Multiresolution Analysis for Linear Canonical Wavelet Transform

Yong Guo, Li-Dong Yang, Bing-Zhao Li

Abstract—Since linear canonical wavelet transform (LCWT) breaks through the limitation of wavelet transform in time-Fourier domain analysis, LCWT has become a useful mathematical tool in the applied mathematics, engineering and signal processing fields. The multi-resolution analysis (MRA) associated with LCWT can not only provides a method for constructing orthogonal wavelet associated LCWT, but also develops a theoretical basis for fast LCWT algorithm, and thus plays a key role for its prospective applications. In this paper, inspired by sampling theorem of band-limited signal in LCT domain, the MRA associated with LCWT is studied firstly. Moreover, the construction method of orthogonal wavelets for LCWT is developed. Finally, two examples of generalized orthogonal Haar and Shannon wavelets for LCWT are deduced.

Index Terms—linear canonical wavelet transform, multiresolution analysis, wavelet transform, linear canonical transform, linear canonical convolution.

I. INTRODUCTION

F OURIER transform (FT) and fractional Fourier transform (FRFT) are important tools in the applied mathematics, engineering and signal processing fields [1], [2]. As a generalization of FT and FRFT, linear canonical transform (LCT) is a three parameters family of linear integral transform [3], [4]. Since LCT has more degrees of freedom than FT and FRFT, it has been applied to many areas, such as signal separation, digital watermarking and filter design [5]–[10].

Owing to its global kernel, LCT is not capable of indicating the time localization of the LCT spectral components, and thus LCT is not suitable to process non-stationary signal whose LCT spectral characteristics change over time. The short time LCT is thus proposed to overcome this drawback [11]. Specifically, the original signal is firstly segmented by a time-localized window, and then performed LCT spectral analysis for each segment. STLCT is capable of offering a joint signal representation in both time and LCT domains, but its fixed window width limits the practical applications , it is impossible to provide good time resolution and spectral resolution simultaneously. Because of this, it is desirable to propose a novel approaches whose window can be adjusted.

The linear canonical wavelet transform (LCWT) is an efficient tool to analyze time-varying LCT spectra. As a

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generalization of wavelet transform in LCT domain, LCWT can represent adaptively signal in both time and LCT domains owing to its adjustable window. Therefore, LCWT not only breaks through the limitation of WT in time-Fourier domain analysis, but also overcomes the limitation of LCT in indicating the signal's local characteristics [12]. LCWT successfully inherits the advantages of multiresolution analysis (MRA) for WT. The MRA and the construction of orthogonal wavelets associated with LCWT serves a crucial role for its perspective applications. Thus, it is necessary to detect the MRA and the construction of orthogonal wavelets associated with LCWT. In this paper, we first investigate the MRA associated with LCWT. In further, the construction of orthogonal wavelet for LCWT is developed and two examples of orthogonal wavelet for LCWT are presented.

The remaining sections of this paper are organized as follows: the preliminaries are summarized in section II. In section III, the definition and physical explanation of LCWT are presented. Furthermore, the MRA and the construction of orthogonal wavelet associated with LCWT are investigated in Section IV and In Section V respectively. In section VI, two examples of orthogonal wavelet for LCWT are presented. The conclusion is concluded in Section VII.

II. PRELIMINARIES

A. Linear canonical transform

The linear canonical transform (LCT) with real matrix A_1 of f(t) is defined as [4]

$$L_f^{A_1}(u) = \mathscr{L}^{A_1}[f(t)](u) = \int_R f(t) K_{A_1}(t, u) dt$$
(1)

with the kernel

$$K_{A_1}(t,u) = \begin{cases} \frac{1}{\sqrt{i2\pi b_1}} e^{\frac{i(a_1t^2 + d_1u^2 - 2tu)}{2b_1}}, & b_1 \neq 0\\ \sqrt{d_1} e^{\frac{ic_1d_1u^2}{2}} \delta(t - d_1u), & b_1 = 0 \end{cases}$$
(2)

where $A_1 = (a_1, b_1, c_1, d_1)$ satisfying $a_1d_1 - b_1c_1 = 1$. $L_f^{A_1}(u)$ and \mathscr{L}^{A_1} denote the LCT of f(t) and the LCT operator respectively. When $b_1 \neq 0$, the inverse LCT is given by

$$f(t) = \mathscr{L}^{A_1^{-1}}[L_f^{A_1}(u)](t) = \int_R L_f^{A_1}(u) K_{A_1}^*(t, u) du$$
(3)

where the kernel $K_{A_1}^*(t,u) = K_{A_1^{-1}}(t,u)$ and A_1^{-1} denotes the inverse matrix of A_1 .

When $A_1 = (0, 1, -1, 0)$, LCT reduces to Fourier transform (FT). The relationship between LCT and FT is shown as follows [18]:

$$L_{f}^{A_{1}}(u) = \mathscr{L}^{A_{1}}[f(t)](u) = \frac{1}{\sqrt{ib_{1}}}e^{\frac{id_{1}}{2b_{1}}u^{2}}\mathscr{F}[f(t)e^{\frac{ia_{1}}{2b_{1}}t^{2}}](u/b_{1})$$
(4)

Lemma 1 The discrete time LCT (DTLCT) of a sequence $f_n \in \ell^2(Z)$ has the chirp periodicity [19], i.e.,

$$\tilde{L}_{f_n}^{A_1}(u+2k\pi b_1)e^{-\frac{i}{2}\frac{d_1}{b_1}(u+2k\pi b_1)^2} = \tilde{L}_{f_n}^{A_1}(u)e^{-\frac{i}{2}\frac{d_1}{b_1}u^2}$$
(5)

where $\tilde{L}_{f_n}^{A_1}(u)$ denotes the DTLCT of f_n , defined as

$$\tilde{L}_{f_n}^{A_1}(u) = \tilde{\mathscr{Z}}^{A_1}[f_n](u) = \sum_{n \in \mathbb{Z}} f_n K_{A_1}(n, u)$$
(6)

Lemma 2 The Parseval identity associated with LCT is given by [4]

$$\int_{R} f(t)g^{*}(t)dt = \int_{R} L_{f}^{A_{1}}(u)[L_{g}^{A_{1}}(u)]^{*}du$$
(7)

In particularly, if f(t) = g(t), then

$$\int_{R} |f(t)|^{2} dt = \int_{R} |L_{f}^{A_{1}}(u)|^{2} du$$
(8)

B. Linear canonical convolution

The linear canonical convolution of f(t) and g(t) is given by [18]

$$f(t)\Theta_{A_1}g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)e^{-\frac{ia_1}{b_1}\tau(t-\tau/2)}d\tau \qquad (9)$$

where Θ_{A_1} denotes the linear canonical convolution operator. **Lemma 3** The convolution theorem associated with LCT is given by [18]

$$\mathscr{L}^{A_1}[f(t)\Theta_{A_1}g(t)](u) = \sqrt{2\pi}L_f^{A_1}(u)F_g(u/b_1)$$
(10)

where $L_f^{A_1}(u)$ denotes the LCT of f(t), $F_g(u/b_1)$ denotes the FT of g(t) with its argument scaled by $1/b_1$.

III. LINEAR CANONICAL WAVELET TRANSFORM

The LCWT with real matrix parameter A_1 of f(t) is defined as [12]

$$W_{f}^{A_{1}}(a,b) = \mathscr{W}^{A_{1}}[f(t)](a,b) = \int_{R} f(t)\psi_{a,b,A_{1}}^{*}(t)dt$$

$$= \frac{1}{\sqrt{a}}\int_{R} f(t)\psi^{*}(\frac{t-b}{a})e^{\frac{i}{2}(t^{2}-b^{2})\cdot\frac{a_{1}}{b_{1}}}dt$$
(11)

where

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$$\psi_{a,b,A_1}(t) = \frac{1}{\sqrt{a}} \psi(\frac{t-b}{a}) e^{-\frac{i}{2}(t^2 - b^2) \cdot \frac{a_1}{b_1}},$$
 (12)

 $\psi(\frac{t-b}{a})(a \in R^+, b \in R)$ is the continuous wavelet function. When $A_1 = (0, 1, -1, 0)$, LCWT reduces to the classical WT. When $A_1 = (\cos \alpha, \sin \alpha, -\sin \alpha, \cos \alpha)$, LCWT reduces to the fractional wavelet transform (FRWT) [13]–[17].

Based on the convolution theorem associated with LCT (see Lemma 3), LCWT can be rewritten as [12]

$$W_f^{A_1}(a,b) = f(t)\Theta_{A_1}\left[\frac{1}{\sqrt{a}}\psi^*(-\frac{t}{a})\right]$$
 (13)

Hence, LCWT can be viewed as a linear canonical convolution of signal with the conjugate of mother wavelet function after scale expansion and time reversal in the time domain.

Then, from Eq. (10) and Eq. (13), LCWT can be expressed in term of the LCT, i.e.,

$$W_f^{A_1}(a,b) = \int_R \sqrt{2\pi a} L_f^{A_1}(u) F_{\psi}^*(au/b_1) K_{A_1}^*(b,u) du \quad (14)$$

The Eq.(14) states that each linear canonical wavelet component can be viewed as a scaled bandpass filter in the LCT domain, and thus the multiplication of $L_f^{A_1}(u)$ and $F_{\psi}^*(au/b_1)$ can provide the local properties of f(t) in the LCT domain. This implies that LCWT can breaks through the limitation of WT in the time-Fourier domain analysis, represents adaptively signal in the time-LCT domain by its adjustable analysis window.

IV. MULTIRESOLUTION ANALYSIS ASSOCIATED WITH LCWT

As a generalization of wavelet transform in the LCT domain, LCWT successfully inherits the advantages of MRA for wavelet transform. In this section, inspired by the sampling theorem of band-limited signal in LCT domain, the MRA associated with LCWT is established.

The sampling theorem of band-limited signal associated with LCT is shown as [20]

$$f(t) = \sum_{n \in \mathbb{Z}} f(nT_s) \operatorname{sinc} \left[\frac{\Omega_{A_1}(t - nT_s)}{b_1 \pi} \right] e^{-\frac{i}{2} (t^2 - (nT_s)^2) \frac{a_1}{b_1}} \quad (15)$$

where T_s is the sampling period and f(t) is band-limited signal in the LCT domain (i.e., $L_f^{A_1}(u) = 0$ when $|u| > \Omega_{A_1}$). The Eq. (15) shows that band-limited signal f(t) can be complete recovered from the sampled values $f(nT_s)$ when $0 < T_s \le \frac{\pi b_1}{\Omega_{A_1}}$.

When $\Omega_{A_1}^{n_1} = b_1 \pi$, the set of band-limited signal in LCT domain is denoted as $V_0^{A_1}$, i.e.,

$$V_0^{A_1} = \{ f(t) | L_f^{A_1}(u) = 0, |u| \ge \Omega_{A_1} = b_1 \pi \}$$
(16)

where sampling period $T_s = 1$. Therefore, according to Eq. (15), $\forall f(t) \in V_0^{A_1}$ can be expressed as

$$f(t) = \sum_{n \in \mathbb{Z}} f(n)\phi_{A_1,0,n}.$$
(17)

where

$$\phi_{A_1,0,n} = sinc(t-n)e^{-\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}$$
(18)

Combined with the orthogonality of $\{\phi_{A_1,0,n}\}_{n\in\mathbb{Z}}$, we can further obtain that $\{\phi_{A_1,0,n} = sinc(t-n)e^{-\frac{1}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ forms a standard orthonormal basis of $V_0^{A_1}$.

When $\Omega_{A_1} = 2b_1\pi$, $T_s = 1/2$, the set of band-limited signal in LCT domain is denoted as $V_1^{A_1}$, i.e.,

$$V_1^{A_1} = \{ f(t) | L_f^{A_1}(u) = 0, |u| \ge \Omega_{A_1} = 2b_1 \pi \}$$
(19)

Therefore, according to Eq. (15), $V_1^{A_1} \subseteq V_0^{A_1}$ and $\forall f(t) \in V_1^{A_1}$ can be expressed as

$$f(t) = \sum_{n \in \mathbb{Z}} f(n)\phi_{A_1,1,n}$$
(20)

where

$$\phi_{A_1,1,n} = 2^{\frac{1}{2}} sinc(2t-n)e^{-\frac{i}{2}[t^2 - (\frac{n}{2})^2]\frac{a_1}{b_1}}$$
(21)

It can also be further obtained that $\{\phi_{A_1,1,n} = 2^{\frac{1}{2}} sinc(2t - n)e^{-\frac{i}{2}[t^2 - (\frac{n}{2})^2]\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$ forms a standard orthonormal basis of $V_1^{A_1}$. Since

$$\mathscr{L}^{A_1}[f(2t)e^{\frac{i}{2}[(2t)^2-t^2]\frac{a_1}{b_1}}](u) = \frac{1}{2}e^{\frac{3i}{8}u^2\frac{a_1}{b_1}}\mathscr{L}^{A_1}[f(t)](\frac{u}{2}) \quad (22)$$

we have that if $f(t) \in V_0^{A_1}$, then

$$f(2t)e^{\frac{i}{2}[(2t)^2 - t^2]\frac{a_1}{b_1}} \in V_1^{A_1}$$
(23)

Generally, let
$$\Omega_{A_1} = 2^k \pi b_1, T_s = 1/2^k,$$

 $V_k^{A_1} = \{f(t) | L_f^{A_1}(u) = 0, |u| \ge 2^k b_1 \pi\}.$ (24)

For $\forall f(t) \in V_k^{A_1}$, we have

$$f(t) = \sum_{n \in \mathbb{Z}} f(\frac{n}{2^k}) 2^{\frac{k}{2}} sinc[(2^k t - n)] e^{-\frac{i}{2} [t^2 - (\frac{n}{2^k})^2] \frac{a_1}{b_1}}$$
(25)

$$=\sum_{n\in\mathbb{Z}}f(\frac{n}{2^{k}})\phi_{A_{1},k,n}$$
(26)

where

$$\phi_{A_1,k,n} = 2^{\frac{k}{2}} sinc[(2^k t - n)] e^{-\frac{i}{2}[t^2 - (\frac{n}{2^k})^2]\frac{a_1}{b_1}}$$
(27)

 $\{\phi_{A_1,k,n}\}_{n\in\mathbb{Z}}$ forms a standard orthonormal basis of $V_k^{A_1}$. For $\forall k \in \mathbb{Z}$, we have

(1)
$$V_k^{A_1} \subseteq V_{k+1}^{A_1};$$

(2) $f(t) \in V_k^{A_1} \Leftrightarrow f(2t)e^{\frac{i}{2}[(2t)^2 - t^2]\frac{a_1}{b_1}} \in V_{k+1}^{A_1};$
(3) $\bigcap_{k \in \mathbb{Z}} V_k^{A_1} = \{0\}, \overline{\bigcup_{k \in \mathbb{Z}} V_k^{A_1}} = L^2(R);$

In summary, the sampling theorem of band-limited signal in LCT domain inspires us to define an orthogonal MRA associated with LCWT.

Definition 1 An orthogonal MRA associated with LCWT is defined as a sequence of closed subspace $V_k^{A_1} \subseteq L^2(R)$ $(k \in$ Z) such that

(1)
$$V_{k}^{A_{1}} \subseteq V_{k+1}^{A_{1}}, \forall k \in \mathbb{Z};$$

(2) $f(t) \in V_{k}^{A_{1}} \Leftrightarrow f(2t)e^{\frac{i}{2}[(2t)^{2}-t^{2}]\frac{a_{1}}{b_{1}}} \in V_{k+1}^{A_{1}}, \forall k \in \mathbb{Z};$
(3) $\bigcap_{k \in \mathbb{Z}} V_{k}^{A_{1}} = \{0\}, \overline{\bigcup_{k \in \mathbb{Z}} V_{k}^{A_{1}}} = L^{2}(\mathbb{R});$
(4) There exists function $A(t) \in L^{2}(\mathbb{R})$ such

(4) There exists a function $\phi(t) \in L^2(R)$ such that $\{\phi_{A_1,0,n} = \phi(t-n)e^{-\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the subspace $V_0^{A_1}$, where $\phi(t)$ is called scaling function of the given MRA for LCWT.

Theorem 1 Assume $\{V_k^{A_1}\}_{k\in\mathbb{Z}}$ is a orthogonal MRA associated with LCWT, $\phi(t)$ is the corresponding scaling function. For any $k \in Z$, these functions

$$\{\phi_{A_1,k,n}(t) = 2^{\frac{k}{2}}\phi(2^kt - n)e^{-\frac{i}{2}(t^2 - (\frac{n}{2^k})^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$$
(28)

form an orthonormal basis of the subspace $V_k^{A_1}$.

Proof see Appendix A.

Theorem 2 Assume $\varphi(t) \in L^2(R)$ and $V_0^{A_1} = \overline{\text{span}}\{\varphi_{A_1,0,n}(t) = \phi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$, then the set of functions $\{\varphi_{A_1,0,n} = \varphi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$ is a Riesz basis of $V_k^{A_1}$ if and only if there exist constants $0 < A \le B < +\infty$ such that

$$A \le \sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1 + 2k\pi)|^2 \le B, \forall u \in [0, 2\pi b_1].$$
(29)

Proof see Appendix B.

Theorem 3 Assume $\{V_k^{A_1}\}_{k\in\mathbb{Z}}$ is a generalized MRA of $L^2(R)$ associated with LCWT, which is generated by $\varphi(t)$. Let

$$F_{\phi}(u/b_1) = \frac{F_{\phi}(u/b_1)}{\sqrt{\sum_{k \in \mathbb{Z}} |F_{\phi}(u/b_1 + 2k\pi)|^2}},$$
 (30)

Proof see Appendix C.

V. CONSTRUCTION OF ORTHOGONAL WAVELETS FOR LCWT

The orthogonal wavelets for LCWT can be constructed based on the developed MRA associated with LCWT. The subspace $W_k^{A_1}$ is defined as the orthogonal complement of $V_k^{A_1}$ in $V_{k+1}^{A_1}$, i.e.,

$$W_k^{A_1} \perp V_k^{A_1}, \quad V_{k+1}^{A_1} = W_k^{A_1} \oplus V_k^{A_1}, \quad \forall k \in \mathbb{Z}$$
 (31)

Then, according to **Definition 1**, it can be obtained that $\{W_k^{A_1}\}_{k\in\mathbb{Z}}$ possess the following properties:

(1) $W_k^{A_1} \perp V_l^{A_1}, \forall k \neq l;$ (2) $\bigoplus_{k \in \mathbb{Z}} W_k^{A_1} = L^2(\mathbb{R});$

(3) $g(t) \in W_k^{A_1} \Leftrightarrow g(2t)e^{\frac{i}{2}[(2t)^2-t^2]\frac{a_1}{b_1}} \in W_{k+1}^{A_1}, \forall k \in \mathbb{Z}.$

The property (2) means that an orthonormal basis of $L^2(R)$ can be constructed by finding out an orthonormal basis of the subspace $W_{l}^{A_1}$. The property (3) implies that a MRA construction of $L^2(R)$ related to LCWT can be changed to construct the orthonormal basis of $W_0^{A_1}$. Therefore, the crucial point is to construct a function $\psi(t) \in L^2(\mathbb{R})$, such that the set of functions $\{\psi_{A_1,0,n} = \psi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{\dot{a}_1}{b_1}}\}_{n\in\mathbb{Z}}$

forms an orthonormal basis of $W_0^{A_1}$. Since $\{\phi_{A_1,1,n} = 2^{\frac{1}{2}}\phi(2t-n)e^{-\frac{i}{2}(t^2-(\frac{n}{2})^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ form an orthonormal basis of the subspace $V_1^{A_1}$ and $\phi_{A_1,0,0} = \phi(t)e^{-\frac{i}{2}t^2\frac{a_1}{b_1}} \in V_0^{A_1} \subseteq V_1^{A_1}$, there exist $\{h_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\phi_{A_1,0,0} = \phi(t)e^{-\frac{t}{2}t^2\frac{\theta_1}{b_1}} = \sum_{n \in \mathbb{Z}} h_n \phi_{A_1,1,n}(t)$$
(32)

where

$$h_n = \sqrt{2}e^{\frac{i}{2}\frac{n^2}{4}\frac{a_1}{b_1}} \int_{\mathbb{R}} \phi(t)\phi(2t-n)dt.$$
(33)

Taking LCT on both sides of above equation, then

$$\begin{aligned} \mathscr{L}^{A_{1}}[\phi(t)e^{-\frac{i}{2}t^{2}\frac{a_{1}}{b_{1}}}](u) \\ &= \mathscr{L}^{A_{1}}[\sum_{n\in\mathbb{Z}}h_{n}2^{\frac{1}{2}}\phi(2t-n)e^{-\frac{i}{2}(t^{2}-(\frac{n}{2})^{2})\frac{a_{1}}{b_{1}}}](u) \\ &\Longrightarrow \int_{R}A_{b_{1}}\phi(t)e^{-\frac{i}{2}t^{2}\frac{a_{1}}{b_{1}}}e^{\frac{i}{2}\frac{a_{1}t^{2}+d_{1}u^{2}-2tu}{b_{1}}}dt \\ &= \int_{R}A_{b_{1}}\sum_{n\in\mathbb{Z}}h_{n}2^{\frac{1}{2}}\phi(2t-n)e^{-\frac{i}{2}(t^{2}-(\frac{n}{2})^{2})\frac{a_{1}}{b_{1}}}e^{\frac{i}{2}\frac{a_{1}t^{2}+d_{1}u^{2}-2tu}{b_{1}}}dt \\ &\Longrightarrow \int_{R}\phi(t)e^{-itu/b_{1}}dt = \int_{R}\sum_{n\in\mathbb{Z}}h_{n}2^{\frac{1}{2}}\phi(2t-n)e^{\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{a^{2}}{a^{2}}-itu/b_{1}}dt \\ &\stackrel{\xi=2t-n}{\Longrightarrow}F_{\phi}(\frac{u}{b_{1}}) = \frac{1}{\sqrt{2}}\int_{R}\sum_{n\in\mathbb{Z}}h_{n}\phi(\xi)e^{\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{a^{2}}{a^{2}}-i(\frac{\xi+n}{2})u/b_{1}}d\xi \\ &\Longrightarrow F_{\phi}(\frac{u}{b_{1}}) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}h_{n}e^{\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{a^{2}}{a^{2}}-i\frac{2u}{2b_{1}}}\int_{R}\phi(\xi)e^{-i\frac{\xiu}{2b_{1}}}d\xi \\ &\Longrightarrow F_{\phi}(\frac{u}{b_{1}}) = \Lambda(\frac{u}{2b_{1}})F_{\phi}(\frac{u}{2b_{1}}) \end{aligned}$$
where

$$\Lambda(\frac{u}{b_1}) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{\frac{i}{2} \frac{a_1}{b_1} \frac{n^2}{4} - i \frac{nu}{b_1}}.$$
 (34)

 $\sqrt{\sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1 + 2k\pi)|^2},$ (30) It can be verified that $\Lambda(u/b_1)$ is a $2\pi b_1$ period function. then, the set of functions $\{\phi_{A_1,0,n}(t) = \phi(t - S_1)\}_{n \in \mathbb{Z}}$ forms an orthonormal basis of $V_0^{A_1}$. Broof are $\{\phi_{A_1,0,n} = \phi(t-n)e^{-\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$ is an orthonormal basis of the subspace $V_0^{A_1}$, according to **Theorem 2**, we have

$$\sum_{k \in \mathbb{Z}} |F_{\phi}(u/b_1 + 2k\pi)|^2 = 1.$$
(35)

Moreover, we have

$$\begin{split} &\sum_{k \in \mathbb{Z}} |F_{\phi}(u/b_{1}+2k\pi)|^{2} \\ &= \sum_{k \in \mathbb{Z}} |\Lambda(\frac{u}{2b_{1}}+k\pi)F_{\phi}(\frac{u}{2b_{1}}+k\pi)|^{2} \\ &= \sum_{l \in \mathbb{Z}} |\Lambda(\frac{u}{2b_{1}}+2l\pi)F_{\phi}(\frac{u}{2b_{1}}+2l\pi)|^{2} + \\ &\sum_{l \in \mathbb{Z}} |\Lambda(\frac{u}{2b_{1}}+2l\pi+\pi)F_{\phi}(\frac{u}{2b_{1}}+2l\pi+\pi)|^{2} \\ &= \sum_{l \in \mathbb{Z}} |\Lambda(\frac{u}{2b_{1}})|^{2}|F_{\phi}(\frac{u}{2b_{1}}+2l\pi)|^{2} + \\ &\sum_{l \in \mathbb{Z}} |\Lambda(\frac{u}{2b_{1}}+\pi)|^{2}|F_{\phi}(\frac{u}{2b_{1}}+2l\pi+\pi)|^{2} \\ &= |\Lambda(\frac{u}{2b_{1}})|^{2} + |\Lambda(\frac{u}{2b_{1}}+\pi)|^{2} \end{split}$$
(36)

It can be deduced that

$$|\Lambda(\frac{u}{b_1})|^2 + |\Lambda(\frac{u}{b_1} + \pi)|^2 = 1$$
(37)

Since $\psi_{A_1,0,0}(t) = \psi(t)e^{-\frac{i}{2}t^2\frac{a_1}{b_1}} \in W_0^{A_1} \subseteq V_1^{A_1}$, there exist $\{g_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\psi_{A_1,0,0} = \psi(t)e^{-\frac{i}{2}t^2\frac{a_1}{b_1}} = \sum_{n \in \mathbb{Z}} g_n \phi_{A_1,1,n}(t)$$
(38)

Taking LCT on both sides of above equation, then

$$F_{\Psi}(\frac{u}{b_1}) = \Gamma(\frac{u}{2b_1})F_{\phi}(\frac{u}{2b_1}),$$
(39)

where

$$\Gamma(\frac{u}{b_1}) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} g_n e^{\frac{i}{2} \frac{a_1}{b_1} \frac{n^2}{4} - i \frac{nu}{b_1}}.$$
 (40)

To make the set of functions $\{\psi_{A_1,0,n} = \psi(t - \psi_{A_1,0,n})\}$ $n e^{\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}$ and n = 0 orthonormal basis of $W_0^{A_1}$, then

$$\sum_{k\in\mathbb{Z}} |F_{\Psi}(\frac{u}{b_1} + 2k\pi)|^2 = 1.$$
(41)

Similar to the derivation of Eq. (36), we have

$$|\Gamma(\frac{u}{b_1})|^2 + |\Gamma(\frac{u}{b_1} + \pi)|^2 = 1$$
(42)

Moreover, since $W_0^{A_1}$ and $V_0^{A_1}$ are orthogonal in $V_1^{A_1}$, then

$$\langle \phi_{A_1,0,m}(t), \psi_{A_1,0,n}(t) \rangle_{L^2} = 0, \ \forall \ m, n \in \mathbb{Z}$$
 (43)

It can be easily deduced the following two results based on the similar derivation of Theorem 2, i.e.,

$$\mathscr{L}^{A_1}[\phi_{A_1,0,m}(t)](u) = \sqrt{2\pi} K_{A_1}(m,u) F_{\phi}(u/b_1)$$
(44)

$$\mathscr{L}^{A_1}[\psi_{A_1,0,n}(t)](u) = \sqrt{2\pi K_{A_1}(n,u)}F_{\psi}(u/b_1)$$
(45)

By using Eq. (44), Eq. (45) and Eq. (8), we have the Eq. (46). Because $\left\{\frac{1}{\sqrt{2\pi}}e^{-inu/b_1}\right\}$ is a orthonormal basis of $L^2[0, 2\pi b_1]$, then

$$\Lambda(\frac{u}{2b_1})\Gamma^*(\frac{u}{2b_1}) + \Lambda(\frac{u}{2b_1} + \pi)\Gamma^*(\frac{u}{2b_1} + \pi) = 0 \quad a.e. \ \forall u \in R$$
(47)

Overall, the following theorem has been proved. Theorem 4 Let

$$\Psi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_n \phi(2t - n) e^{\frac{i}{2} \frac{n^2}{4} \frac{a_1}{b_1}}$$
(48)

then, the set of functions $\{\psi_{A_1,0,n}(t)\}_{n\in\mathbb{Z}}$ is an orthogonal basis of $W_0^{A_1}$ if and only if $\mathbf{M}(u/b_1)$ is a unitary matrix, i.e.,

$$\mathbf{M}(u/b_1)\mathbf{M}^*(u/b_1) = \mathbf{I}, \ a.e. \ \forall u \in R$$
(49)

where $\mathbf{M}^*(u/b_1)$ denotes the conjugate transpose of $\mathbf{M}(u/b_1)$, I denotes the identity matrix and

$$\mathbf{M}(u/b_1) = \begin{pmatrix} \Lambda(u/b_1) & \Lambda(u/b_1 + \pi) \\ \Gamma(u/b_1) & \Gamma(u/b_1 + \pi) \end{pmatrix}$$
(50)

If Eq. (49) holds, then there exists a function $\lambda(u)$ such that

$$(\Gamma^{*}(\frac{u}{b_{1}}),\Gamma^{*}(\frac{u}{b_{1}}+\pi)) = (\lambda(\frac{u}{b_{1}})\Lambda(\frac{u}{b_{1}}+\pi),-\lambda(\frac{u}{b_{1}})\Lambda(\frac{u}{b_{1}}))$$
(51)

Since $\Gamma^*(u/b_1)$ and $\Lambda(u/b_1 + \pi)$ are $2\pi b_1$ periodic function, $\lambda(u/b_1)$ is a $2\pi b_1$ periodic function. Therefore, $\lambda(u/b_1)$ can be expanded as a Fourier series, i.e.,

$$\lambda(u/b_1) = \sum_{k \in \mathbb{Z}} c_k e^{iku/b_1},$$
(52)

where

$$c_{k} = \frac{1}{2\pi b_{1}} \int_{0}^{2\pi b_{1}} \lambda(u) e^{-iku/b_{1}} du$$

= $\frac{1}{2\pi b_{1}} [\int_{0}^{\pi b_{1}} \lambda(u) e^{-iku/b_{1}} du + \int_{\pi b_{1}}^{2\pi b_{1}} \lambda(u) e^{-iku/b_{1}} du]$
= $\frac{1 - (-1)^{k}}{2\pi b_{1}} \int_{0}^{\pi b_{1}} \lambda(u) e^{-iku/b_{1}} du$

Hence, $\lambda(u/b_1)$ can be rewritten as

$$\begin{split} \lambda(\frac{u}{b_1}) &= \sum_{l \in \mathbb{Z}} c_{2l+1} e^{i(2l+1)u/b_1} \\ &= e^{iu/b_1} \sum_{l \in \mathbb{Z}} c_{2l+1} e^{i2lu/b_1} \\ &= e^{iu/b_1} \gamma(\frac{2u}{b_1}) \end{split}$$

where $\gamma(u/b_1) = \sum_{l \in Z} c_{2l+1} e^{ilu/b_1}$. Moreover, followed by the Eq. (40), we have

$$\begin{split} \Lambda^*(\frac{u}{b_1} + \pi) &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n^* e^{-\frac{i}{2} \frac{a_1}{b_1} \frac{n^2}{4} + i \frac{n(u+b_1\pi)}{b_1}} \\ &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} (-1)^n h_n^* e^{-\frac{i}{2} \frac{a_1}{b_1} \frac{n^2}{4} + i \frac{nu}{b_1}} \end{split}$$

Thus, let $\gamma^*(\frac{2u}{b_1}) = 1$, then

$$\begin{split} &\Gamma(\frac{u}{b_{1}}) = \lambda^{*}(\frac{u}{b_{1}})\Lambda^{*}(\frac{u}{b_{1}} + \pi) \\ &= e^{-iu/b_{1}}\gamma^{*}(\frac{2u}{b_{1}})\frac{1}{\sqrt{2}}\sum_{n\in Z}(-1)^{n}h_{n}^{*}e^{-\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{n^{2}}{4} + i\frac{nu}{b_{1}}} \\ &\Longrightarrow \frac{1}{\sqrt{2}}\sum_{n\in Z}g_{n}e^{\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{n^{2}}{4} - i\frac{nu}{b_{1}}} = \frac{e^{-iu/b_{1}}}{\sqrt{2}}\sum_{n\in Z}(-1)^{n}h_{n}^{*}e^{-\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{n^{2}}{4} + i\frac{nu}{b_{1}}} \\ &\Longrightarrow g_{n}e^{\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{n^{2}}{4}} = \int_{R}e^{-iu/b_{1}}\sum_{m\in Z}(-1)^{m}h_{m}^{*}e^{-\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{m^{2}}{4} + i\frac{nu}{b_{1}}}e^{i\frac{nu}{b_{1}}}du \\ &\Longrightarrow g_{n}e^{\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{n^{2}}{4}} = \sum_{m\in Z}(-1)^{m}h_{m}^{*}e^{-\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{m^{2}}{4}}\int_{R}e^{i(n+m-1)\frac{u}{b_{1}}}du \\ &\Longrightarrow g_{n}e^{\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{n^{2}}{4}} = (-1)^{1-n}h_{1-n}^{*}e^{-\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{(1-n)^{2}}{4}}e^{-\frac{i}{2}\frac{a_{1}}{b_{1}}\frac{n^{2}}{4}} \end{split}$$

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$$\begin{split} \langle \phi_{A_{1},0,m}(t),\psi_{A_{1},0,n}(t)\rangle_{L^{2}} &= \langle \mathscr{L}^{A_{1}}[\phi_{A_{1},0,m}(t)](u),\mathscr{L}^{A_{1}}[\psi_{A_{1},0,n}(t)](u)\rangle_{L^{2}} \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{R} F_{\phi}(u/b_{1})F_{\psi}^{*}(u/b_{1})e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{R} \Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}})|F_{\phi}(\frac{u}{2b_{1}})|^{2}e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \sum_{k\in\mathbb{Z}} \int_{4k\pi b_{1}}^{4(k+1)\pi b_{1}} \Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}}) \cdot |F_{\phi}(\frac{u}{2b_{1}}+2k\pi)|^{2}e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{4k\pi b_{1}}^{4(k+1)\pi b_{1}} \Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}}) \cdot \sum_{k\in\mathbb{Z}} |F_{\phi}(\frac{u}{2b_{1}}+2k\pi)|^{2}e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{0}^{4\pi b_{1}} \Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}})e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{0}^{2\pi b_{1}} \left[\Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}}) + \Lambda(\frac{u}{2b_{1}}+\pi)\Gamma^{*}(\frac{u}{2b_{1}}+\pi)\right]e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{0}^{2\pi b_{1}} \left[\Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}}) + \Lambda(\frac{u}{2b_{1}}+\pi)\Gamma^{*}(\frac{u}{2b_{1}}+\pi)\right]e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{0}^{2\pi b_{1}} \left[\Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}}) + \Lambda(\frac{u}{2b_{1}}+\pi)\Gamma^{*}(\frac{u}{2b_{1}}+\pi)\right]e^{-i(m-n)u/b_{1}}du \\ &= \frac{1}{b_{1}}e^{\frac{i}{2}(m^{2}-n^{2})\frac{a_{1}}{b_{1}}} \int_{0}^{2\pi b_{1}} \left[\Lambda(\frac{u}{2b_{1}})\Gamma^{*}(\frac{u}{2b_{1}}) + \Lambda(\frac{u}{2b_{1}}+\pi)\Gamma^$$

VI. TWO EXAMPLES OF ORTHOGONAL WAVELET ASSOCIATED WITH LCWT

In this section, two examples of orthogonal wavelet for LCWT are given.

Example 1 Let $\phi(t) = \chi_{[0,1)}$, where $\chi_{[m,n)}$ denotes the characteristic function of [m,n). Followed by Eq. (59), it can be verified that $\{\phi_{A_1,0,n} = \phi(t-n)e^{-\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis of the subspace $V_0^{A_1}$. In further, we have that $\{V_k^{A_1}\}_{k\in\mathbb{Z}}$ is an orthogonal MRA for LCWT. Hence,

$$h_n = \sqrt{2}e^{-\frac{i}{2}\frac{n^2}{4}\frac{a_1}{b_1}} \int_R \phi(t)\phi^*(2t-n)dt = \begin{cases} \frac{\sqrt{2}}{2} & n=0\\ \frac{\sqrt{2}}{2}e^{-\frac{i}{2}\frac{a_1}{4b_1}} & n=1\\ 0 & \text{other} \end{cases}$$

so that

$$g_n = (-1)^{1-n} h_{1-n}^* e^{-\frac{i}{2} \frac{a_1}{b_1} \frac{(1-n)^2}{4}} e^{-\frac{i}{2} \frac{a_1}{b_1} \frac{n^2}{4}} = \begin{cases} -\frac{\sqrt{2}}{2} & n = 0\\ \frac{\sqrt{2}}{2} e^{-\frac{i}{2} \frac{a_1}{4b_1}} & n = 1\\ 0 & \text{otherwise} \end{cases}$$

Overall, from the Eq. (38), the orthogonal wavelet associated with LCWT is shown as follows:

$$\Psi(t) = \sum_{n \in \mathbb{Z}} g_n \sqrt{2} \phi(2t - n) e^{\frac{i}{2} \frac{a_1}{b_1} \frac{n^2}{4}} = -\chi_{[0, \frac{1}{2})} + \chi_{[\frac{1}{2}, 1)} \quad (53)$$

According to the Eq. (34), Eq. (40) and Eq. (50), we have

$$\mathbf{M}(u/b_1) = \frac{1}{2} \begin{pmatrix} 1 + e^{-iu/b_1} & 1 - e^{-iu/b_1} \\ -1 + e^{-iu/b_1} & -1 - e^{-iu/b_1} \end{pmatrix}$$
(54)

It can be easily verified that $\mathbf{M}(u/b_1)$ is a unitary matrix. Example 2 Let $\phi(t) = sinc(t) = \frac{\sin \pi t}{\pi t}$, then

$$F_{\phi}(u) = \mathscr{F}[\phi(t)](u) = \begin{cases} 1 & |u| \le \pi \\ 0 & \text{otherwise} \end{cases}$$
(55)

Therefore, we have

$$\sum_{k \in \mathbb{Z}} |F_{\phi}(u/b_1 + 2k\pi)|^2 = 1.$$
(56)

It means that $\{\phi_{A_1,0,n} = sinc(t-n)e^{-\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis of the subspace $V_0^{A_1}$. In further, it can be verified that $\{V_k^{A_1}\}_{k\in\mathbb{Z}}$ forms an orthogonal MRA of $L^2(R)$.

Hence,

$$h_n = \sqrt{2}e^{-\frac{i}{2}\frac{n^2}{4}\frac{a_1}{b_1}} \int_R \phi(t)\phi^*(2t-n)dt$$
$$= \begin{cases} \frac{\sqrt{2}}{2} & n=0\\ 0 & n=2k, k \neq 0\\ (-1)^k \frac{\sqrt{2}}{(2k+1)\pi} e^{-\frac{i}{2}\frac{(2k+1)^2}{4}\frac{a_1}{b_1}} & n=2k+1 \end{cases}$$

so that

$$\Lambda(\frac{u}{b_1}) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{\frac{i}{2} \frac{a_1}{b_1} \frac{n^2}{4} - i\frac{nu}{b_1}} = \begin{cases} 1 & 0 \le |u| < \frac{|b_1|}{2} \pi \\ 0 & \frac{|b_1|}{2} \pi \le |u| < |b_1| \pi \end{cases}$$
$$\Gamma(\frac{u}{b_1}) = e^{-iu/b_1} \Lambda^*(\frac{u}{b_1} + \pi) = \begin{cases} 0 & 0 \le |u| < \frac{|b_1|}{2} \pi \\ e^{-iu/b_1} & \frac{|b_1|}{2} \pi \le |u| < |b_1| \pi \end{cases}$$

Followed by the Eq. (39), then

$$F_{\psi}(\frac{u}{b_1}) = \Gamma(\frac{u}{2b_1})F_{\phi}(\frac{u}{2b_1}) = e^{-\frac{iu}{2b_1}}F_{\phi}(\frac{u}{2b_1}) - e^{-i\frac{u}{2b_1}}F_{\phi}(\frac{u}{b_1})$$

Apply the inverse Fourier on the two sides of above equation, then

$$\psi(t) = 2\phi(2t-1) - \phi(t-\frac{1}{2}) = \frac{\sin[\pi(2t-1)] - \sin\pi(t-\frac{1}{2})}{\pi(t-\frac{1}{2})}$$

According to Eq. (50), it can be obtained that $\mathbf{M}(u/b_1)$ is a unitary matrix.

VII. CONCLUSION

The MRA and the construction of orthogonal wavelets for LCWT serve a useful tool for its perspective applications. In this paper, we first develop a MRA associated with LCWT. Then, the corresponding orthogonal wavelets for LCWT is constructed based on the newly developed MRA. Finally, the generalized Haar and Shannon wavelets associated with LCWT are investigated.

APPENDIX A

Proof First, $\{\phi_{A_1,k,n}(t) = 2^{\frac{k}{2}}\phi(2^kt-n)e^{-\frac{i}{2}(t^2-(\frac{n}{2^k})^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$ is an orthonormal system owing to

$$\langle \phi_{A_1,k,m}, \phi_{A_1,k,n} \rangle = 2^{-2k} e^{\frac{i}{2}(n^2 - m^2)\frac{a_1}{b_1}} \int_R \phi(2^k t - m)\phi(2^k t - n)dt$$

$$=2^{-k}e^{\frac{i}{2}2^{-2k}(n^2-m^2)\frac{a_1}{b_1}}\int_R\phi(\eta-m)\phi(\eta-n)d\eta \text{ Owing to} \\ =\delta_{m,n}$$

Next, for $\forall f(t) \in V_k^{A_1}$, according to condition (2) of **Definition 1**, we have

$$f(2^{-k}t)e^{\frac{i}{2}[(2^{-k}t)^2 - t^2]\frac{a_1}{b_1}} \in V_0^{A_1}.$$
(57)

Hence, $f(2^{-k}t)e^{\frac{i}{2}[(2^{-k}t)^2-t^2]\frac{a_1}{b_1}}$ can be expressed as a combination of the basis $\{\phi_{A_1,0,n} = \phi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$ of $V_0^{A_1}$, i.e.,

$$f(2^{-k}t)e^{\frac{i}{2}[(2^{-k}t)^2 - t^2]\frac{a_1}{b_1}} = \sum_{n \in \mathbb{Z}} c_n \phi(t-n)e^{\frac{i}{2}(t^2 - n^2)\frac{a_1}{b_1}}$$
$$\Longrightarrow f(2^{-k}t) = \sum_{n \in \mathbb{Z}} c_n \phi(t-n)e^{\frac{i}{2}((2^{-k}t)^2 - n^2)\frac{a_1}{b_1}}$$
(58)
$$\Longrightarrow f(t) = \sum_{n \in \mathbb{Z}} c'_n 2^{\frac{k}{2}} \phi(t-n)e^{\frac{i}{2}(t^2 - (\frac{n}{2^k})^2)\frac{a_1}{b_1}}$$

where $c'_n = c_n 2^{-\frac{k}{2}} e^{\frac{i}{2}n^2(1-2^{-2k})\frac{a_1}{b_1}}$. It means that f(t) can be expressed as a combination of the $\{\phi_{A_1,k,n}\}_{n\in\mathbb{Z}}$. Therefore, the subspace $V_k^{A_1}$ is generated by $\phi(t)$ with dilation, translation and chirp modulation, i.e.,

$$V_k^{A_1} = \overline{\text{span}}\{\phi_{A_1,k,n}(t) = 2^{\frac{k}{2}}\phi(2^kt - n)e^{-\frac{i}{2}(t^2 - (\frac{n}{2^k})^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$$
(59)

If the condition (4) of **Definition 1** is relaxed by assuming that the set of functions $\{\phi_{A_1,0,n}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V_0^{A_1}$, the $\phi(t)$ generates a generalized MRA of the $L^2(R)$ associated with LCWT. Then, the following **Theorem 2** and **Theorem 3** can be obtained.

APPENDIX B

Proof Necessity: for any $f(t) \in V_0^{A_1}$, we have

$$f(t) = \sum_{n \in \mathbb{Z}} c_n \varphi(t-n) e^{-\frac{i}{2}(t^2 - n^2)\frac{a_1}{b_1}}$$
(60)

By taking LCT on both sides of above equation, then

$$\begin{split} L_{f}^{A_{1}}(u) &= \mathscr{L}^{A_{1}}[\sum_{n \in \mathbb{Z}} c_{n} \varphi(t-n) e^{-\frac{i}{2}(t^{2}-n^{2})\frac{a_{1}}{b_{1}}}](u) \\ &= \int_{R} A_{b_{1}} \sum_{n \in \mathbb{Z}} c_{n} \varphi(t-n) e^{-\frac{i}{2}(t^{2}-n^{2})\frac{a_{1}}{b_{1}}} e^{\frac{i}{2}\frac{a_{1}t^{2}+d_{1}u^{2}-2tu}{b_{1}}} dt \\ &= \int_{R} A_{b_{1}} \sum_{n \in \mathbb{Z}} c_{n} \varphi(t-n) e^{\frac{i}{2}\frac{a_{1}n^{2}+d_{1}u^{2}-2tu}{b_{1}}} dt \\ \xi^{=t-n} \sum_{n \in \mathbb{Z}} A_{b_{1}} c_{n} \int_{R} \varphi(\xi) e^{\frac{i}{2}\frac{a_{1}n^{2}+d_{1}u^{2}-2\xi}{b_{1}}} d\xi \\ &= \sum_{n \in \mathbb{Z}} A_{b_{1}} c_{n} e^{\frac{i}{2}\frac{a_{1}n^{2}+d_{1}u^{2}-2nu}{b_{1}}} \int_{R} \varphi(\xi) e^{-i\xi\frac{u}{b_{1}}} d\xi \\ &= \sqrt{2\pi} \tilde{L}_{c_{n}}^{A_{1}}(u) F_{\varphi}(u/b_{1}) \end{split}$$

Owing to the Parseval's identity of LCT (see Eq. (8)), we have

$$\begin{split} \|f(t)\|_{L^2}^2 &= \|L_f^{A_1}(u)\|_{L^2}^2 = 2\pi \int_R |\tilde{L}_{c_n}^{A_1}(u)F_{\varphi}(u/b_1)|^2 du \\ &= 2\pi \sum_{k \in \mathbb{Z}} \int_0^{2\pi b_1} |\tilde{L}_{c_n}^{A_1}(u+2k\pi b_1)F_{\varphi}(u/b_1+2k\pi)|^2 du \\ &= 2\pi \int_0^{2\pi b_1} |\tilde{L}_{c_n}^{A_1}(u)|^2 \sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1+2k\pi)|^2 du \end{split}$$

$$\|c_n\|_{\ell^2}^2 = \frac{1}{2\pi} \int_0^{2\pi b_1} |\tilde{L}_{c_n}^{A_1}(u)|^2 du$$
$$A \le \sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1 + 2k\pi)|^2 \le B$$

then

$$A\|c_n\|_{\ell^2}^2 \le \|f(t) = \sum_{n \in \mathbb{Z}} c_n \varphi_{A_1,0,n}\|_{L^2}^2 \le B\|c_n\|_{\ell^2}^2 \tag{61}$$

Followed by the definition of the Riesz basis, the set of functions $\{\varphi_{A_1,0,n} = \varphi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V_0^{A_1}$.

Sufficiency: if the set of functions $\{\varphi_{A_1,0,n}(t) = \varphi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V_k^{A_1}$, similar to the deduction of necessity, we have

$$\begin{split} A \int_{\Omega_1} |\tilde{L}_{c_n}^{A_1}(u)|^2 du &\leq \int_{\Omega_1} |\tilde{L}_{c_n}^{A_1}(u)|^2 \sum_{k \in Z} |F_{\varphi}(u/b_1 + 2k\pi)|^2 du \\ &\leq B \int_{\Omega_1} |\tilde{L}_{c_n}^{A_1}(u)|^2 du \end{split}$$

where $\Omega_1 = (0, 2\pi b_1)$. Owing to the arbitrariness of $\tilde{L}_{c_n}^{A_1}(u)$, then

$$A \le \sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1 + 2k\pi)|^2 \le B \quad a.e. \ \forall u \in [0, 2\pi b_1].$$
 (62)

In particular, $\{\varphi_{A_1,0,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $V_0^{A_1}$ if and only if A = B = 1.

APPENDIX C

Proof Since the set of functions $\{\varphi_{A_1,0,n} = \varphi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{a_1}{b_1}}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $V_k^{A_1}$, there exist constants $0 < A \leq B < +\infty$ such that

$$A \le \sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1 + 2k\pi)|^2 \le B, \forall u \in [0, 2\pi b_1].$$
(63)

It can be deduced that $\sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1 + 2k\pi)|^2$ is a function with a period of $2\pi b_1$. Thus, there exists a sequence of $\{d_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\frac{e^{-iun_0/b_1}}{\sqrt{\sum_{k\in Z} |F_{\varphi}(u/b_1 + 2k\pi)|^2}} = \sum_{n\in Z} d_n e^{-iun/b_1}$$
(64)

Substitute Eq. (30) into above equation, we have

$$F_{\phi}(u/b_1)e^{-iun_0/b_1} = \sum_{n \in \mathbb{Z}} d_n F_{\phi}(u/b_1)e^{-iun/b_1}$$
(65)

By taking inverse FT on both sides of above equation, we obtain

$$\begin{split} \phi(t-n_0) &= \sum_{n \in Z} d_n \varphi(t-n) \\ \iff & \phi(t-n_0) e^{\frac{i}{2}(t^2 - n_0^2)\frac{a_1}{b_1}} = \sum_{n \in Z} d_n \varphi(t-n) e^{\frac{i}{2}(t^2 - n_0^2)\frac{a_1}{b_1}} \\ \iff & \phi(t-n_0) e^{\frac{i}{2}(t^2 - n_0^2)\frac{a_1}{b_1}} = \sum_{n \in Z} d'_n \varphi(t-n) e^{\frac{i}{2}(t^2 - n^2)\frac{a_1}{b_1}} \end{split}$$

where $d'_{n} = c_{n}e^{\frac{i}{2}(n^{2}-n_{0}^{2})\frac{a_{1}}{b_{1}}}$. Therefore, it can be obtained that $\phi(t-n)e^{\frac{i}{2}(t^{2}-n^{2})\frac{a_{1}}{b_{1}}} \in V_{0}^{A_{1}}.$ (66)

In addition to, on the one hand, because $V_0^{A_1}$ is a closed

space, then

$$\overline{\text{span}}\{\phi_{A_1,0,n}(t) = \phi(t-n)e^{\frac{i}{2}(t^2-n^2)\frac{d_1}{b_1}}\}_{n\in\mathbb{Z}} \subseteq V_0^{A_1}.$$
 (67)

On the other hand, based on the Eq. (30), we have

$$F_{\varphi}(u/b_1) = F_{\phi}(u/b_1) \sqrt{\sum_{k \in \mathbb{Z}} |F_{\varphi}(u/b_1 + 2k\pi)|^2}, \qquad (68)$$

then,

$$V_0^{A_1} \subseteq \overline{\text{span}}\{\phi_{A_1,0,n}(t) = \phi(t-n)e^{\frac{t}{2}(t^2-n^2)\frac{d_1}{b_1}}\}_{n \in \mathbb{Z}}.$$
 (69)

It means that

$$V_0^{A_1} = \overline{\text{span}} \{ \phi_{A_1,0,n} = \phi(t-n) e^{\frac{i}{2}(t^2 - n^2) \frac{a_1}{b_1}} \}_{n \in \mathbb{Z}}.$$
 (70)

Moreover, we have

$$\sum_{k\in\mathbb{Z}} |F_{\phi}(u/b_1 + 2k\pi)|^2 = \frac{\sum_{k\in\mathbb{Z}} |F_{\phi}(u/b_1 + 2k\pi)|^2}{\sum_{k\in\mathbb{Z}} |F_{\phi}(u/b_1 + 2k\pi)|^2} = 1.$$
(71)

Then, the set of functions $\{\phi_{A_1,0,n}(t) = \phi(t - n)e^{\frac{i}{2}(t^2 - n^2)\frac{a_1}{b_1}}\}_{n \in \mathbb{Z}}$ form an orthonormal basis of $V_0^{A_1}$.

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