Stochastic Synchronization between Different Networks and Its Application in Bilayer Coupled Public Traffic Network

Wenju Du, Yinzhen Li, Jiangang Zhang and Jianning Yu

Abstract—**The conventional bus network and subway network are constructed respectively by using space R method in this paper, then regards these two networks as the sub-networks, and a new bilayer public traffic coupled network is presented based on the transfer relationship between subway and conventional bus. By using the synchronization theory of two different complex networks with stochastic disturbance, the paper investigated the synchronization of bilayer public traffic coupled network. Finally, the impact of stochastic disturbance on the balance of bilayer public traffic coupled network is analyzed through numerical simulation.**

Index Terms—**Complex network, Stochastic disturbance, Synchronization, Bilayer coupled public traffic network, Balance**

I. Introduction

With the continuous development of economy and the improvement of people's living standard, the quantity of automobile is increasing rapidly. At present, there exist a series of problems in the development of urban transportation in China, such as the difficulty of travel, the increasing traffic time costs and so on. The problem of urban traffic development should be solved urgently. And the traffic jams and congestion in big cities such as Beijing and Shanghai is becoming more and more serious, which also causes inconvenience to people's travel and causes urban environmental pollution and frequent traffic accidents. Therefore, reducing and alleviating traffic congestion is a problem we urgently need to solve. However, there is no obvious effect to solve these problems only by regulating and optimizing the conventional bus, and the emergence of rail traffic has greatly compensated for many shortcomings of conventional bus. Both conventional bus and rail transit are belongs to urban public transport system, each has its own unique advantages. The conventional bus is low in cost, wide in coverage and flexible in mobility, and the rail transit has fast speed, big transport volume and good punctuality. But they all have their own shortcomings. Therefore, strengthening the effective coordination and transfer connection between conventional bus and rail transit is helpful to improve the operation efficiency of the whole urban public transport system, so as to maximize the demand of the passengers and achieve the coordinated and continuous development of them. However, most of the current studies only focus on a single conventional bus network or a single rail transit network. And it can't well reflect the characteristics of complex urban public transport system.

The synchronization of complex network is an important topic in study of complex network dynamics. In recent years, many scholars have studied the synchronization problem of complex network [1-10]. However, these studies are just focused on the internal synchronization problem between single networks, and there is not much research on synchronization between two networks. Li et al. [11] derived a criterion for the synchronization of two unidirectionally coupled networks. Tang et al. [12] designed an effective adaptive controller and investigated the synchronization problem between two complex networks with nonidentical topological structures. Chen et al. [13] presented a general network model for two complex networks with time-varying delay coupling and derived a synchronization criterion by using adaptive controllers. Wang et al. [14] designed an adaptive controller to achieve synchronization between two different complex networks with time-varying delay coupling. Sun et al. [15] investigated the linear generalized synchronization between two complex networks. Besides, the research of synchronization between two networks is basically aimed at two networks with the same number of nodes. However, the synchronization problem of bilayer coupled networks with different number of nodes has more practical significance.

There are many uncertainties in nature, and these uncertainties are the randomness of external incentives or the randomness of internal parameters of structure. Many practical systems are affected by stochastic perturbations, which are often the main causes of instability. Because of the extensive research background of random disturbance and its research in the complex network has attracted the attention of many scholars. Due to the inevitable existence of random interference in real life, many synchronization phenomena are affected by the random interference. Therefore, it is very important to study the synchronization between the bilayer coupled networks under random disturbance. Guo et al. [16]

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focused on a time-varying complex dynamical network, and the stochastic synchronization problem of the network is investigated. Wang et al. [17] investigated the stochastic synchronization of complex network by using the properties of Weiner process. Hao et al. [18] studied the stochastic synchronization of complex dynamical networks with unknown periodic time-varying couplings and stochastic noise perturbations. He et al. [19] investigated the global synchronization problem of switched complex dynamical networks with stochastic disturbances. Zhang et al. [20] focused on a class of chaotic neural networks, and the synchronization problem of the networks is investigated under stochastic perturbations. Zhou et al. [21] investigated the exponential synchronization of a new class of stochastic neural networks driven by fractional Brownian motion. Sakthivel et al. [22] investigated the synchronization and state estimation problems of a coupled discrete-time stochastic complex network. Li et al. [23] studied the synchronization problem of stochastic discrete-time complex networks with partial mixed impulsive effects.

As one of the important research tool, the complex network has been widely applied in urban traffic system [24-28]. The urban public traffic network is a real and typical complex network, which has been studied by many scholars. However, most of the research is only aimed at the static statistical features of the network, such as, the research on topological property of traffic network, reliability or robustness and structure optimization. However, there are few researches on the dynamic characteristics of urban public traffic network. As a typical complex network, it is necessary to analyze the dynamic characteristics of the urban public traffic network because of its own characteristics. In this paper, we mainly focused on two coupled complex networks with different sizes under stochastic disturbance, and the proper controller is designed to make these two networks achieve globally asymptotically synchronized in mean square. In addition, a new type of bus-subway bilayer coupled public traffic network model is established based on space R modeling method. And the synchronization problem of bilayer coupled public traffic network under random disturbance is studied by using the synchronization theory of coupled network. Finally, the balance problem of bilayer coupled public traffic network is studied under stochastic disturbance.

The paper is organized as follows. In Section 2, the synchronization theory of two different complex networks with stochastic disturbance is given. A new bilayer public traffic coupled network model is established in Section 3. In Section 4, the balance of bilayer public traffic coupled network is investigated under the stochastic disturbance. Simulation results are given to show the validity of the controllers in Section 5. In Section 6, we conclude the paper.

II. Synchronization between Two Different Complex Networks with Stochastic Disturbance

Considering two coupled complex dynamic networks, each complex network is composed of the same linear coupling nodes, and the network models are described as follows:

$$dx_i(t)=\Big[f\big(x_i(t)\big)+\varepsilon_1\sum_{j=1}^{N_1}a_{ij}\boldsymbol{\Gamma}_1x_j(t)+\mu\sum_{j=1}^{N_2}c_{ij}y_j(t)\Big]dt,\quad i=1,2,\ldots,N_1 \tag{1}$$

$$dy_i(t)=\Big[g\big(y_i(t)\big)+\varepsilon_2\sum_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}_2y_j(t)+\mu\sum_{j=1}^{N_1}d_{ij}x_j(t)+u_i(t)\Big]dt+\delta_i\big(y_i(t)-x_i(t)\big)d\omega_i(t),\quad i=1,2,\ldots,N_2 \tag{2}$$

where $x_i=(x_{i1},x_{i2},\ldots,x_{in})^{\mathrm T}\in R^n, y_i=(y_{i1},y_{i2},\ldots,y_{in})^{\mathrm T}\in R^n$ are the node's state variables of the networks (1) and (2), respectively, $\dot{x}_i(t)$ and $\dot{y}_i(t)$ are the dynamic equations of a single node, $f(\cdot),g(\cdot):R^n\to R^n$ are the nonlinear continuous differentiable functions, N_1,N_2 are the number of nodes of networks (1) and (2), respectively, $\boldsymbol{\Gamma}_1,\boldsymbol{\Gamma}_2\in R^{n\times n}$ are the internal coupling function between the state variables of each node in two networks, $\varepsilon_1,\varepsilon_2$ are the internal coupling strength of networks (1) and (2), respectively, μ is the external coupling strength between two networks, the matrixes $A=(a_{ij})\in R^{N_1\times N_1}, B=(b_{ij})\in R^{N_2\times N_2}$ are the topology of networks (1) and (2), respectively, where $a_{ii}=-\sum_{j=1,i\neq j}^{N_1}a_{ij}, b_{ii}=-\sum_{j=1,i\neq j}^{N_2}b_{ij}$ and $a_{ij}(b_{ij})$ are defined as follows: if there is a connection from node j to node i, then $a_{ij}(b_{ij})>0(i\neq j)$; otherwise $a_{ij}(b_{ij})=0(i\neq j)$. The matrices $C=(c_{ij})\in R^{N_1\times N_2}, D=(d_{ij})\in R^{N_2\times N_1}$ are the coupling matrix between two networks, where c_{ij},d_{ij} are defined as follows: if there is a connection from node i (belongs to network (1)) to node j (belongs to network (2)), then $c_{ij}>0$; otherwise $c_{ij}=0$; if there is a connection from node i (belongs to network (2)) to node j (belongs to network (1)), then $d_{ij}>0$; otherwise $d_{ij}=0$. And the coefficients $\delta_i:R^n\times\cdots\times R^n\to R^{n\times n}$ represents the noise intensity function matrix, it is used to describe the strength of external stochastic disturbance, $\omega_i(t)=\big(\omega_{i1}(t),\omega_{i2}(t),\ldots,\omega_{in}(t)\big)^{\mathrm T}\in R^n$ is an n-dimensional Brownian motion which defined on a complete probability space (Ω,F,P), which satisfies $E\{d\omega(t)\}=0, E\Big\{\big(d\omega(t)\big)^2\Big\}=dt$, where Ω is a sample space which generated by $\omega_i(t)$, F is a σ-algebra, P is a probability measure. In this paper, we assume that $\omega_i(t)$ is independent of $\omega_j(t)$ when $i\neq j$. And $u_i(t)$ is the controller for node i to be designed. Without loss of generality, we assuming that $N_1>N_2$, that is network (1) and network (2) has the different number of nodes.

Definition 1. Let $x_i(t,X_0)(i=1,2,\ldots,N_1)$ and $y_i(t,Y_0,u_i)$ $(i=1,2,\ldots,N_2)$ be the solutions of the network (1) and (2), where $X_0=\big(x_1^0,x_2^0,\ldots,x_{N_1}^0\big)^{\mathrm T}\in R^{nN_1}, Y_0=\big(y_1^0,y_2^0,\ldots,y_{N_2}^0\big)^{\mathrm T}\in R^{nN_2}$, and $f,g:\Omega\to R^n$ are the continuously differentiable mappings with $\Omega \subseteq R^n$. If there exist a nonempty open subset $\Lambda \subseteq \Omega$, with $x_i^0, y_i^0 \in \Lambda$, so when $t \geq 0$, such that $x_i(t, X_0)(1 \leq i \leq N_1)$, $y_i(t, Y_0, u_i)(1 \leq i \leq N_2) \in \Omega$, and

$$\lim_{t\to\infty} E\left\{\left\|y_i(t, Y_0, u_i) - x_i(t, X_0)\right\|^2\right\} = 0, \quad (i = 1, 2, \cdots, N_2) \quad (3)$$

then the complex networks (1) and (2) realized globally asymptotically synchronization in mean square, where $\|.\|$ represents the Euclidean vector norm, and $E\{.\}$ represents the mathematical expectation.

Assumption 1. For the function $f(x)$, there exist a positive constant $l > 0$, such that

$$\begin{aligned} &\left[y_i(t) - x_i(t)\right]^{\mathrm{T}}\left[f(y_i(t)) - f(x_i(t))\right] \\ &\leq l\left[y_i(t) - x_i(t)\right]^{\mathrm{T}}\left[y_i(t) - x_i(t)\right], \end{aligned} \quad (4)$$

Assumption 2. Assume that the noise matrix $\delta_i(e_i(t))$ satisfy the Lipschitz condition, and there exist a constant $\rho > 0$, satisfying

$$tr\left(\delta_i(e_i(t))^{\mathrm{T}} \delta_i(e_i(t))\right) \leq 2\rho e_i(t)^{\mathrm{T}} e_i(t), \quad i = 1, 2, \cdots, N_2 \quad (5)$$

Definition 2. (*Itô* formula) Consider the following n-dimensional stochastic differential equation

$$dx(t) = f(t)dt + \delta(t)d\omega(t), \quad (6)$$

If there is $V(x(t), t) \in C^{2,1}(R^n \times R_+; R_+)$, where $C^{2,1}(R^n \times R_+; R_+)$ is the family of all nonnegative functions which are twice continuously differentiable in x and once differentiable in t, then the operator $LV(x(t), t)$ can be defined as follows:

$$\begin{aligned} LV(x(t), t) = &V_t(x(t), t) + V_x(x(t), t) + \\ &\frac{1}{2} tr\left[\delta^{\mathrm{T}}(x) V_{xx}(x(t), t)\delta(x)\right], \end{aligned} \quad (7)$$

where

$$V_t(x(t), t) = \frac{\partial V(x(t), t)}{\partial t}, V_{xx}(x(t), t) = \left(\frac{\partial^2 V(x(t), t)}{\partial x_i \partial x_j}\right)_{n\times n},$$

$$V_x(x(t), t) = \left(\frac{\partial V(x(t), t)}{\partial x_1}, \frac{\partial V(x(t), t)}{\partial x_2}, \cdots, \frac{\partial V(x(t), t)}{\partial x_n}\right).$$

Lemma 1. [29] Consider the stochastic differential equations (6), if there exist the positive constants $\alpha_1, \alpha_2, \alpha_3$, for any $t \in [t_0, +\infty]$, we have

$$\begin{aligned} &\alpha_1(\|x(t)\|) \leq V(x(t), t) \leq \alpha_2(\|x(t)\|), \\ &LV(x(t), t) \leq -\alpha_3(\|x(t)\|), \end{aligned} \quad (8)$$

then the equilibrium $x = 0$ is globally asymptotic stable in the mean square sense. Besides, for any $0 \leq t_0 \leq t \leq \infty$, one has

$$E\{V(x(t), t)\} = E\{V(x(t_0), t_0)\} + E\left\{\int_{t_0}^{t} LV(x(\tau), \tau)d\tau\right\}.$$

Theorem 1. Suppose that Assumption 1, 2 holds. We select the controllers as follows:

$$\begin{aligned} u_i(t) = &f(y_i(t)) - g(y_i(t)) + \varepsilon_1\sum_{j=1}^{N_2} a_{ij}\boldsymbol{\Gamma}_1 y_j(t) + \mu\sum_{j=1}^{N_2} c_{ij}x_j(t) \\ &- \varepsilon_2\sum_{j=1}^{N_2} b_{ij}\boldsymbol{\Gamma}_2 x_j(t) - \mu\sum_{j=1}^{N_2} d_{ij}y_j(t) + \varepsilon_1\sum_{j=N_2+1}^{N_1} a_{ij}\boldsymbol{\Gamma}_1 x_j(t) \\ &- \mu\sum_{j=N_2+1}^{N_1} d_{ij}x_j(t) - k_i(t)e_i(t), \quad i = 1, 2, \cdots, N_2 \end{aligned} \quad (9)$$

then the complex networks (1) and (2) realized globally asymptotically synchronization in mean square under the controllers (9), where $\dot{k}_i(t) = d_i\|e_i\|^2$, d_i are the positive constants, $i = 1, 2, \ldots, N_2$.

Proof. Define the errors vectors by $e_i(t) = y_i(t) - x_i(t)$, $i = 1, 2, \ldots, N_2$, and then the error systems can be described by

$$\begin{aligned} de_i(t) = &\Big[g(y_i(t)) - f(x_i(t)) + \mu\sum_{j=1}^{N_1} d_{ij}x_j(t) - \mu\sum_{j=1}^{N_2} c_{ij}y_j(t) \\ &- \varepsilon_1\sum_{j=1}^{N_1} a_{ij}\boldsymbol{\Gamma}_1 x_j(t) + \varepsilon_2\sum_{j=1}^{N_2} b_{ij}\boldsymbol{\Gamma}_2 y_j(t) + u_i(t)\Big]dt \\ &+ \delta_i(y_i(t) - x_i(t))d\omega_i(t) \\ = &\Big[f(y_i(t)) - f(x_i(t)) - \mu\sum_{j=1}^{N_2}(c_{ij} + d_{ij})e_j(t) \\ &+ \varepsilon_1\sum_{j=1}^{N_2} a_{ij}\boldsymbol{\Gamma}_1 e_j(t) + \varepsilon_2\sum_{j=1}^{N_2} b_{ij}\boldsymbol{\Gamma}_2 e_j(t) + k_i(t)e_i(t)\Big]dt \\ &+ \delta_i(e_i(t))d\omega_i(t), \quad i = 1, 2, \ldots, N_2 \end{aligned} \quad (10)$$

Choose the Lyapunov candidate as follows:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{N_2}\frac{1}{d_i}\left(k_i(t) - \tilde{k}\right)^2, \quad (11)$$

where $\tilde{k}$ is a sufficiently larger positive constant which is to be determined. By derivation of Eq. (11), we get

$$\begin{aligned} LV(e(t), t) = &\sum_{i=1}^{N_2} e_i^{\mathrm{T}}(t)\Big[f(y_i(t)) - f(x_i(t)) + \varepsilon_1\sum_{j=1}^{N_2} a_{ij}\boldsymbol{\Gamma}_1 e_j(t) \\ &+ \varepsilon_2\sum_{j=1}^{N_2} b_{ij}\boldsymbol{\Gamma}_2 e_j(t) - \mu\sum_{j=1}^{N_2}(c_{ij} + d_{ij})e_j(t) + k_i(t)e_i(t)\Big] \\ &+ \sum_{i=1}^{N_2}\frac{1}{d_i}\left(k_i(t) - \tilde{k}\right)^2 \dot{k}_i(t) + \frac{1}{2}\sum_{i=1}^{N_2} tr\left(\delta_i^{\mathrm{T}}(e_i(t))\delta_i(e_i(t))\right), \end{aligned}$$

According to Assumption 1and 2, we have

$$\begin{aligned} LV(e(t), t) \leq &\sum_{i=1}^{N_2} l e_i^{\mathrm{T}}(t)e_i(t) + \varepsilon_1\sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)a_{ij}\boldsymbol{\Gamma}_1 e_j(t) + \\ &\varepsilon_2\sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)b_{ij}\boldsymbol{\Gamma}_2 e_j(t) - \mu\sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)(c_{ij} + d_{ij})e_j(t) \\ &- \sum_{i=1}^{N_2}\tilde{k}e_i^{\mathrm{T}}(t)e_i(t) + \sum_{i=1}^{N_2}\rho e_i^{\mathrm{T}}(t)e_i(t), \end{aligned}$$

Due to

$$\sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)a_{ij}\boldsymbol{\Gamma}_1 e_j(t) = e^{\mathrm{T}}(t)A'\otimes\boldsymbol{\Gamma}_1 e(t) \leq \lambda_{\max}\left(Q_1^s\right)e^{\mathrm{T}}(t)e(t),$$

$$\sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)b_{ij}\boldsymbol{\Gamma}_2 e_j(t) = e^{\mathrm{T}}(t)B\otimes\boldsymbol{\Gamma}_2 e(t) \leq \lambda_{\max}\left(Q_2^s\right)e^{\mathrm{T}}(t)e(t),$$

$$\sum_{i=1}^{N_2}\sum_{j=1}^{N_2} e_i^{\mathrm{T}}(t)(c_{ij} + d_{ij})e_j(t) = e^{\mathrm{T}}(t)(C' + D')e(t) \leq \lambda_{\max}\left(Q_3^s\right)e^{\mathrm{T}}(t)e(t),$$

So, we can get

$$\begin{aligned} LV(e(t), t) \leq &\sum_{i=1}^{N_2} l e_i^{\mathrm{T}}(t)e_i(t) + \varepsilon_1\lambda_{\max}\left(Q_1^s\right)e^{\mathrm{T}}(t)e(t) \\ &+ \varepsilon_2\lambda_{\max}\left(Q_2^s\right)e^{\mathrm{T}}(t)e(t) - \mu\lambda_{\max}\left(Q_3^s\right)e^{\mathrm{T}}(t)e(t) \\ &- \sum_{i=1}^{N_2}\tilde{k}e_i^{\mathrm{T}}(t)e_i(t) + \sum_{i=1}^{N_2}\rho e_i^{\mathrm{T}}(t)e_i(t) \\ = &\Big[l + \varepsilon_1\lambda_{\max}\left(Q_1^s\right) + \varepsilon_2\lambda_{\max}\left(Q_2^s\right) - \mu\lambda_{\max}\left(Q_3^s\right) \\ &- \tilde{k} + \rho\Big]e^{\mathrm{T}}(t)e(t), \end{aligned}$$

where $e(t)=\left(e_1(t),e_2(t),\ldots,e_{N_2}(t)\right)^{\mathrm{T}}\in R^{nN_2}$, $Q_1=A'\otimes\boldsymbol{\Gamma}_1$, $Q_2=B\otimes\boldsymbol{\Gamma}_2$, $Q_3=C'+D'$, $Q_1^s=\dfrac{Q_1+Q_1^{\mathrm{T}}}{2}$, $Q_2^s=\dfrac{Q_2+Q_2^{\mathrm{T}}}{2}$, $Q_3^s=\dfrac{Q_3+Q_3^{\mathrm{T}}}{2}$, and A',C',D' are N_2 order principal minor determinant of matrixes A,C,D, respectively.

Obviously, there exist a sufficiently large positive constant $\tilde{k}$, such that

$$l+\varepsilon_1\lambda_{\max}\left(Q_1^s\right)+\varepsilon_2\lambda_{\max}\left(Q_2^s\right)-\mu\lambda_{\max}\left(Q_3^s\right)-\tilde{k}+\rho<0.$$

Let $k=\tilde{k}+\mu\lambda_{\max}\left(Q_3^s\right)-l-\varepsilon_1\lambda_{\max}\left(Q_1^s\right)-\varepsilon_2\lambda_{\max}\left(Q_2^s\right)-\rho$, then we obtain

$$LV\left(e(t),t\right)\le -ke^{\mathrm{T}}(t)e(t), \tag{12}$$

According to formula (12) and lemma 1, we can get

$$\begin{aligned} E\left\{V\left(e(t),t\right)\right\} &= E\left\{V\left(e(t_0),t_0\right)\right\}+E\left\{\int_{t_0}^{t}LV\left(e(\tau),\tau\right)d\tau\right\} \\ &\le E\left\{V\left(e(t_0),t_0\right)\right\}-kE\left\{\int_{t_0}^{t}e^{\mathrm{T}}(\tau),e(\tau)d\tau\right\}. \end{aligned} \tag{13}$$

By the definition of $V\left(e(t),t\right)$ in formula (11), there exist positive constants α_1, such that

$$V\left(e(t),t\right)\ge\alpha_1\sum_{i=1}^{N_2}e_i^{\mathrm{T}}(t)e_i(t), \tag{14}$$

And based on formula (13) we known that $V\left(e(t),t\right)$ and $\sum_{i=1}^{N_2}e_i^{\mathrm{T}}(t)e_i(t)$ is bounded, namely there exist positive constants α_2 such that

$$V\left(e(t),t\right)\le\alpha_2\sum_{i=1}^{N_2}e_i^{\mathrm{T}}(t)e_i(t), \tag{15}$$

Therefore, $V\left(e(t),t\right)$ satisfy

$$\alpha_1\sum_{i=1}^{N_2}e_i^{\mathrm{T}}(t)e_i(t)\le V\left(e(t),t\right)\le\alpha_2\sum_{i=1}^{N_2}e_i^{\mathrm{T}}(t)e_i(t), \tag{16}$$

and

$$LV\left(e(t),t\right)\le -k\sum_{i=1}^{N_2}e_i^{\mathrm{T}}(t)e_i(t). \tag{17}$$

So, from Lemma 1 we know that the error systems (10) are stable at $e(t)=0$ in the mean square sense. Thus, the networks (1) and (2) realized globally asymptotically synchronization in mean square.

III. A NEW BILAYER PUBLIC TRAFFIC COUPLED NETWORK MODEL

The urban public traffic network is the complex network composed of different bus stops and lines. There are mainly three modeling methods to establish the urban traffic network: Space L method, space P method and space R method [30, 31]. In this paper, a new bus-subway bilayer coupled public traffic network model is proposed, and the detailed modeling method of the new bilayer coupled public traffic network is described as follows:

(1) Firstly, take the conventional bus line and the subway line as the network's node, and then construct the sub-networks A and B based on the space R method.

(2) If there is an opportunity to transfer between conventional bus and subway, we link these two different types of nodes and constitute the coupling edges of bilayer coupled public traffic network. The coupling edges reflect the transfer relationship between subway and conventional bus. The conventional bus network, subway network and its coupling edges form the bilayer coupled public traffic network.

Without loss of generality, taking three subway lines (subway line 1, subway line 2 and subway line 3) and eight conventional bus lines (bus no. 4, 12, 19, 36, 102, 117, 181, 511) at Xi'an as the network nodes, we established a new bilayer coupled public traffic network model as show in Fig. 1.

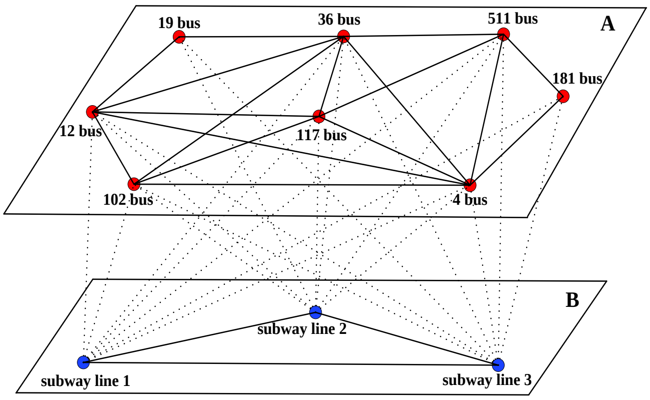

Fig. 1. The topology map of bilayer public traffic network model

IV. BALANCE ANALYZE OF BILAYER PUBLIC TRAFFIC NETWORK WITH STOCHASTIC DISTURBANCE

Next, the balance problem of bilayer coupled public traffic network is analyzed by using the synchronization theory of the coupled network with stochastic disturbance. Wu et al. [32] draws the conclusion that the passenger flow of urban public traffic fulfills the nonlinear properties. Assuming that the passenger flow of three subway lines and eight conventional bus lines all meet the nonlinear Lorenz system, and the nodes dynamical equations of the two sub-networks are

$$\begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{bmatrix}=\begin{bmatrix}-10&10&0\\28&-1&0\\0&0&-8/3\end{bmatrix}\begin{bmatrix}x_{i1}\\x_{i2}\\x_{i3}\end{bmatrix}+\begin{bmatrix}0\\-x_{i1}x_{i3}\\x_{i1}x_{i2}\end{bmatrix}, \tag{18}$$

$$\begin{bmatrix}\dot{y}_{i1}\\ \dot{y}_{i2}\\ \dot{y}_{i3}\end{bmatrix}=\begin{bmatrix}-10&10&0\\28&-1&0\\0&0&-8/3\end{bmatrix}\begin{bmatrix}y_{i1}\\y_{i2}\\y_{i3}\end{bmatrix}+\begin{bmatrix}0\\-y_{i1}y_{i3}\\y_{i1}y_{i2}\end{bmatrix}, \tag{19}$$

For the bilayer coupled public traffic network described in Fig. 1, we have

$a_{12}=1,\ a_{13}=0,\ a_{14}=1,\ a_{15}=1,\ a_{16}=1,\ a_{17}=1,\ a_{18}=1,$
$a_{23}=1,\ a_{24}=1,\ a_{25}=1,\ a_{26}=1,\ a_{27}=0,\ a_{28}=0,\ a_{34}=0,$
$a_{35}=0,\ a_{36}=0,\ a_{37}=0,\ a_{38}=0,\ a_{45}=1,\ a_{46}=1,\ a_{47}=0,$
$a_{48}=1,\ a_{56}=1,\ a_{57}=0,\ a_{58}=0,\ a_{67}=0,\ a_{68}=1,\ a_{78}=1,$
$a_{ji}=a_{ij}(i\ne j,i,j=1,2,\ldots,8),\ a_{ii}=-\sum_{i=1,i\ne j}^{8}a_{ij}(i,j=1,2,\ldots,8),$

$$b_{11}=-2,\ b_{12}=1,\ b_{13}=1,\ b_{21}=1,\ b_{22}=-2,\ b_{23}=1,$$
$$b_{31}=1,\ b_{32}=1,\ b_{23}=-2,$$

$c_{11}=1,\ c_{12}=1,\ c_{13}=1,\ c_{21}=1,\ c_{22}=1,\ c_{23}=1,\ c_{31}=0,$
$c_{32}=1,\ c_{33}=1,\ c_{41}=1,\ c_{42}=1,\ c_{43}=1,\ c_{51}=1,\ c_{52}=1,$
$c_{53}=1,\ c_{61}=1,\ c_{62}=1,\ c_{63}=0,\ c_{71}=1,\ c_{72}=0,\ c_{73}=1,$
$c_{81}=1,\ c_{82}=1,\ c_{83}=1,\ d_{ji}=c_{ij}(i=1,2,\dots,8,\ j=1,2,3)$

Assuming that $\boldsymbol{\Gamma}_1=\boldsymbol{\Gamma}_2=\operatorname{diag}\{1,1,1\}$ and the controllers designed as follows:

$$\begin{aligned} u_i(t)=&\ f(y_i(t))-g(y_i(t))+\varepsilon_1\sum_{j=1}^{3}a_{ij}y_j(t)\\ &-\varepsilon_2\sum_{j=1}^{3}b_{ij}x_j(t)+\mu\sum_{j=1}^{3}c_{ij}x_j(t)-\mu\sum_{j=1}^{3}d_{ij}y_j(t)\\ &+\varepsilon_1\sum_{j=4}^{8}a_{ij}x_j(t)-\mu\sum_{j=4}^{8}d_{ij}x_j(t)-k_i(t)e_i(t), \end{aligned} \tag{20}$$

where $\dot{k}_i(t)=d_i\|e_i\|^2$, $d_i(i=1,2,3)$ are the positive constants.

According to Eq. (1), the dynamical equation of each node $i(1\le i\le 8)$ in conventional bus network A is

$$dx_i(t)=\left[f(x_i(t))+\varepsilon_1\sum_{j=1}^{8}a_{ij}x_j(t)+\mu\sum_{j=1}^{3}c_{ij}y_j(t)\right]dt, \tag{21}$$

And from Eq. (2) we get the dynamical equation of each node $i(1\le i\le 3)$ in subway network B as follows

$$\begin{aligned} dy_i(t)=&\Big[g(y_i(t))+\varepsilon_2\sum_{j=1}^{3}b_{ij}y_j(t)+\mu\sum_{j=1}^{8}d_{ij}x_j(t)\\ &+u_i(t)\Big]dt+\delta_i(y_i(t)-x_i(t))d\omega_i(t), \end{aligned} \tag{22}$$

where

$$f(x_i(t))=\begin{bmatrix}-10&10&0\\28&-1&0\\0&0&-8/3\end{bmatrix}\begin{bmatrix}x_{i1}(t)\\x_{i2}(t)\\x_{i3}(t)\end{bmatrix}+\begin{bmatrix}0\\-x_{i1}(t)x_{i3}(t)\\x_{i1}(t)x_{i2}(t)\end{bmatrix},$$

$$g(y_i(t))=\begin{bmatrix}-10&10&0\\28&-1&0\\0&0&-8/3\end{bmatrix}\begin{bmatrix}y_{i1}(t)\\y_{i2}(t)\\y_{i3}(t)\end{bmatrix}+\begin{bmatrix}0\\-y_{i1}(t)y_{i3}(t)\\y_{i1}(t)y_{i2}(t)\end{bmatrix}.$$

For any vectors x_i and y_i of Lorenz system, there exist a positive constant R such that $\|x_{im}\|\le R,\|y_{im}\|\le R(m=1,2,3)$, since the Lorenz system is bounded in a certain region. So, we have

$$\begin{aligned} &\|f(y_i)-f(x_i)\|\\ &=\sqrt{(-y_{i1}y_{i3}-(-x_{i1}x_{i3}))^2+(y_{i1}y_{i2}-x_{i1}x_{i2})^2}\\ &=\sqrt{(-y_{i3}(y_{i1}-x_{i1})-x_{i1}(y_{i3}-x_{i3}))^2+(y_{i2}(y_{i1}-x_{i1})+x_{i1}(y_{i2}-x_{i2}))^2}\\ &\le\sqrt{2}R\|y_i-x_i\|, \end{aligned} \tag{23}$$

For the convenience of calculation, we let $\delta_i(e_i(t))=\sqrt{2\rho_0}e_i(t), i=1,2,\dots,N_2$. And we also assume that $\dot{\omega}(t)$ is one dimensional white noise, so $\delta_i(e_i(t))$ satisfy the Assumption 2. And form (27), Assumption 1 is established. According to Theorem 1, the conventional bus network A and subway network B achieved synchronization, that is, the whole bilayer coupled public traffic network is reached stable.

V. NUMERICAL SIMULATIONS

The synchronization effect of urban public traffic network is the dynamic balance between the running vehicle and the traveling passenger, that is, the operation time of the bus is most close to the preset time (the shortest time of traffic jam), and passengers stay at the bus station for the shortest time. In this paper, we mainly investigate the impact of stochastic disturbance (such as traffic accident, traffic signal and vehicle failures etc.) on the balance of bilayer coupled public traffic network. In numerical simulation processes, we select the initial value conditions as follows:

$$x_i(0)=(0.1+0.3i,0.2+0.3i,0.3+0.3i)^{\mathrm{T}},(1\le i\le 8),$$

$$y_i(0)=(2.5+0.3i,2.6+0.3i,2.7+0.3i)^{\mathrm{T}},(1\le i\le 3),$$

$$g_i(0)=3.6+0.1i,(1\le i\le 3).$$

Fixed $\varepsilon_1=\varepsilon_2=0.3,\mu=0.6,k_i=1,(1\le i\le 3)$, and we get the synchronization errors of the bilayer coupled public traffic network is shown in Fig. 2. As shown in Fig. 2, the bilayer coupled public traffic network achieves balance in 20 time units, namely, the operating vehicles and the travel passengers reach a dynamic balance.

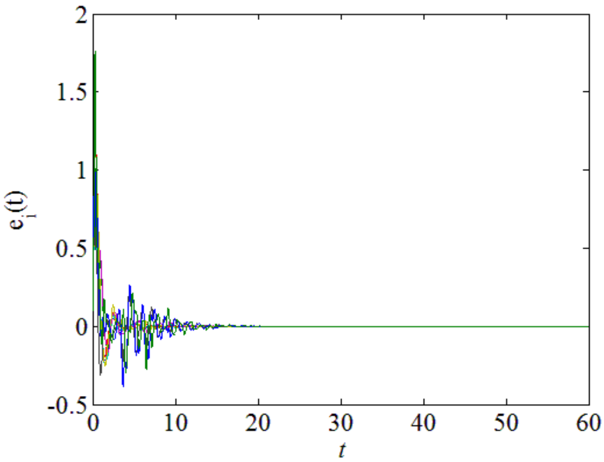

Fig. 2. Synchronization errors for bilayer coupled public traffic network

Next, let's consider the balance problem of the bilayer coupled public traffic network under random disturbances. Suppose that the bus no. 4 and 12 are inevitably running with the stochastic disturbance at $30\le t\le 40$ and the synchronization errors are given in Fig. 3, the bus no. 4 and subway line 1 are inevitably running with the stochastic disturbance at $30\le t\le 40$ and the synchronization errors are given in Fig. 4, and the Fig. 5 are the synchronization errors when the subway lines 1 and 3 are inevitably running with the stochastic disturbance at $30\le t\le 40$. According to Fig. 3, 4, 5 we can see that all the buses and subways are affected by stochastic disturbance, but the influence of stochastic disturbance on the whole networks is effectively suppressed when adding the controller. As seen in Fig. 3, the bilayer coupled public traffic network tends to stable at 44 time units when the stochastic disturbance is imposed on the conventional bus lines. When the stochastic disturbance is applied to both the conventional bus line and the subway line, the bilayer coupled public traffic network reach stable at 47 time units, as shown in Fig.4. According to Fig. 5, the bilayer coupled public traffic network tends to stable at 52 time units when the stochastic disturbance is imposed on the subway lines. That is to say, the stochastic disturbance of subway line caused by random events has a great influence on the stability of the bilayer coupled public traffic network, while the influence of the stochastic disturbance of the conventional bus line on the stability of the bilayer coupled public traffic network is relatively small.

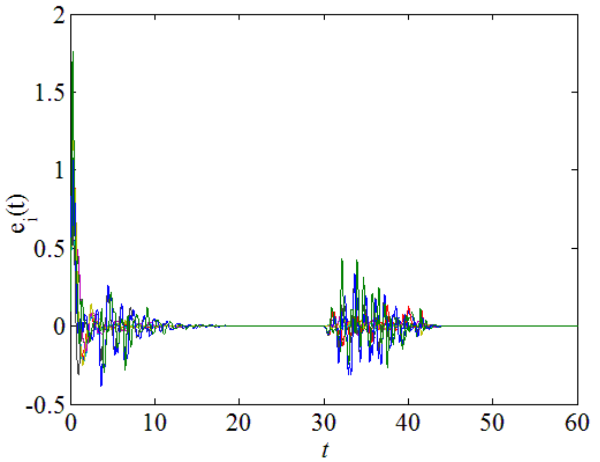

Fig. 3. Synchronization errors for bilayer coupled public traffic network after add stochastic disturbance to the conventional bus lines

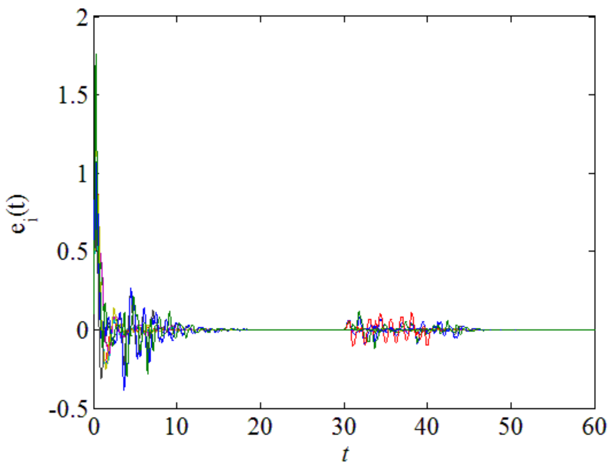

Fig. 4. Synchronization errors for bilayer coupled public traffic network after add stochastic disturbance to the conventional bus lines and subway lines

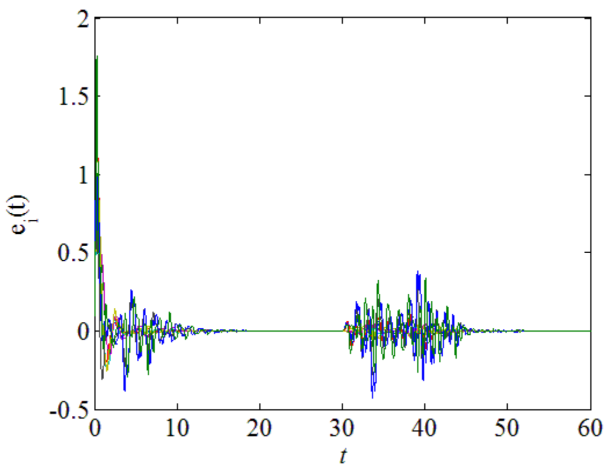

Fig. 5. Synchronization errors for bilayer coupled public traffic network after add stochastic disturbance to the subway lines

VI. Conclusion

In this paper, a new bilayer coupled public traffic network is proposed based on the space R modeling method. And in this network, the conventional bus network has a larger scale but the transmission performance of the network is poor, and the subway network has a small network size but the transmission performance is better. The two networks are coupled by the transfer relationship between some stations, and cooperate to complete the transmission task of the whole urban public traffic network, so that the passengers can complete the trip quickly and conveniently through the mixed traffic mode. Based on the synchronization theory of stochastic coupled network, the synchronization problem of bilayer coupled public traffic network under stochastic disturbance is studied. Through numerical simulation, the impact of stochastic disturbance, such as traffic accidents, traffic signals, vehicle failures and other random events on the balance of the bilayer coupled public traffic network are obtained.

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