \approx e_2$. By Proposition 3, HC_{k-1} and HC' can be combined into a Hamiltonian cycle HC of $\Delta(k+1,k+1)$ such that HC contains all boundary edges of vertical and horizontal boundaries, and it contains at least one boundary edge of skewed boundary. Thus, $\Delta(k+1,k+1)$ contains a canonical Hamiltonian cycle and the lemma holds true when $n = k+1$. By induction, $\Delta(n,n)$ contains a canonical Hamiltonian cycle for $n \geqslant 2$. ■

Next, we will study the Hamiltonian connectivity of triangular supergrid graphs. We first observe two conditions for that $HP(\Delta(n,n),s,t)$ does not exist. These two forbidden conditions are described as follows:

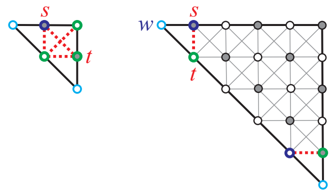

Fig. 4. Triangular supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) condition (F1), and (b) condition (F2), where dotted lines indicate the forbidden edges (s,t).

(F1) $\Delta(n,n)$ is a 3-triangle, and (s,t) is a nonboundary edge of $\Delta(n,n)$ (see Fig. 4(a)).

(F2) $\Delta(n,n)$ satisfies $n \geqslant 3$, and (s,t) is an edge of $\Delta(n,n)$ such that s and t are adjacent to a triangular corner w of $\Delta(n,n)$ (see Fig. 4(b)).

The conditions of (F1) and (F2) are called *forbidden* for $HP(\Delta(n,n),s,t)$. Note that $|V(\Delta(n,n))| = \frac{n(n+1)}{2}$. The following lemma computes the longest (s,t)-path with length $L(\Delta(n,n),s,t)$ when $(\Delta(n,n),s,t)$ satisfies condition (F1) or (F2).

Lemma 7. *Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geqslant 3$, and let s and t be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n),s,t)$ satisfies condition* (F1) *or* (F2)*, then $L(\Delta(n,n),s,t) = \frac{n(n+1)}{2} - 1$.*

Proof: By inspection, the lemma holds true when $n = 3$. In the following, assume that $n \geqslant 4$. Then, $(\Delta(n,n),s,t)$ satisfies condition (F2), and, hence, (s,t) is an edge of $\Delta(n,n)$ such that s and t are adjacent to a triangular corner w of $\Delta(n,n)$. By Lemma 6, $\Delta(n,n)$ contains a canonical Hamiltonian cycle HC. Since $deg(w) = 2$, edges (s,w) and (w,t) are in HC. By removing w from HC, we obtain a (s,t)-path P with length $\frac{n(n+1)}{2} - 1$. Clearly, $HP(\Delta(n,n),s,t)$ does not exist, and, hence, the length of any (s,t)-path is less than $\frac{n(n+1)}{2}$. Thus, P is the longest (s,t)-path. In addition, P contains all boundary edges (except (s,w) or (w,t)) of vertical and horizontal boundaries, and it contains at least one boundary edge of skewed boundary in $\Delta(n,n)$. Then, $L(\Delta(n,n),s,t) = |V(\Delta(n,n))| - 1 = \frac{n(n+1)}{2} - 1$, and, hence, the lemma holds true. ■

We have computed the longest (s,t)-path of $\Delta(n,n)$ when $(\Delta(n,n),s,t)$ satisfies forbidden condition (F1) or (F2). When $(\Delta(n,n),s,t)$ does not satisfy conditions (F1) and (F2), we will construct a canonical Hamiltonian (s,t)-path of $\Delta(n,n)$ as follows.

Lemma 8. *Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geqslant 3$, and let s and t be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n),s,t)$ does not satisfy conditions* (F1) *and* (F2)*, then $\Delta(n,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(\Delta(n,n),s,t)$ does exist.*

Proof: We will prove this lemma by induction on n, $n \geqslant 3$. Initially, let $n = 3$ or 4. By inspecting every case, we can verify the lemma when $n = 3$ and 4. Fig. 5(a) and Fig. 5(b) depict the possible constructed canonical Hamiltonian

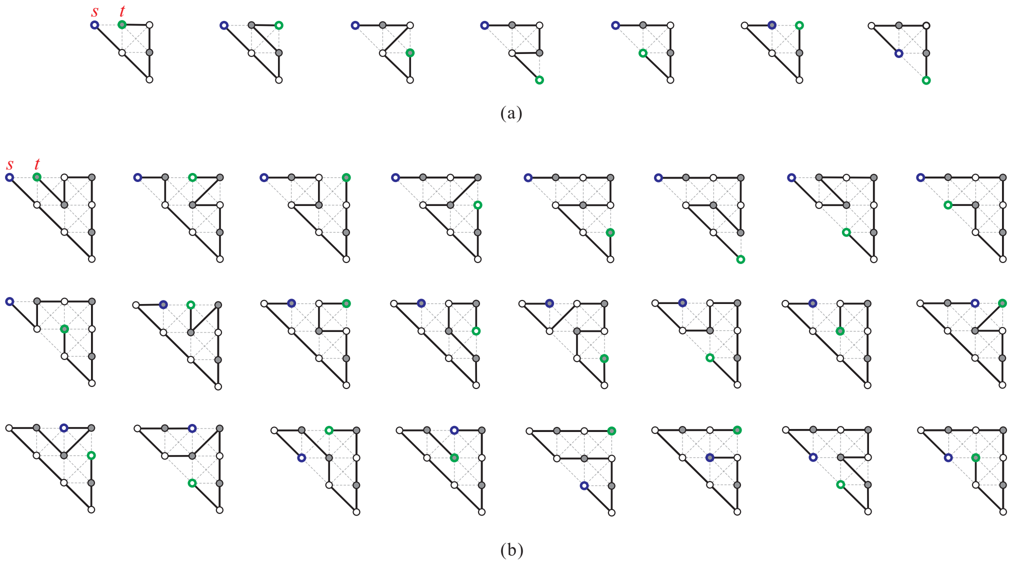

Fig. 5. The possible canonical Hamiltonian (s,t)-path of (a) $\Delta(3,3)$ and (b) $\Delta(4,4)$ when forbidden conditions (F1) and (F2) are not satisfied, where solid lines indicate the edges in the Hamiltonian (s,t)-path.

(s,t)-path of $\Delta(3,3)$ and $\Delta(4,4)$, respectively.

Now, assume that the lemma holds true when $n = k \geqslant 4$. Then, there exists a canonical Hamiltonian (s^*,t^*)-path P^* of $\Delta(k-1,k-1)$ if $(\Delta(k-1,k-1),s^*,t^*)$ does not satisfy conditions (F1) and (F2). Consider that $n = k+1$. Let w and w' be two triangular corners of $\Delta(k+1,k+1)$ such that w' is in vertical boundary. Let s and t be two distinct vertices of $\Delta(k+1,k+1)$ such that $(\Delta(k+1,k+1),s,t)$ does not satisfy forbidden condition (F2). We then make a vertical cut on $\Delta(k+1,k+1)$ to partition it into two disjoint subgraphs $\Delta(k-1,k-1)$ and $T' = R(2,k) \cup \{w'\}$, where T' is a 2-rectangle attached by a 2-triangle. Then, $w \in \Delta(k-1,k-1)$ and $w' \in T'$. Let w^* be a triangular corner of $\Delta(k-1,k-1)$ different from w. Consider the following three cases:

Case 1: $s,t \in \Delta(k-1,k-1)$. By visiting all boundary edges of T', we can obtain a Hamiltonian cycle HC' of T'. There are two subcases:

Case 1.1: $(\Delta(k-1,k-1),s,t)$ satisfies condition (F1) or (F2). Since $(\Delta(k+1,k+1),s,t)$ does not satisfy condition (F2), $s \nsim w$ or $t \nsim w$. Suppose that $(\Delta(k-1,k-1),s,t)$ satisfies condition (F1). Then, $k-1 = 3$. By inspecting every case, we can construct a Hamiltonian (s,t)-path P of $\Delta(k-1,k-1) - w^*$ such that there is a vertical boundary edge e in P. Since there exists a boundary edge (u,v) in HC' such that $u \sim w^*$ and $v \sim w^*$, by Proposition 5, HC' and w^* can be merged into a Hamiltonian cycle HC^* of $T' \cup \{w^*\}$. Then, there exists an edge e^* in HC^* such that $e^* \approx e$. By Proposition 4, P and HC^* can be combined into a canonical Hamiltonian (s,t)-path of $\Delta(k+1,k+1)$. On the other hand, suppose that $(\Delta(k-1,k-1),s,t)$ satisfies condition (F2). Then, $s \sim w^*$, $t \sim w^*$, and there exists a boundary edge (u,v) in HC' such that $u \sim w^*$ and $v \sim w^*$. By Proposition 5, HC' and w^* can be combined into a Hamiltonian cycle HC^* of $T' \cup \{w^*\}$. By the proof of Lemma 7, there exists a Hamiltonian (s,t)-path P of $\Delta(k-1,k-1) - w^*$ such that P visits all boundary edges of vertical and horizontal boundaries, and P contains at least one boundary edge of skewed boundary in $\Delta(k-1,k-1) - w^*$. Then, there exist two edges $e \in P$ and $e^* \in HC^*$ such that $e \approx e^*$. By Proposition 4, P and HC^* can be combined into a canonical Hamiltonian (s,t)-path of $\Delta(k+1,k+1)$.

Case 1.2: $(\Delta(k-1,k-1),s,t)$ does not satisfy conditions (F1) and (F2). By induction hypothesis, there exists a canonical Hamiltonian (s,t)-path P of $\Delta(k-1,k-1)$. Then, there exist two edges $e \in P$ and $e' \in HC'$ such that $e \approx e'$. By Proposition 4, P and HC' can be combined into a canonical Hamiltonian (s,t)-path of $\Delta(k+1,k+1)$.

Case 2: $s \in \Delta(k-1,k-1)$ and $t \in T'$. Let p be a vertex in $\Delta(k-1,k-1)$ such that $p = w^*$ if $s \neq w^*$, and (p,w^*) is a vertical boundary edge of $\Delta(k-1,k-1)$ otherwise. Then, $(\Delta(k-1,k-1),s,p)$ does not satisfy conditions (F1) and (F2). Let q be a vertex in T' such that $q \sim p$ and (q,t) is not a horizontal edge of T'. Since T' is a 2-rectangle attached by a triangle, such a vertex q can be easily found. Then, $HP(T',q,t)$ does exist (see [18]). Let P' be the canonical Hamiltonian (q,t)-path of T'. By induction hypothesis, there exists a canonical Hamiltonian (s,p)-path P^* of $\Delta(k-1,k-1)$. Then, $P^* \Rightarrow P'$ forms a canonical Hamiltonian (s,t)-path of $\Delta(k+1,k+1)$.

Case 3: $s,t \in T'$. Let $T' = R' \cup \{w'\}$, where $R' = R(2,k)$. Then, R' is a 2-rectangle. Since $(\Delta(k+1,k+1),s,t)$ does not satisfy condition (F2), $s \nsim w'$ or $t \nsim w'$. Suppose that $s \nsim t$ or (s,t) is not a horizontal and nonboundary edge in R'. In [18], there exists a canonical Hamiltonian (s,t)-path P' of R'. By Proposition 5, w' can be merged into P' to form a canonical Hamiltonian (s,t)-path P^* of T'. By Lemma 6, $\Delta(k-1,k-1)$ contains a canonical Hamiltonian cycle HC. Then, there exist two edges $e \in HC$ and $e^* \in P^*$ such that $e \approx e^*$. By Propsoition 4, P^* and HC can be combined into a canonical Hamiltonian (s,t)-path of $\Delta(k+1,k+1)$. On the other hand, suppose that (s,t) is a horizontal and nonboundary edge in R'. Without loss of generality, assume that $s_x < t_x$. Let $p_1,p_2 \in R'$ and $q_1,q_2 \in \Delta(k-1,k-1)$ such that (s,p_1) and (s,p_2) are two vertical and boundary edges in R', p_1 is to the upper of s, $p_1 \sim q_1$, and $p_2 \sim q_2$. Then, we can easily construct two disjoint (s,p_1)-path P_1 and (p_2,t)-path P_2 of T' so that $P_1 \cup P_2$ visits all vertices of T' and contains at least one boundary edge in each boundary

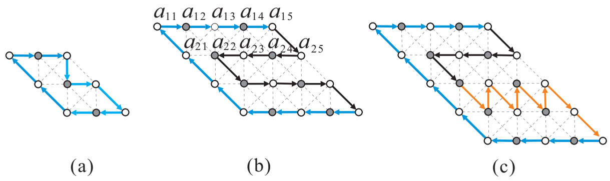

Fig. 6. The canonical Hamiltonian cycle of (a) $P(3,3)$, (b) $P(5,4)$, and (c) $P(5,5)$, where arrow lines indicate the edges in the Hamiltonian cycle.

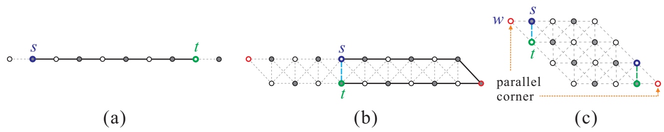

Fig. 7. Parallelogram supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) condition (F3), (b) condition (F4), and (c) condition (F5), where the solid lines indicate the longest (s,t)-path.

of T'. We can see that $(\Delta(k-1,k-1),q_1,q_2)$ does not satisfy conditions (F1) and (F2). By induction hypothesis, there exists a canonical Hamiltonian (q_1,q_2)-path P^* of $\Delta(k-1,k-1)$. Then, $P_1 \Rightarrow P^* \Rightarrow P_2$ forms a canonical Hamiltonian (s,t)-path of $\Delta(k+1,k+1)$.

It immediately follows from the above cases that $\Delta(k+1,k+1)$ contains a canonical Hamiltonian (s,t)-path. Thus, the lemma holds true when $n=k+1$. By induction, $\Delta(n,n)$ contains a canonical Hamiltonian (s,t)-path for $n \geqslant 3$, and hence $HP(\Delta(n,n),s,t)$ does exist. This completes the proof of the lemma. ■

B. The Hamiltonicity and Hamiltonian connectivity of parallelogram supergrid graphs

In this subsection, we will prove the Hamiltonicity and Hamiltonian connectivity of parallelogram supergrid graphs. In a parallelogram supergrid graph $P(m,n)$, we only consider that $V(P(m,n)) = \{v=(v_x,v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y+m-1\}$. The other type of parallelograms can be verified by the same arguments. Note that there are two horizontal and two skewed boundaries in $P(m,n)$. We first provide a constructive proof to show that any parallelogram supergrid graph $P(m,n)$ with $m \geqslant n \geqslant 2$ contains a Hamiltonian cycle. We then prove that $P(m,n)$ always contains a Hamiltonian (s,t)-path except three trivial conditions. The following lemma first appears in [15] and shows the Hamiltonicity of parallelogram supergrid graphs.

Lemma 9. *Let $P(m,n)$ be a parallelogram supergrid graph with $m \geqslant n \geqslant 2$. Then, $P(m,n)$ contains a canonical Hamiltonian cycle.*

Proof: By inspection, the lemma can be easily verified when $3 \geqslant m$. For example, Fig. 6(a) shows a canonical Hamiltonian cycle of $P(3,3)$. In the following, assume that $m \geqslant n \geqslant 4$. Note that $P(m,n)$ consists of m columns and n rows of vertices. Let a_{ij} be the vertex located at i-th row and j-th column of $P(m,n)$, where $n \geqslant i \geqslant 1$ and $m \geqslant j \geqslant 1$. Consider the following two cases:

Case 1: n is even. Let $P_1 = a_{11} \to a_{12} \to \cdots \to a_{1(m-1)} \to a_{1m}$, $P_\imath = a_{\imath 2} \to a_{\imath 3} \to \cdots \to a_{\imath(m-1)} \to a_{\imath m}$ for $n \geqslant \imath \geqslant 2$, and let $P_{n+1} = a_{21} \to a_{31} \to \cdots \to a_{(n-1)1} \to a_{n1}$. Let $HC = P_1 \Rightarrow \text{rev}(P_2) \Rightarrow P_3 \Rightarrow \text{rev}(P_4) \Rightarrow \cdots \Rightarrow P_\jmath \Rightarrow \text{rev}(P_{\jmath+1}) \Rightarrow \cdots \Rightarrow P_{n-1} \Rightarrow \text{rev}(P_n) \Rightarrow \text{rev}(P_{n+1})$, where $\jmath$ is an odd and $n-1 \geqslant \jmath \geqslant 1$. Then, HC is a canonical Hamiltonian cycle of $P(m,n)$. For instance, Fig. 6(b) depicts a canonical Hamiltonian cycle of $P(5,4)$.

Case 2: n is odd. In this case, $n \geqslant 5$. Let $P_1 = a_{11} \to a_{12} \to \cdots \to a_{1(m-1)} \to a_{1m}$, $P_\imath = a_{\imath 2} \to a_{\imath 3} \to \cdots \to a_{\imath(m-1)} \to a_{\imath m}$ for $n-3 \geqslant \imath \geqslant 2$, $P_{n-2} = a_{(n-2)2} \to a_{(n-1)2} \to a_{(n-2)3} \to a_{(n-1)3} \to \cdots \to a_{(n-2)\hat{\jmath}} \to a_{(n-1)\hat{\jmath}} \to \cdots \to a_{(n-2)m} \to a_{(n-1)m}$, where $m \geqslant \hat{\jmath} \geqslant 2$, $P_{n-1} = a_{n2} \to a_{n3} \to \cdots \to a_{nm}$, and let $P_n = a_{21} \to a_{31} \to \cdots \to a_{(n-1)1} \to a_{n1}$. Let $HC^* = P_1 \Rightarrow \text{rev}(P_2) \Rightarrow P_3 \Rightarrow \text{rev}(P_4) \Rightarrow \cdots \Rightarrow P_\jmath \Rightarrow \text{rev}(P_{\jmath+1}) \Rightarrow \cdots \Rightarrow P_{n-2} \Rightarrow \text{rev}(P_{n-1}) \Rightarrow \text{rev}(P_n)$, where $\jmath$ is an odd and $n-2 \geqslant \jmath \geqslant 1$. Then, HC^* is a canonical Hamiltonian cycle of $P(m,n)$. For example, Fig. 6(c) depicts a canonical Hamiltonian cycle of $P(5,5)$.

It immediately follows from the above cases that the lemma holds true. ■

Now, we will investigate the Hamiltonian connectivity of parallelogram supergrid graphs. We first observe three forbidden conditions for $HP(P(m,n),s,t)$. Then, we prove that $HP(P(m,n),s,t)$ does exist except the forbidden conditions. We first consider 1-parallelogram $(P(m,1),s,t)$. The following condition implies $HP(P(m,1),s,t)$ does not exist.

(F3) $P(m,n)$ is a 1-parallelogram, but s or t is not a corner vertex (see Fig. 7(a)).

Since the possible path between s and t in $P(m,1)$ is unique, the longest (s,t)-path in $(P(m,1),s,t)$ is unique and its length equals $t_x - s_x + 1$. Note that $s_x < t_x$, i.e., s is to the left of t. Then, $HP(P(m,1),s,t)$ does exist if $(P(m,1),s,t)$ does not satisfy condition (F3).

Next, we consider $(P(m,2),s,t)$ with $m \geqslant 2$. By inspection, the following condition implies that $P(m,2)$ contains no Hamiltonian (s,t)-path.

(F4) $P(m,n)$ is a 2-parallelogram with $m \geqslant 2$, and (s,t) is a vertical edge of $P(m,n)$ (see Fig. 7(b)).

Consider that $(R(m,2),s,t)$ satisfies condition (F4). In this case, $s_x = t_x$. Note that the left parallel corner is coordinated as $(1,1)$. Without loss of generality, assume that $s_y \leqslant t_y$. We can easily see that the longest (s,t)-path $L(P(m,2),s,t)$ is either $2s_x-1$ or $2(m-s_x+1)+1$. Then, $L(P(m,2),s,t) = \max\{2s_x-1, 2m-2s_x+3\}$. When $(P(m,2),s,t)$ does not satisfy condition (F4), it is not difficult to verify that $HP(P(m,2),s,t)$ does exist. Thus, we have the following lemma.

Lemma 10. *Let $P(m,2)$ be a 2-parallelogram with $m \geqslant 2$, and let s and t be its two distinct vertices with $s_x \leqslant t_x$. Then, $L(P(m,2),s,t) = \max\{2s_x-1, 2m-2s_x+3\}$ if $(P(m,2),s,t)$ satisfies condition* (F4)*; and $L(P(m,2),s,t) = 2m$, i.e., $HP(P(m,2),s,t)$ does exist, otherwise.*

The third forbidden condition for $HP(P(m,n),s,t)$ is as follows:

(F5) $P(m,n)$ satisfies $m \geqslant n \geqslant 2$, and (s,t) is an edge of $P(m,n)$ such that $s \sim w$ and $t \sim w$ for any parallel corner w of $P(m,n)$, where $s \neq w$, $t \neq w$, and $deg(w) = 2$ (see Fig. 7(c)).

When $(P(m,n),s,t)$ satisfies condition (F5), we can compute the longest (s,t)-path by removing the vertex w from the canonical Hamiltonian cycle of $P(m,n)$ constructed in Lemma 9. Thus, we have the following lemma.

Lemma 11. *Let $P(m,n)$ be a parallelogram supergrid graph with $m \geqslant n \geqslant 2$, and let s and t be its two distinct vertices. If $(P(m,n),s,t)$ satisfies condition* (F5), *then $L(P(m,n),s,t) = mn-1$, and the longest (s,t)-path contains at least one boundary edge of each boundary in $P(m,n)$ when $n \geqslant 3$.*

In the following, we consider that $(P(m,n),s,t)$ does not satisfy conditions (F3)–(F5). Then, we will construct a canonical Hamiltonian (s,t)-path of $P(m,n)$. We first consider 3-parallelogram $P(m,3)$ as follows.

Lemma 12. *Let $P(m,n)$ be a 3-parallelogram with $n = 3$ and $m \geqslant 3$, and let s and t be two distinct vertices of $P(m,n)$ with $s_x \leqslant t_x$. If $(P(m,n),s,t)$ does not satisfy condition* (F5), *then $P(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(P(m,3),s,t)$ does exist.*

Proof: Let w and w' be two parallel corners of $P(m,3)$. Since $(P(m,3),s,t)$ does not satisfy condition (F5), we get that $s \nsim \tilde{w}$ or $t \nsim \tilde{w}$ for $\tilde{w} = w$ or w'. Consider the following cases:

Case 1: $m = n = 3$. We first make a vertical cut on $P(3,3)$ to obtain two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(3,3)$ and $\Delta_2 = \Delta(2,2)$, as depicted in Fig. 8(a). Without loss of generality, assume that $w \in \Delta_1$ and $w' \in \Delta_2$. Let w_1 and w_2 be respectively parallel corners of Δ_1 and Δ_2 different from w and w'. There are three subcases:

Case 1.1: $s,t \in \Delta_1$. By visiting all boundary edges of Δ_2, we obtain a Hamiltonian cycle HC_2 of Δ_2. Suppose that (Δ_1,s,t) does not satisfy condition (F1). By Lemma 8, Δ_1 contains a canonical Hamiltonian (s,t)-path P_1 (see Fig. 5(a)). Then, there exist two edges $e_1 \in P_1$ and $e_2 \in HC_2$ such that $e_1 \approx e_2$. By Proposition 4, P_1 and HC_2 can be combined into a canonical Hamiltonian (s,t)-path of $P(3,3)$. On the other hand, suppose that (Δ_1,s,t) satisfies condition (F1). Then, (s,t) is a nonboundary edge of Δ_1, and $s \nsim w$ or $t \nsim w$ (see Fig. 4(a)). By inspecting every case, we can construct a Hamiltonian (s,t)-path P_1^* of $\Delta_1 - w_1$ such that it contains a vertical boundary edge e_1 of Δ_1. Let w^* be the vertex of $\Delta_2 - \{w', w_2\}$. Then, HC_2 contains vertical boundary edge (w^*, w_2) of Δ_2 such that $w_1 \sim w^*$ and $w_1 \sim w_2$. By Proposition 5, w_1 can be merged into HC_2 to form a Hamiltonian cycle HC_2^* of $\Delta_2 \cup \{w_1\}$. Then, there exists an edge $e_2 \in HC_2^*$ such that $e_1 \approx e_2$. By Proposition 4, P_1^* and HC_2^* can be combined into a canonical Hamiltonian (s,t)-path of $P(3,3)$. Fig. 8(a) depicts such a constructed Hamiltonian (s,t)-path.

Case 1.2: $s,t \in \Delta_2$. By Lemma 6, Δ_1 contains a canonical Hamiltonian cycle HC_1. Since $(P(3,3),s,t)$ does not satisfy condition (F5), $s \nsim w'$ or $t \nsim w'$. Thus, $w' = s$ or t. Since $s_x \leqslant t_x$, $w' = t$. Then, Δ_2 contains a Hamiltonian (s,t)-path P_2 such that it contains the vertical boundary edge e_2 of Δ_2. Thus, there exist two edges $e_1 \in HC_1$ and $e_2 \in P_2$ with $e_1 \approx e_2$. By Proposition 4, we can combine P_2 and HC_1 into a canonical Hamiltonian (s,t)-path of $P(3,3)$.

Case 1.3: $s \in \Delta_1$ and $t \in \Delta_2$. Let p be a vertex in Δ_1 such that $p = w_1$ if $s \neq w_1$, and (p, w_1) is a vertical boundary edge of Δ_1 otherwise. Let $q \in \Delta_2$ such that $q \neq t$, $q \sim p$, and (q,t) is not a skewed edge of Δ_2. Then, (Δ_1,s,p) does not satisfy condition (F1), and Δ_2 contains a Hamiltonian (q,t)-path P_2 which visits the skewed edge of Δ_2. By Lemma 8, Δ_1 contains canonical Hamiltonian (s,p)-path P_1. Then, $P_1 \Rightarrow P_2$ forms a canonical Hamiltonian (s,t)-path of $P(3,3)$.

Case 2: $m = n + 1 = 4$. In this case, we first make a vertical cut on $P(4,3)$ to get two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(3,3)$ and $\Delta_2 = \Delta(3,3)$, as depicted in Fig. 8(b). There are the following two subcases:

Case 2.1: $s,t \in \Delta_1$ or Δ_2. By symmetry, we can only consider that $s,t \in \Delta_1$. By similar arguments in proving Case 1.1, a canonical Hamiltonian (s,t)-path of $P(4,3)$ can be constructed.

Case 2.2: $s \in \Delta_1$ and $t \in \Delta_2$. Let $p \in \Delta_1$ and $q \in \Delta_2$ such that $p \neq s$, $q \neq t$, (Δ_1,s,p) and (Δ_2,q,t) do not satisfy condition (F1), and $p \sim q$. Consider that p and q do exist. By Lemma 8, Δ_1 and Δ_2 contain canonical Hamiltonian (s,p)-path P_1 and (q,t)-path P_2, respectively. Then, $P_1 \Rightarrow P_2$ forms a canonical Hamiltonian (s,t)-path of $P(4,3)$. On the other hand, consider that p or q does not exist. By inspecting every case for the locations of s and t, only one case occurs about that p and q do not exist. The location of s and t is shown in Fig. 8(b). Then, a canonical Hamiltonian (s,t)-path of $P(4,3)$ can be easily constructed, as depicted in Fig. 8(b).

Case 3: $m = n + 2 = 5$. We first perform two vertical cuts on $P(5,3)$ to partition it into three disjoint supergrid subgraphs, $\Delta_1 = \Delta(3,3)$, $\Delta_2 = \Delta(2,2)$, and $R = R(2,3)$, as depicted in Fig. 8(c). Let w_1 and w_2 be respectively parallel corners of Δ_1 and Δ_2 different from w and w'. There are four subcases:

Case 3.1: $s,t \in \Delta_1$ or Δ_2. Consider that $s,t \in \Delta_1$. By visiting all boundary edges of $R = R(2,3)$, we get a Hamiltonian cycle HC_R of R such that it contains four flat paths of R. By visiting all boundary edges of Δ_2, we obtain a canonical Hamiltonian cycle HC_2 of Δ_2. Then, there exist two edges $e_R \in HC_R$ and $e_2 \in HC_2$ such that $e_R \approx e_2$. By Proposition 3, HC_R and HC_2 can be combined into a Hamiltonian cycle HC^* of $R \cup \Delta_2$ such that HC^* contains one flat face of R that is placed to face Δ_1. Suppose that (Δ_1,s,t) does not satisfy condition (F1). By Lemma 8, Δ_1 contains a canonical Hamiltonian (s,t)-path P_1. Then, there exist two edges $e_1 \in P_1$ and $e^* \in HC^*$ such that $e_1 \approx e^*$. By Proposition 4, P_1 and HC^* can be combined into a canonical Hamiltonian (s,t)-path of $P(m,n)$. On the other hand, suppose that (Δ_1,s,t) satisfies condition (F1). Then, (s,t) is a nonboundary edge of Δ_1, and $s \nsim w$ or $t \nsim w$ (see Fig. 4(a)). By inspecting every case, we can construct a Hamiltonian (s,t)-path P_1^* of $\Delta_1 - w_1$ such that it contains a vertical boundary edge e_1 of Δ_1. Let w^* be the down-left corner of R and let (w^*, p) be the vertical edge in R. Then, $w_1 \sim w^*$ and $w_1 \sim p$. By Proposition 5, w_1 can be merged into HC^* to form a Hamiltonian cycle HC' of $\Delta_2 \cup R \cup \{w_1\}$. By Lemma 7, $\Delta_1 - \{w_1\}$ contains a canonical Hamiltonian (s,t)-path P_1^*. Then, there exist two edges $e_1^* \in P_1^*$ and $e' \in HC'$ such that $e_1^* \approx e'$. By Proposition 4, P_1^* and HC' can be combined into a canonical Hamiltonian (s,t)-path of $P(5,3)$. The subcase of $s,t \in \Delta_2$ can be proved similarly.

Case 3.2: $s,t \in R$. By Lemma 6, Δ_1 contains a canonical Hamiltonian cycle HC_1. By visiting all boundary edges of Δ_2, Δ_2 has a Hamiltonian cycle HC_2 which contains all boundary edges. Suppose that (s,t) is a horizontal and nonboundary edge of R. Then, $HP(R,s,t)$ does not exist. We then perform a horizontal cut on R to obtain two disjoint subgraphs R_1 and R_2, as illustrated in Fig. 8(c). By visiting all boundary edges of R_1 except (s,t), we obtain a Hamiltonian (s,t)-path P_R of R_1. For every vertex $v \in R_2$, v is incident to one edge of HC_1 or HC_2. By Proposition 5, the vertices of R_2 can be merged into HC_1 or HC_2. Let the merged cycles of R_2 into HC_1 and HC_2 be HC_1' and HC_2', respectively. Then, there exist four edges $e_1' \in HC_1'$, $e_2' \in HC_2'$, and $e_1^*, e_2^* \in P_R$ such that $e_1' \approx e_1^*$ and $e_2' \approx e_2^*$. By Proposition 4, P_R, HC_1', and HC_2' can be combined into a canonical Hamiltonian (s,t)-path of $P(5,3)$. For example, Fig. 8(c) shows a such canonical Hamiltonian (s,t)-path of $P(5,3)$. On the other hand, suppose that $s \nsim t$ or (s,t) is not a horizontal and nonboundary edge of R. Then, R contains a canonical Hamiltonian (s,t)-path P_R constructed in [18]. Thus, there exist four edges $e_1 \in HC_1$, $e_2 \in HC_2$, and $e_1^*, e_2^* \in P_R$ such that $e_1 \approx e_1^*$ and $e_2 \approx e_2^*$. By Proposition 4, P_R, HC_1, and HC_2 can be combined into a canonical Hamiltonian (s,t)-path of $P(5,3)$.

Case 3.3: s and t are in the different partitioned subgraphs. Note that $s_x \leqslant t_x$. We have the following subcases:

Case 3.3.1: ($s \in \Delta_1$ and $t \in R$) or ($s \in R$ and $t \in \Delta_2$). Consider that $s \in \Delta_1$ and $t \in R$. Let p be a vertex in Δ_1 such that $p = w_1$ if $s \neq w_1$, and (p, w_1) is a vertical boundary edge of Δ_1 otherwise. Let $q \in R$ such that $q \neq t$, $q \sim p$, and (q,t) is not a horizontal nonboundary edge of R. Then, (Δ_1, s, p) does not satisfy condition (F1), and R contains a canonical Hamiltonian (q,t)-path P_R constructed in [18]. By visiting all boundary edges of Δ_2, we get a Hamiltonian cycle HC_2 of Δ_2 which contains all boundary edges. Then, there exist two edges $e_R \in P_R$ and $e_2 \in HC_2$ such that $e_R \approx e_2$. By Proposition 4, P_R and HC_2 can be combined into a Hamiltonian (q,t)-path P_R' of $R \cup \Delta_2$. By Lemma 8, Δ_1 contains a canonical Hamiltonian (s,p)-path P_1. Then, $P_1 \Rightarrow P_R'$ forms a canonical Hamiltonian (s,t)-path of $P(5,3)$. The subcase of $s \in R$ and $t \in \Delta_2$ can be verified by the same arguments.

Case 3.3.2: $s \in \Delta_1$ and $t \in \Delta_2$. Let p be a vertex in Δ_1 such that $p = w_1$ if $s \neq w_1$, and (p, w_1) is a vertical boundary edge of Δ_1 otherwise. Let q be a vertex in Δ_2 such that $q = w_2$ if $t \neq w_2$, and (q, w_2) is a vertical boundary edge of Δ_2 otherwise. Let $r_1, r_2 \in R$ such that $r_1 \sim p$, $r_2 \sim q$, and (r_1, r_2) is not a horizontal nonboundary edge of R. By inspecting any case, p, q, and r_1, r_2 do exist. Then, (Δ_1, s, p) does not satisfy condition (F1), $HP(\Delta_2, q, t)$ does exist, and $HP(R, r_1, r_2)$ does exist. By Lemma 8, Δ_1 contains a canonical Hamiltonian (s,p)-path P_1. Let P_R be the canonical Hamiltonian (r_1, r_2)-path of R constructed in [18], and let P_2 be the Hamiltonian (q,t)-path of Δ_2. Then, $P_1 \Rightarrow P_R \Rightarrow P_2$ forms a canonical Hamiltonian (s,t)-path of $P(5,3)$.

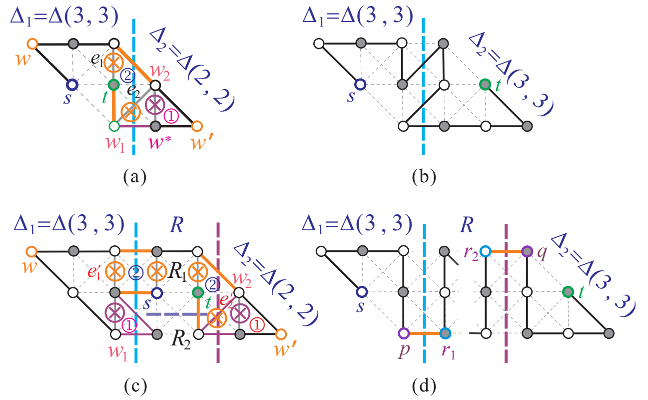

Fig. 8. The constructed canonical Hamiltonian (s,t)-path of $P(m,3)$ for (a) $m = 3$, $s,t \in \Delta_1$, and (Δ_1, s, t) satisfies condition (F1), (b) $m = 4$, and $s \in \Delta_1$, $t \in \Delta_2$, (c) $m = 5$, $s,t \in R = R(2,3)$, and (s,t) is a horizontal and nonboundary edge of R, and (d) $m \geqslant 6$, and $s \in \Delta_1$, $t \in \Delta_2$, where bold dashed lines represent the cut operations on $P(m,3)$, solid lines indicate the constructed Hamiltonian (s,t)-path, and $\otimes$ represents the destruction of an edge while constructing the Hamiltonian (s,t)-path.

Case 4: $m \geqslant n+3 = 6$. We first make two vertical cuts on $P(m,3)$ to partition it into three disjoint supergrid subgraphs, $\Delta_1 = \Delta(3,3)$, $R = R(m-3,3)$, and $\Delta_2 = \Delta(3,3)$, as depicted in Fig. 8(d). By Lemma 2, R is Hamiltonian connected. Then, a canonical Hamiltonian (s,t)-path of $P(m,3)$ can be constructed by similar arguments in proving Case 3. For instance, Fig. 8(d) depicts a canonical Hamiltonian (s,t)-path of $P(m,3)$ when $s \in \Delta_1$ and $t \in \Delta_2$.

We have considered any case to construct a canonical Hamiltonian (s,t)-path of $P(m,3)$ for $m \geqslant 3$. This completes the proof of the lemma. ■

By similar arguments in proving Lemma 12, we can prove the Hamiltonian connectivity of parallelogram supergrid graph $P(m,n)$ with $m \geqslant n \geqslant 4$ as follows.

Lemma 13. *Let $P(m,n)$ be a parallelogram supergrid graph with $m \geqslant n \geqslant 4$, and let s and t be two distinct vertices of $P(m,n)$ with $s_x \leqslant t_x$. If $(P(m,n), s, t)$ does not satisfy condition* (F5)*, then $P(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(P(m,n), s, t)$ does exist.*

Proof: We will prove this lemma by constructing a canonical Hamiltonian (s,t)-path of $P(m,n)$. Let w and w' be the two parallel corners of $P(m,n)$. Since $(P(m,n), s, t)$ does not satisfy condition (F5), $s \nsim \tilde{w}$ or $t \nsim \tilde{w}$ for $\tilde{w} = w$ or w'. Since $n \geqslant 4$, $n-1 \geqslant 3$. The considered cases are the same as Lemma 12 and are discussed as follows:

Case 1: $m = n$. We first make a vertical cut on $P(m,n)$ to get two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(n,n)$ and $\Delta_2 = \Delta(n-1, n-1)$, as depicted in Fig. 9(a). Without loss of generality, assume that $w \in \Delta_1$ and $w' \in \Delta_2$. Let w_1 and w_2 be respectively corners of Δ_1 and Δ_2 different from w and w'. There are three subcases::

Case 1.1: $s,t \in \Delta_1$ or Δ_2. By Lemma 6, Δ_1 and Δ_2 contain canonical Hamiltonian cycles. Then, a canonical Hamiltonian (s,t)-path of $P(m,n)$ can be constructed by similar arguments in proving Case 1.1 of Lemma 12.

Case 1.2: $s \in \Delta_1$ and $t \in \Delta_2$. Let q be a vertex in Δ_2 such that $q = w_2$ if $t \neq w_2$, and (q, w_2) is a vertical boundary edge of Δ_2 otherwise. Then, (Δ_2, q, t) does not

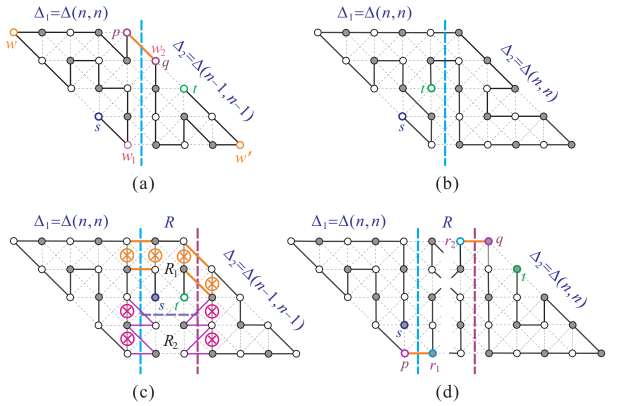

Fig. 9. The constructed canonical Hamiltonian (s,t)-path of $P(m,n)$ with $m \geqslant n \geqslant 4$ for (a) $m = n$, and $s \in \Delta_1$, $t \in \Delta_2$, (b) $m = n+1$, and $s,t \in \Delta_1$, (c) $m = n+2$, $s,t \in R(2,n)$, and $HP(R,s,t)$ does not exist, and (d) $m \geqslant n+3$, and $s \in \Delta_1$, $t \in \Delta_2$, where bold dashed lines represent the cut operations on $P(m,n)$, solid lines indicate the constructed Hamiltonian (s,t)-path, and $\otimes$ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path.

satisfy conditions (F1) and (F2). Let $p \in \Delta_1$ such that $p \neq s$, $p \sim q$, and (Δ_1, s, p) does not satisfy condition (F2). Since $n \geqslant 4$, p and q do exist. By Lemma 8, Δ_1 and Δ_2 contain canonical Hamiltonian (s,p)-path P_1 and (q,t)-path P_2, respectively. Then, $P_1 \Rightarrow P_2$ forms a canonical Hamiltonian (s,t)-path of $P(m,n)$. Fig. 9(a) depicts a such canonical Hamiltonian (s,t)-path.

Case 2: $m = n+1$. In this case, we first perform a vertical cut on $P(m,n)$ to partition it into two disjoint triangular supergrid subgraphs $\Delta_1 = \Delta(n,n)$ and $\Delta_2 = \Delta(n,n)$, as depicted in Fig. 9(b). By similar arguments in proving Case 1, we can construct a canonical Hamiltonian (s,t)-path of $P(m,n)$. For instance, Fig. 9(b) shows a constructed canonical Hamiltonian (s,t)-path when $s,t \in \Delta_1$ and (Δ_1, s, t) does not satisfy condition (F2).

Case 3: $m = n+2$. We first make two vertical cuts on $P(m,n)$ to partition it into three disjoint supergrid subgraphs, $\Delta_1 = \Delta(n,n)$, $R = R(2,n)$, and $\Delta_2 = \Delta(n-1,n-1)$, as depicted in Fig. 9(c). There are the following three subcases:

Case 3.1: $s,t \in \Delta_1$ or Δ_2. By Lemma 6, Δ_1 and Δ_2 contain canonical Hamiltonian cycles. By similar arguments in proving Case 3.1 of Lemma 12, a canonical Hamiltonian (s,t)-path of $P(m,n)$ can be constructed.

Case 3.2: $s,t \in R$. By similar arguments in proving Case 3.2 of Lemma 12, we can construct a canonical Hamiltonian (s,t)-path of $P(m,n)$. For instance, Fig. 9(c) depicts a constructed canonical Hamiltonian (s,t)-path of $P(m,n)$ when $HP(R,s,t)$ does not exist.

Case 3.3: s and t are not in the same partitioned subgraph. This subcase can be verified by similar arguments in proving Case 3.3 of Lemma 12.

Case 4: $m \geqslant n+3$. We first perform two vertical cuts on $P(m,n)$ to partition it into two disjoint supergrid subgraphs, $\Delta_1 = \Delta(n,n)$, $R = R(m-n,n)$, and $\Delta_2 = \Delta(n,n)$, as depicted in Fig. 9(d). Then, a canonical Hamiltonian (s,t)-path of $P(m,n)$ can be constructed by similar arguments in proving Case 3.3 of Lemma 12. For instance, Fig. 9(d) shows a canonical Hamiltonian (s,t)-path of $P(m,n)$ when $s \in \Delta_1$ and $t \in \Delta_2$.

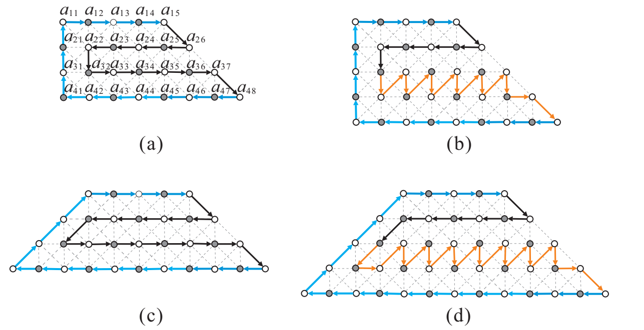

Fig. 10. The canonical Hamiltonian cycle of (a) $T_1(8,4)$, (b) $T_1(9,5)$, (c) $T_2(11,4)$, and $T_2(13,5)$, where arrow lines indicate the edges in the Hamiltonian cycle.

It follows from the above cases that a canonical Hamiltonian (s,t)-path of $P(m,n)$ with $m \geqslant n \geqslant 4$ can be constructed, and, hence, $HP(P(m,n),s,t)$ does exist. ■

It immediately follows from Lemmas 12 and 13 that we conclude the following theorem.

Theorem 14. *Let $P(m,n)$ be a parallelogram supergrid graph with $m \geqslant n \geqslant 1$, and let s and t be two distinct vertices of $P(m,n)$. If $(P(m,n),s,t)$ does not satisfy conditions* (F3)–(F5)*, then $P(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(P(m,n),s,t)$ does exist.*

IV. The Hamiltonicity and Hamiltonian Connectivity of Trapezoid Supergrid Graphs

In this section, we will prove the Hamiltonicity and Hamiltonian connectivity (except two trivial conditions) of trapezoid supergrid graphs. There are two types of trapezoid supergrid graphs $T_1(m,n)$ and $T_2(m,n)$. By similar arguments in proving Lemma 9, we can verify the Hamiltonicity of trapezoid supergrid graphs as follows.

Lemma 15. *Let $T(m,n)$ be a trapezoid supergrid graph. Then, $T(m,n)$ contains a canonical Hamiltonian cycle.*

Proof: Consider $T(m,n) = T_1(m,n)$, i.e., $T(m,n)$ is a n_{T_1}-trapezoid. By inspection, the lemma can be easily verified when $3 \geqslant n$. In the following, assume that $n \geqslant 4$. By definition of $T_1(m,n)$, $m \geqslant n+1 \geqslant 5$. Note that $T_1(m,n)$ consists of m columns and n rows of vertices. Let a_{ij} be the vertex located at i-th row and j-th column of $T_1(m,n)$, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m-i+1$. Consider the following two cases:

Case 1: n is even. Let $P_1 = a_{11} \rightarrow a_{12} \rightarrow \cdots \rightarrow a_{1(m-n)} \rightarrow a_{1(m-n+1)}$, $P_\imath = a_{\imath 2} \rightarrow a_{\imath 3} \rightarrow \cdots \rightarrow a_{\imath(m-n+\imath-1)} \rightarrow a_{\imath(m-n+\imath)}$ for $n \geqslant \imath \geqslant 2$, and let $P_{n+1} = a_{21} \rightarrow a_{31} \rightarrow \cdots \rightarrow a_{(n-1)1} \rightarrow a_{n1}$. Let $HC = P_1 \Rightarrow \text{rev}(P_2) \Rightarrow P_3 \Rightarrow \text{rev}(P_4) \Rightarrow \cdots \Rightarrow P_\jmath \Rightarrow \text{rev}(P_{\jmath+1}) \Rightarrow \cdots \Rightarrow P_{n-1} \Rightarrow \text{rev}(P_n) \Rightarrow \text{rev}(P_{n+1})$, where $\jmath$ is an odd and $n-1 \geqslant \jmath \geqslant 1$. Then, HC is a canonical Hamiltonian cycle of $T_1(m,n)$. For example, Fig. 10(a) depicts a canonical Hamiltonian cycle of $T_1(8,4)$.

Case 2: n is odd. In this case, $n \geqslant 5$. Let $P_1 = a_{11} \rightarrow a_{12} \rightarrow \cdots \rightarrow a_{1(m-n)} \rightarrow a_{1(m-n+1)}$, $P_\imath = a_{\imath 2} \rightarrow a_{\imath 3} \rightarrow \cdots \rightarrow a_{\imath(m-n+\imath-1)} \rightarrow a_{\imath(m-n+\imath)}$ for $n-3 \geqslant \imath \geqslant 2$, $P_{n-2} = a_{(n-2)2} \rightarrow a_{(n-1)2} \rightarrow a_{(n-2)3} \rightarrow a_{(n-1)3} \rightarrow \cdots \rightarrow a_{(n-2)\hat{\jmath}} \rightarrow a_{(n-1)\hat{\jmath}} \rightarrow$

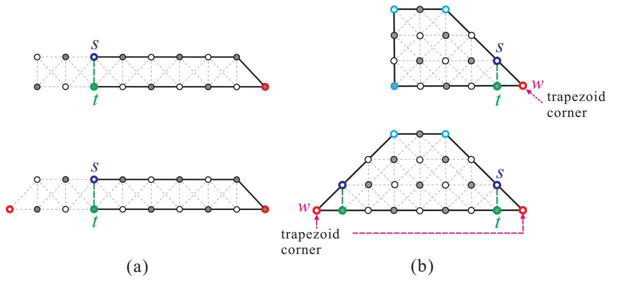

Fig. 11. Trapezoid supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) condition (F6), and (b) condition (F7), where the solid lines indicate the longest (s,t)-path.

$\cdots \to a_{(n-2)(m-2)} \to a_{(n-1)(m-2)m} \to a_{(n-1)(m-1)}$, where $m-2 \geqslant \jmath \geqslant 2$, $P_{n-1} = a_{n2} \to a_{n3} \to \cdots \to a_{nm}$, and let $P_n = a_{21} \to a_{31} \to \cdots \to a_{(n-1)1} \to a_{n1}$. Let $HC^* = P_1 \Rightarrow \text{rev}(P_2) \Rightarrow P_3 \Rightarrow \text{rev}(P_4) \Rightarrow \cdots \Rightarrow P_\jmath \Rightarrow \text{rev}(P_{\jmath+1}) \Rightarrow \cdots \Rightarrow P_{n-2} \Rightarrow \text{rev}(P_{n-1}) \Rightarrow \text{rev}(P_n)$, where $\jmath$ is an odd and $n-2 \geqslant \jmath \geqslant 1$. Then, HC^* is a canonical Hamiltonian cycle of $T_1(m,n)$. For example, Fig. 10(b) shows a canonical Hamiltonian cycle of $T_1(9,5)$.

We have proved the lemma holds true for $T(m,n) = T_1(m,n)$. For the type of trapezoid supergrid graphs $T_2(m,n)$, we can verify their Hamiltonicity by the same construction. For instance, Fig. 10(c) and Fig. 10(d) depict the canonical Hamiltonian cycles of $T_2(11,4)$ and $T_2(13,5)$, respectively. Thus, the lemma holds true. ■

Next, we will study the Hamiltonian connectivity of trapezoid supergrid graphs. Let $T(m,n)$ be a trapezoid supergrid graph, where $T(m,n) = T_1(m,n)$ or $T(m,n) = T_2(m,n)$. We first observe the conditions so that $HP(T(m,n),s,t)$ does not exist. For a 2_{T_1}-trapezoid or 2_{T_2}-trapezoid, the following condition implies that $HP(T(m,2),s,t)$ does not exist.

(F6) $T(m,n)$ is a 2_{T_1}-trapezoid or 2_{T_2}-trapezoid, and (s,t) is a vertical and nonboundary edge of $T(m,n)$ (see Fig. 11(a)).

For a trapezoid corner w of $T(m,n)$, we can easily see that $HP(T(m,n),s,t)$ does not exist when $s,t \neq w$, $s \sim w$, and $t \sim w$.

(F7) $T(m,n)$ is a trapezoid supergrid graph for $n \geqslant 2$, w is a trapezoid corner of $T(m,n)$, $s,t \neq w$, $s \sim w$, and $t \sim w$ (see Fig. 11(b)).

By similar arguments in proving Lemma 7, the following lemma can be verified.

Lemma 16. *Let $T(m,n)$ be a trapezoid supergrid graph with $n \geqslant 2$, and let s and t be two distinct vertices of $T(m,n)$. Then, the following statements hold true:*
(1) *if $(T(m,n),s,t)$ satisfies condition* (F6), *then $L(T_1(m,n),s,t) = \max\{2(m-s_x+1)-1, 2s_x\}$ and $L(T_2(m,n),s,t) = \max\{2(m-s_x+1)-1, 2s_x+1\}$.*
(2) *if $(T(m,n),s,t)$ satisfies condition* (F7), *then $L(T(m,n),s,t) = |V(T(m,n))|-1$.*

In the following, we will assume that $(T(m,n),s,t)$ does not satisfy conditions (F6) and (F7). Then, we will construct a canonical Hamiltonian (s,t)-path of $T(m,n)$. We first prove $T_1(m,n)$ to be Hamiltonian connected as follows.

Lemma 17. *Let $T_1(m,n)$ be a trapezoid supergrid graph with $m-1 \geqslant n \geqslant 2$, and let s and t be two distinct vertices of $T_1(m,n)$. If $(T_1(m,n),s,t)$ does not satisfy conditions* (F6)–(F7), *then $T_1(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(T_1(m,n),s,t)$ does exist.*

Proof: When $n=2$, i.e., $T_1(m,n)$ is a 2_{T_1}-trapezoid, $HP(R_1,s,t)$ does exist [18] and hence $HP(T_1(m,n),s,t)$ can be easily constructed. In the following, assume that $n \geqslant 3$. By definition of $T_1(m,n)$, $m \geqslant n+1 \geqslant 4$ and, hence, $m-n+1 \geqslant 2$. We first make a vertical cut on $T_1(m,n)$ to obtain two disjoint subgraphs $R_1 = R(m-n+1,n)$ and $\Delta_1 = \Delta(n-1,n-1)$, as shown in Fig. 12(a). Depending on the locations of s and t, we consider the following three cases:

Case 1: $s,t \in R_1$. In this case, we consider whether R_1 is a 2-rectangle as follows:

Case 1.1: $m-n+1 = 2$. In this subcase, R_1 is a 2-rectangle. Suppose that (s,t) is not a horizontal and nonboundary edge of R_1. In [18], R_1 contains a canonical Hamiltonian (s,t)-path P_1. By Lemma 6, Δ_1 contains a canonical Hamiltonian cycle C_1. Then, there exist two edges $e_1 \in P_1$ and $e_2 \in C_1$ such that $e_1 \approx e_2$. By Proposition 4, P_1 and C_1 can be merged into a Hamiltonian (s,t)-path of $T_1(m,n)$. On the other hand, suppose that (s,t) is a horizontal and nonboundary edge of R_1. Then, R_1 contains no Hamiltonian (s,t)-path. We next preform two horizontal cuts on R_1 to get three disjoint rectangular supergrid subgraphs R_{11}, R_{12} and R_{13} so that R_{12} contains only s and t, as depicted in Fig. 12(b). Let $p_1, q_1 \in R_{11}$, $p_2, q_2 \in R_{13}$, $r_1, r_2 \in \Delta_1$ such that (s,p_1) and (s,p_2) are vertical edges, (t,q_1) and (t,q_2) are vertical edges, and (q_1,r_1) and (q_2,r_2) are horizontal edges in $T_1(m,n)$, as shown in Fig. 12(b). We can easily construct a Hamiltonian (p_1,q_1)-path P_1 of R_{11} and a Hamiltonian (q_2,p_2)-path P_2 of R_{13} such that P_1 (resp., P_2) visits all boundary edges of R_{11} (resp. R_{13}) except (p_1,q_1) (resp., (q_2,p_2)) if $|V(R_{11})| > 2$ (resp., $|V(R_{13})| > 2$). We can see that (Δ_1, r_1, r_2) does not satisfy conditions (F1) and (F2). By Lemma 8, Δ_1 contains a canonical Hamiltonian (r_1,r_2)-path P_3. Then, $s \Rightarrow P_1 \Rightarrow P_3 \Rightarrow P_2 \Rightarrow t$ forms a canonical Hamiltonian (s,t)-path of $T_1(m,n)$.

Case 1.2: $m-n+1 > 2$. By Lemma 2, R_1 contains a canonical Hamiltonian (s,t)-path P_1. By Lemma 6, Δ_1 contains a canonical Hamiltonian cycle C_1. Then, there exist two edges $e_1 \in P_1$ and $e_2 \in C_1$ such that $e_1 \approx e_2$. By Proposition 4, P_1 and C_1 can be combined into a canonical Hamiltonian (s,t)-path of $T_1(m,n)$.

Case 2: $s,t \in \Delta_1$. Let w be the trapezoid corner of $T_1(m,n)$, and let w' be a trapezoid corner of Δ_1 different from w. Since $(T_1(m,n),s,t)$ does not satisfy conditions (F6)–(F7), we get that $s \nsim w$ or $t \nsim w$. Suppose that (Δ_1,s,t) satisfies condition (F1) or (F2). Let $\Delta_1' = \Delta_1 - \{w'\}$ and let $R_1' = R_1 \cup \{w'\}$. By Lemma 7, Δ_1' contains a canonical Hamiltonian (s,t)-path P_1'. By Lemma 1, R_1 contains a canonical Hamiltonian cycle C_1. Then, there exists an edge (u,v) in C_1 such that $u \sim w'$ and $v \sim w'$. By Proposition 5, C_1 and w' can be merged into a Hamiltonian cycle C_1' of R_1'. We can easily find two edges $e_1 \in C_1'$ and $e_2 \in P_1'$ such that $e_1 \approx e_2$. By Proposition 4, P_1' and

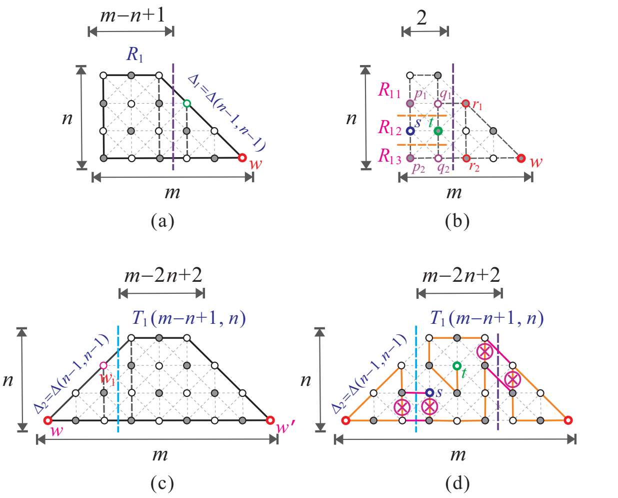

Fig. 12. (a) A vertical cut on $T_1(m,n)$ to get disjoint subgraphs $R_1 = R(m-n+1,n)$ and $\Delta_1 = \Delta(n-1,n-1)$, (b) two horizontal cuts on R_1 for (s,t) is a horizontal and nonboundary edge of R_1, (c) a vertical cut on $T_2(m,n)$ to obtain disjoint subgraphs $T_1(m-n+1,n)$ and $\Delta_2 = \Delta(n-1,n-1)$, and (d) the Hamiltonian (s,t)-path of $T_2(m,n)$ for $s,t \in T_1(m-n+1,n)$, where bold dashed lines indicate the cut operations on $T_1(m,n)$ or $T_2(m,n)$, solid lines indicate the constructed Hamiltonian (s,t)-path, and $\otimes$ represents the destruction of an edge while constructing the Hamiltonian (s,t)-path.

C_1' can be combined into a canonical Hamiltonian (s,t)-path of $T_1(m,n)$. On the other hand, suppose that (Δ_1,s,t) does not satisfy conditions (F1) and (F2). By Lemma 8, Δ_1 contains a canonical Hamiltonian (s,t)-path for $n-1 \geqslant 3$. When $n-1 = 2$, it is easy to construct a canonical Hamiltonian (s,t)-path of Δ_1. Thus, Δ_1 contains a canonical Hamiltonian (s,t)-path P_1. By Lemma 1, R_1 contains a canonical Hamiltonian cycle C_1. Then, there exist two edges $e_1 \in P_1$ and $e_2 \in C_1$ such that $e_1 \approx e_2$. By Proposition 4, P_1 and C_1 can be combined into a canonical Hamiltonian (s,t)-path of $T_1(m,n)$.

Case 3: $s \in R_1$ and $t \in \Delta_1$. In this case, we first find two vertices $p \in R_1$ and $q \in \Delta_1$ to satisfy that $HP(R_1,s,p)$ and $HP(\Delta_1,q,t)$ do exist, and $p \sim q$. The vertices p and q can be easily computed. Let $P_1 = HP(R_1,s,p)$ and $Q_1 = HP(\Delta_1,q,t)$ be canonical Hamiltonian (s,p)-path and (q,t)-path of R_1 and Δ_1, respectively. Then, $P_1 \Rightarrow Q_1$ forms a canonical Hamiltonian (s,t)-path of $T_1(m,n)$.

We have considered any case to construct a canonical Hamiltonian (s,t)-path of $T_1(m,n)$. This completes the proof of the lemma. ■

Next, we consider the other type of trapezoid supergrid graph $T_2(m,n)$ as follows.

Lemma 18. *Let $T_2(m,n)$ be a trapezoid supergrid graph with $\frac{m}{2} \geqslant n \geqslant 2$, and let s and t be two distinct vertices of $T_2(m,n)$. If $(T_2(m,n),s,t)$ does not satisfy conditions* (F6)–(F7), *then $T_2(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(T_2(m,n),s,t)$ does exist.*

Proof: By inspection, $HP(T_2(m,2),s,t)$ does exist when $(T_2(m,n),s,t)$ does not satisfy condition (F6). In the following, assume that $n \geqslant 3$. We first perform a vertical cut on $T_2(m,n)$ to partition it into two disjoint subgraphs $\Delta_2 = \Delta(n-1,n-1)$ and $T_1(m-n+1,n)$, as depicted in Fig. 12(c). Let w and w' be the trapezoid corners of $T_2(m,n)$ such that $w \in \Delta_2$ and $w' \in T_1(m-n+1,n)$. Depending on the locations of s and t, there are three cases:

Case 1: $s,t \in \Delta_2$. Let w_1 be a triangular corner of Δ_2 different from w. Suppose that (Δ_2,s,t) satisfies condition (F1) or (F2). Since $(T_2(m,n),s,t)$ does not satisfy condition (F7), we get that $s \sim w_1$ and $t \sim w_1$. Let $\Delta_2' = \Delta_2 - \{w_1\}$, and let $T_1' = T_1(m-n+1,n) \cup \{w_1\}$. By Lemma 7, Δ_2' contains a canonical Hamiltonian (s,t)-path P'. By Lemma 15, $T_1(m-n+1,n)$ contains a canonical Hamiltonian cycle C_1. Then, there exists an edge (u,v) in C_1 such that $u \sim w_1$ and $v \sim w_1$. By Proposition 5, w_1 can be combined into C_1 to form a canonical Hamiltonian cycle C_1' of T_1'. Then, there exist two edges $e' \in P'$ and $e_1 \in C_1'$ such that $e' \approx e_1$. By Proposition 4, P' and C_1' can be combined into a canonical Hamiltonian (s,t)-path of $T_2(m,n)$. On the other hand, suppose that (Δ_2,s,t) does not satisfy conditions (F1) and (F2). By Lemma 8, Δ_2 contains a canonical Hamiltonian (s,t)-path P. By Lemma 15, $T_1(m-n+1,n)$ contains a canonical Hamiltonian cycle C_1. Then, there exist two edges $e \in P$ and $e_1 \in C_1$ such that $e \approx e_1$. By Proposition 4, P and C_1 can be combined into a canonical Hamiltonian (s,t)-path of $T_2(m,n)$.

Case 2: $s \in \Delta_2$ and $t \in T_1(m-n+1,n)$. Let $p \in \Delta_2$ and $q \in T_1(m-n+1,n)$ such that $HP(\Delta_2,s,p)$ and $HP(T_1(m-n+1,n),q,t)$ do exist, and $p \sim q$. The vertices p and q can be easy to compute. Let P and Q be the constructed canonical Hamiltonian (s,p)-path and Hamiltonian (q,t)-path of Δ_2 and $T_1(m-n+1,n)$, respectively. Then, $P \Rightarrow Q$ forms a canonical Hamiltonian (s,t)-path of $T_2(m,n)$.

Case 3: $s,t \in T_1(m-n+1,n)$. By Lemma 17, $T_1(m-n+1,n)$ contains a canonical Hamiltonian (s,t)-path P. By Lemma 6, Δ_2 contains a canonical Hamiltonian cycle C. Then, there exist two edges $e \in P$ and $e_1 \in C$ such that $e \approx e_1$. By Proposition 4, P and C can be combined into a canonical Hamiltonian (s,t)-path of $T_2(m,n)$. For instance, Fig. 12(d) depicts the construction of such a canonical Hamiltonian (s,t)-path.

It follows from the above cases that a canonical Hamiltonian (s,t)-path of $T_2(m,n)$ is constructed. Thus, the lemma holds true. ■

It immediately follows from Lemmas 17–18 that the following theorem holds true.

Theorem 19. *Let $T(m,n)$ be a trapezoid supergrid graph with $n \geqslant 2$, and let s and t be two distinct vertices of $T(m,n)$, where $T(m,n) = T_1(m,n)$ or $T_2(m,n)$. If $(T(m,n),s,t)$ does not satisfy conditions* (F6)–(F7), *then $T(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(T(m,n),s,t)$ does exist.*

V. Concluding Remarks

In this paper, we provide constructive proofs to show that some shaped supergrid graphs, including triangular, parallelogram, and trapezoid, are Hamiltonian and Hamiltonian connected except few trivial conditions. These constructive proofs give linear time algorithms to construct the longest paths or Hamiltonian paths between any two distinct vertices of shaped supergrid graphs. A supergrid graph is called alphabet if its boundaries form an alphabet. There are 26 types of alphabet supergrid graphs. We can see from the structures of alphabet supergrid graphs that they can be decomposed into triangular, parallelogram, and trapezoid supergrid subgraphs. In the future, we would like to apply our results to study the Hamiltonian connectivity of alphabet supergrid graphs.

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