Parameter Estimation for the α-Stable Vasicek Model Based on Discrete Observations

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Abstract—**This paper is concerned with parameter estimation for Vasicek model driven by α-stable motion from discrete observation. The contrast function is used to obtain the least squares estimator. The strong consistency and asymptotic distribution of the estimators are studied by using ergodic theorem, Hölder inequality and Markov inequality. Some numerical calculus and simulations are given to verify the effectiveness of estimator.**

Index Terms—**parameter estimation, parameter estimation, α-stable noises, discrete observations, consistency.**

I. Introduction

Itô type stochastic differential equations are widely used in the modeling of stochastic phenomena in the fields of physics, chemistry, medicine([5], [16]). Recently, they are applied to describe the dynamics of a financial asset, such as Cox-Ingersoll-Ross([9], [10]), Chan-Karloyi-Longstaff-Sanders ([7]) and Hull-White model ([8]). However, part or all of the parameters in stochastic model are always unknown. In the past few decades, some methods have been put forward to estimate the parameters for Itô type stochastic differential equations, such as maximum likelihood estimation([24], [25]), least squares estimation([6], [14], [21]) and Bayes estimation([12], [15]). But, in fact, non-Gaussian noise can more accurately reflect the practical random perturbation. α-stable noise, as a kind of important non-Gaussian noise, has attracted wide attention in the research and practice in the fields of engineering, economy and society and has been studied by some authors such as Bertoin([3]) and Applebaum([1]. From a practical point of view in parametric inference, it is more realistic and interesting to consider parameter estimation for stochastic differential equations driven by α-stable motion. Recently, a number of literatures have been devoted to the parameter estimation for the models with α-stable noises. When the coefficient is constant, drift parameter estimation has been investigated ([17]–[19]).

Vasicek model, which was introduced by Oldrich Alfons Vasicek in 1977([23]), is a mathematical model describing the evolution of interest rates. It is a type of one-factor short rate model as it describes interest rate movements as driven by only one source of market risk. The model can be used in the valuation of interest rate derivatives, and has also been adapted for credit markets. It is known that parameter estimation for Vasicek model driven by Brownian motion has been well developed([22], [27]). However, some features of the financial processes cannot be captured by the Vasicek model, for example, discontinuous sample paths and heavy tailed properties. Therefore, it is natural to replace the Brownian motion by non-Gaussian noise. Recently, the parameter estimation problems for Vasicek model driven by small Lévy noises have been studied by some authors. For example, Davis([11]) used Malliavin calculus and Monte Carlo estimation to study the estimator of the Vasicek model driven by jump process, Bao([2]) developed the approximate bias of the ordinary least squares estimator of the Vasicek model driven by continuous-time Lévy processes. But, the explicit expression of the estimation error and the consistency of the estimators have not been discussed and there are few literature about the parameter estimation problem for Vasicek model driven by α-stable noises.

In this paper, we consider the parameter estimation problem for Vasicek model with α-stable noises from discrete observations. The contrast function is introduced to obtain the least squares estimator. The strong consistency and asymptotic distribution of the estimator are proved by using ergodic theorem, Hölder inequality and Markov inequality. Some numerical calculus and simulations are given to verify the effectiveness of estimator.

This paper is organized as follows. In Section 2, the Vasicek model driven by α-stable noises is introduced, the contrast function is given and the explicit formula of the least squares estimators is obtained. In Section 3, the strong consistency of the estimators are proved. In Section 4, some simulation results are made. The conclusion is given in Section 5.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $\{\mathscr{F}_t\}_{t\geq 0}$ and $Z = \{Z_t, t \geq 0\}$ be a strictly symmetric α-stable Lévy motion.

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$ and location parameter $\mu \in (-\infty, \infty)$ if it has the following characteristic function:

$$\phi_\eta(u) = \begin{cases} -\sigma^\alpha |u|^\alpha (1 - i\beta sgn(u) \tan \frac{\alpha\pi}{2}) + i\mu u, if \alpha \neq 1, \\ -\sigma |u| (1 + i\beta \frac{2}{\pi} sgn(u) \log |u|) + i\mu u, if \alpha = 1. \end{cases}$$

We denote $\eta \sim S_\alpha(\sigma, \beta, \mu)$. When $\mu = 0$, we say η is strictly α-stable, if in addition $\beta = 0$, we call η symmetrical α-stable. Throughout this paper, it is assumed that α-stable motion is strictly symmetrical and $\alpha \in (1, 2)$.

In this paper, we investigate the parameter estimation problem for α-stable Vasicek model described by the following

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stochastic differential equation:

$$\begin{cases} dX_t = (\theta + \gamma X_t)dt + dZ_t \\ X_0 = x_0, \end{cases} \tag{1}$$

where θ and γ are unknown parameters with $\gamma < 0$ and Z is a strictly symmetric α-stable motion on $\mathbb{R}$ with the index $\alpha \in (1, 2)$.

It is assumed that the process $\{X_t, t \geq 0\}$ can be observed at discrete point $\{t_i = ih, i = 0, 1, 2, ..., n\}$ with $h > 0$. We introduce the following contrast function:

$$\rho_n(\theta, \gamma) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}} - (\theta + \gamma X_{t_{i-1}})\Delta t_{i-1}|^2, \tag{2}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = h$.

Then, we can obtain the estimators as follows

$$\begin{cases} \widehat{\theta}_n = \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}}}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}^2}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ \widehat{\gamma}_n = \dfrac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) \sum_{i=1}^n X_{t_{i-1}}}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \dfrac{n\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2}. \end{cases} \tag{3}$$

Since the α-stable Vasicek model can be written as

$$X_t = X_0 \exp^{\gamma t} - \frac{\theta}{\gamma}(1 - \exp^{\gamma t}) + \int_0^t \exp^{\gamma(t-s)} \sqrt{X_s} dZ_s. \tag{4}$$

The expression of $\widehat{\theta}_n$ and $\widehat{\gamma}_n$ can be changed as

$$\begin{cases} \widehat{\theta}_n = \dfrac{\theta(e^{\gamma h} - 1)}{\gamma h} \\ \quad + \dfrac{\sum_{i=1}^n X_{t_{i-1}}^2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \dfrac{\sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ \widehat{\gamma}_n = \dfrac{e^{\gamma h} - 1}{h} \\ \quad + \dfrac{\sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ \quad - \dfrac{n\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ . \end{cases} \tag{5}$$

III. Main Results and Proofs

In the following theorem, the strong consistency of the least square estimators $\widehat{\theta}_n$ and $\widehat{\gamma}_n$ are proved.

Theorem 1: When $h \to 0$ and $nh \to \infty$,

$$\widehat{\theta}_n \overset{a.s.}{\to} \theta,$$

$$\widehat{\gamma}_n \overset{a.s.}{\to} \theta.$$

Proof: Since X_t is ergodic, by the ergodic theorem and Corollary 3.1 in , it can be checked that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |X_{t_{i-1}}| = \mathbb{E}|X_\infty| = -\frac{1}{2\theta}, a.s. \tag{6}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |X_{t_{i-1}}|^p = \mathbb{E}|X_\infty|^p = \infty, \quad p \geq \alpha, a.s. \tag{7}$$

$$\lim_{n\to\infty} \sup \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |e^{\gamma(t_i - s)}|^\alpha ds} = 0, a.s. \tag{8}$$

and

$$\lim_{n\to\infty} \sup \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_{t_{i-1}} e^{\gamma(t_i - s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |X_{t_{i-1}} e^{\gamma(t_i - s)}|^\alpha ds} = 0, a.s. \tag{9}$$

Moreover,

$$\begin{aligned} &|\frac{\sum_{i=1}^n X_{t_{i-1}}^2 \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s}{h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2}| \\ &\leq \frac{\sum_{i=1}^n X_{t_{i-1}}^2 |\sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s|}{|h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2|} \\ &+ \frac{\sum_{i=1}^n |X_{t_{i-1}}| |\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s|}{|h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2|}. \end{aligned}$$

According to (6), (7) and (8), it follows that

$$\begin{aligned} &\frac{\sum_{i=1}^n X_{t_{i-1}}^2 |\sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s|}{|h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2|} \\ &= \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |e^{\gamma(t_i - s)}|^\alpha ds} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |e^{\gamma(t_i - s)}|^\alpha ds \\ &\times \frac{\sum_{i=1}^n X_{t_{i-1}}^2}{|h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2|} \\ &\leq \sup_n \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |e^{\gamma(t_i - s)}|^\alpha ds} \frac{e^{\alpha\gamma h} - 1}{\alpha\gamma} \\ &\times \frac{n\sum_{i=1}^n X_{t_{i-1}}^2}{|h(\sum_{i=1}^n X_{t_{i-1}})^2 - nh\sum_{i=1}^n X_{t_{i-1}}^2|} \\ &\leq \sup_n \frac{|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{\gamma(t_i - s)} dZ_s|}{\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |e^{\gamma(t_i - s)}|^\alpha ds} \frac{e^{\alpha\gamma h} - 1}{h\alpha\gamma} \\ &\times \frac{1}{\frac{(\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}|)^2}{\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}|^2} - 1} \\ &\overset{a.s.}{\to} 0. \end{aligned}$$

when $h \to 0$ and $nh \to \infty$, by Hölder inequality and (6), (7), (9), we obtain

$$\begin{aligned}
&\frac{\sum_{i=1}^{n}|X_{t_{i-1}}||\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s|}{|h(\sum_{i=1}^{n}X_{t_{i-1}})^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2|}\\
\leq&\frac{|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_{t_{i-1}}e^{\gamma(t_i-s)}dZ_s|}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|X_{t_{i-1}}e^{\gamma(t_i-s)}|^{\alpha}ds}\\
&\times\frac{\sum_{i=1}^{n}|X_{t_{i-1}}|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|X_{t_{i-1}}e^{\gamma(t_i-s)}|^{\alpha}ds}{h(\sum_{i=1}^{n}|X_{t_{i-1}}|)^2-nh\sum_{i=1}^{n}|X_{t_{i-1}}|^2}\\
\leq&\sup_n\frac{|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_{t_{i-1}}e^{\gamma(t_i-s)}dZ_s|}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|X_{t_{i-1}}e^{\gamma(t_i-s)}|^{\alpha}ds}\frac{e^{h\alpha\gamma}-1}{\alpha\gamma}\\
&\times\frac{\sum_{i=1}^{n}|X_{t_{i-1}}|\sum_{i=1}^{n}|X_{t_{i-1}}|^{\alpha}}{h(\sum_{i=1}^{n}|X_{t_{i-1}}|)^2-nh\sum_{i=1}^{n}|X_{t_{i-1}}|^2}\\
\leq&\sup_n\frac{|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_{t_{i-1}}e^{\gamma(t_i-s)}dZ_s|}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|X_{t_{i-1}}e^{\gamma(t_i-s)}|^{\alpha}ds}\frac{e^{h\alpha\gamma}-1}{h\alpha\gamma}\\
&\times\frac{\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|}{\frac{(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|)^2}{(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|^2)^{\frac{\alpha}{2}}}-(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|^2)^{1-\frac{\alpha}{2}}}\\
\overset{a.s.}{\to}&\ 0.
\end{aligned}$$

When $h \to 0$, it is obvious that

$$\frac{\theta(e^{\gamma h}-1)}{\gamma h}\to\theta. \tag{10}$$

Therefore, with above results, we obtain that

$$\widehat{\theta}_n \overset{a.s.}{\to} \theta. \tag{11}$$

Since

$$\begin{aligned}
&|\frac{\sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s}{h(\sum_{i=1}^{n}X_{t_{i-1}})^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2}\\
&-\frac{n\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s}{h(\sum_{i=1}^{n}X_{t_{i-1}})^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2}|\\
\leq&\frac{\sum_{i=1}^{n}|X_{t_{i-1}}||\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s|}{|h(\sum_{i=1}^{n}X_{t_{i-1}})^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2|}\\
&+\frac{n|\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s|}{|h(\sum_{i=1}^{n}X_{t_{i-1}})^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2|}.
\end{aligned}$$

Moreover, by Hölder inequality and (6), (7), (9), we obtain that

$$\begin{aligned}
&\frac{\sum_{i=1}^{n}|X_{t_{i-1}}||\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s|}{|h(\sum_{i=1}^{n}X_{t_{i-1}})^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2|}\\
\leq&\frac{\sum_{i=1}^{n}|X_{t_{i-1}}|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|e^{\gamma(t_i-s)}|^{\alpha}ds}{h(\sum_{i=1}^{n}|X_{t_{i-1}}|)^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2}\\
&\times\frac{|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s|}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|e^{\gamma(t_i-s)}|^{\alpha}ds}\\
\leq&\frac{e^{h\alpha\gamma}-1}{h\alpha\gamma}\frac{\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|}{(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|)^2-\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|^2}\\
&\times\sup_n\frac{|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s|}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|e^{\gamma(t_i-s)}|^{\alpha}ds}\\
\overset{a.s.}{\to}&\ 0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{n|\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}e^{\gamma(t_i-s)}dZ_s|}{|h(\sum_{i=1}^{n}X_{t_{i-1}})^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2|}\\
\leq&\frac{e^{h\alpha\gamma}-1}{h\alpha\gamma}\frac{\sum_{i=1}^{n}|X_{t_{i-1}}|^{\alpha}}{h(\sum_{i=1}^{n}|X_{t_{i-1}}|)^2-nh\sum_{i=1}^{n}X_{t_{i-1}}^2}\\
&\times\frac{|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_{t_{i-1}}e^{\gamma(t_i-s)}dZ_s|}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|X_{t_{i-1}}e^{\gamma(t_i-s)}|^{\alpha}ds}\\
\leq&\frac{e^{h\alpha\gamma}-1}{h\alpha\gamma}\\
&\times\frac{1}{\frac{(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|)^2}{(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|^2)^{\frac{\alpha}{2}}}-(\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}|)^{1-\frac{\alpha}{2}}}\\
&\times\sup_n\frac{|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_{t_{i-1}}e^{\gamma(t_i-s)}dZ_s|}{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|X_{t_{i-1}}e^{\gamma(t_i-s)}|^{\alpha}ds}\\
\overset{a.s.}{\to}&\ 0,
\end{aligned}$$

When $h \to 0$, it is obvious that

$$\frac{e^{\gamma h}-1}{h}\to\gamma. \tag{12}$$

Therefore, with above results, we obtain that

$$\widehat{\gamma}_n \overset{a.s.}{\to} \gamma. \tag{13}$$

The proof is complete. ∎

Remark 1: If the Vasicek model is driven by the small α-stable noises described by the following stochastic differential equation:

$$\begin{cases} dX_t =(\theta+\gamma X_t)dt+\varepsilon dZ_t, & t\in[0,1]\\ X_0 =x_0, \end{cases} \tag{14}$$

where θ and γ are unknown parameters. Without loss of generality, it is assumed that $\varepsilon \in (0,1]$.

Consider the following contrast function

$$\rho_{n,\varepsilon}(\theta,\gamma)=\sum_{i=1}^{n}|X_{t_i}-X_{t_{i-1}}-(\theta+\gamma X_{t_{i-1}})\Delta t_{i-1}|^2, \tag{15}$$

where $\Delta t_{i-1}=t_i-t_{i-1}=\frac{1}{n}$.

It is easy to obtain the least square estimators

$$\begin{cases} \widehat{\theta}_{n,\varepsilon} =\frac{n\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})X_{t_{i-1}}\sum_{i=1}^{n}X_{t_{i-1}}}{(\sum_{i=1}^{n}X_{t_{i-1}})^2-n\sum_{i=1}^{n}X_{t_{i-1}}^2}\\ \qquad -\frac{-n\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\sum_{i=1}^{n}X_{t_{i-1}})^2-n\sum_{i=1}^{n}X_{t_{i-1}}^2}\\ \widehat{\gamma}_{n,\varepsilon} =\frac{n^2\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})X_{t_{i-1}}}{(\sum_{i=1}^{n}X_{t_{i-1}})^2-n\sum_{i=1}^{n}X_{t_{i-1}}^2}\\ \qquad -\frac{n\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})\sum_{i=1}^{n}X_{t_{i-1}}}{(\sum_{i=1}^{n}X_{t_{i-1}})^2-n\sum_{i=1}^{n}X_{t_{i-1}}^2}\\ \qquad . \end{cases} \tag{16}$$

Let $X^0=(X_t^0, t\geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameters:

$$dX_t^0=(\theta+\gamma X_t^0)dt, \quad X_0^0=x_0. \tag{17}$$

Note that

$$X_{t_i} - X_{t_{i-1}} = \frac{1}{n}\theta + \gamma\int_{t_{i-1}}^{t_i} X_s ds + \varepsilon\int_{t_{i-1}}^{t_i} dZ_s. \quad (18)$$

Then, we can give a more explicit decomposition for $\widehat{\theta}_{n,\varepsilon}$ and $\widehat{\gamma}_{n,\varepsilon}$ as follows

$$\begin{aligned}\widehat{\theta}_{n,\varepsilon} &= \theta + \frac{n\gamma\sum_{i=1}^n\int_{t_{i-1}}^{t_i} X_s ds\sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\gamma\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds\sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{n\varepsilon\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s\sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i} dZ_s\sum_{i=1}^n X_{t_{i-1}}^2}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &= \theta + \frac{\gamma\sum_{i=1}^n\int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\gamma\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{\varepsilon\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i} dZ_s\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}.\end{aligned}$$

$$\begin{aligned}\widehat{\gamma}_{n,\varepsilon} &= \frac{n\gamma\sum_{i=1}^n\int_{t_{i-1}}^{t_i} X_s ds\sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n^2\gamma\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{n^2\varepsilon\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{n\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i} dZ_s\sum_{i=1}^n X_{t_{i-1}}}{(\sum_{i=1}^n X_{t_{i-1}})^2 - n\sum_{i=1}^n X_{t_{i-1}}^2} \\ &= \frac{\gamma\sum_{i=1}^n\int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\gamma\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &+ \frac{\varepsilon\sum_{i=1}^n X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &- \frac{\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i} dZ_s\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}.\end{aligned}$$

Lemma 1: When $\varepsilon \to 0$ and $n \to \infty$, we have

$$\sup_{0\le t\le 1}|X_t - X_t^0| \xrightarrow{P} 0.$$

Proof: Observe that

$$X_t - X_t^0 = \gamma\int_0^t (X_s - X_s^0)ds + \varepsilon\int_0^t dZ_s. \quad (19)$$

By using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}&|X_t - X_t^0|^2 \\ \le\ & 2\gamma^2|\int_0^t (X_s - X_s^0)ds|^2 + 2\varepsilon^2|\int_0^t dZ_s|^2 \\ \le\ & 2\gamma^2 t\int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2\sup_{0\le t\le 1}|\int_0^t dZ_s|^2\end{aligned}$$

.

According to the Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \le 2\varepsilon^2 e^{2\gamma^2 t^2}\sup_{0\le t\le 1}|\int_0^t dZ_s|^2. \quad (20)$$

Then, it follows that

$$\sup_{0\le t\le 1}|X_t - X_t^0| \le \sqrt{2}\varepsilon e^{\gamma^2}\sup_{0\le t\le 1}|\int_0^t dZ_s|. \quad (21)$$

By the Markov inequality, for any given $\delta > 0$, when $\varepsilon \to 0$, we have

$$\begin{aligned}&\mathbb{P}(\sqrt{2}\varepsilon e^{\gamma^2}\sup_{0\le t\le 1}|\int_0^t dZ_s| > \delta) \\ &\le \delta^{-1}\sqrt{2}\varepsilon e^{\gamma^2}\mathbb{E}[\sup_{0\le t\le 1}|\int_0^t dZ_s|] \\ &\le \delta^{-1}\sqrt{2}\varepsilon e^{\gamma^2}\mathbb{E}[(\int_0^1 ds)^{\frac{1}{\alpha}}] \\ &= \delta^{-1}\sqrt{2}\varepsilon e^{\gamma^2} \\ &\to 0.\end{aligned}$$

Therefore, it is easy to check that

$$\sup_{0\le t\le 1}|X_t - X_t^0| \xrightarrow{P} 0. \quad (22)$$

The proof is complete. ■

Proposition 1: When $\varepsilon \to 0$ and $n \to \infty$, we have,

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt.$$

Proof: Since

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 = \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 + \frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2). \quad (23)$$

It is clear that

$$\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^0)^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \quad (24)$$

For $\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2)$, according to Lemma 2 and the fact that $\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0| \xrightarrow{P} \int_0^1 |X_t^0|dt$, When $\varepsilon \to 0$ and $n \to \infty$, we have

$$\begin{aligned}&|\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}}^2 - (X_{t_{i-1}}^0)^2)| \\ &= |\frac{1}{n}\sum_{i=1}^n (X_{t_{i-1}} + X_{t_{i-1}}^0)(X_{t_{i-1}} - X_{t_{i-1}}^0)| \\ &\le \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|(|X_{t_{i-1}}| + |X_{t_{i-1}}^0|) \\ &\le \frac{1}{n}\sum_{i=1}^n (|X_{t_{i-1}} - X_{t_{i-1}}^0|^2 + 2|X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0|) \\ &= \frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|^2 \\ &+ 2\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0||X_{t_{i-1}} - X_{t_{i-1}}^0| \\ &\le (\sup_{0\le t\le 1}|X_t - X_t^0|)^2 \\ &+ 2\sup_{0\le t\le 1}|X_t - X_t^0|\frac{1}{n}\sum_{i=1}^n |X_{t_{i-1}}^0| \\ &\xrightarrow{P} 0.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt. \quad (25)$$

The proof is complete. ∎

In the following theorem, the consistency of the least squares estimators are proved.

Theorem 2: When $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$, the least squares estimators $\widehat{\theta}_{n,\varepsilon}$ and $\widehat{\gamma}_{n,\varepsilon}$ are consistent, namely

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta, \quad \widehat{\gamma}_{n,\varepsilon} \xrightarrow{P} \gamma.$$

Proof: According to Proposition 1, it is clear that

$$(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2 \xrightarrow{P} (\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt. \quad (26)$$

With the results that $\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 (X_t^0)^2 dt$ and $\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_t^0 dt$, when $\varepsilon \to 0$ and $n \to \infty$, it can be checked that

$$\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2 \xrightarrow{P} \gamma\int_0^1 X_t dt \int_0^1 (X_t^0)^2 dt, \quad (27)$$

and

$$\gamma\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}} \xrightarrow{P} \gamma\int_0^1 X_t X_t^0 dt \int_0^1 X_t^0 dt.$$

According to Lemma 2, we have

$$\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2 - \gamma\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}} \xrightarrow{P} 0.$$

Since

$$\begin{aligned}
&|\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s| \\
&\le \varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}||\int_{t_{i-1}}^{t_i} dZ_s| \\
&\le \varepsilon\sum_{i=1}^{n}(|X_{t_{i-1}}^0| + |X_{t_{i-1}} - X_{t_{i-1}}^0|)|\int_{t_{i-1}}^{t_i} dZ_s| \\
&\le \varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0||\int_{t_{i-1}}^{t_i} dZ_s| \\
&+\varepsilon\sup_{0\le t\le 1}|X_t - X_t^0||\int_{t_{i-1}}^{t_i} dZ_s|.
\end{aligned}$$

By the Markov inequality, we have

$$\begin{aligned}
&P(|\varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0||\int_{t_{i-1}}^{t_i} dZ_s|| > \delta) \\
&\le \delta^{-1}\varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0|\mathbb{E}|\int_{t_{i-1}}^{t_i} dZ_s| \\
&\le \delta^{-1}\varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0|n^{-\frac{1}{\alpha}} \\
&= \delta^{-1}\varepsilon n^{1-\frac{1}{\alpha}}\frac{1}{n}\sum_{i=1}^{n}|X_{t_{i-1}}^0|,
\end{aligned}$$

which implies that $\varepsilon\sum_{i=1}^{n}|X_{t_{i-1}}^0||\int_{t_{i-1}}^{t_i} dZ_s| \xrightarrow{P} 0$ as $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$.

According to Lemma 2, when $\varepsilon \to 0$ and $n \to \infty$, it is obvious that

$$\varepsilon\sup_{0\le t\le 1}|X_t - X_t^0||\int_{t_{i-1}}^{t_i} dZ_s| \xrightarrow{P} 0. \quad (28)$$

Then, we have

$$\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s \xrightarrow{P} 0. \quad (29)$$

With the results of Proposition 1, (13) and (18), we have

$$\frac{\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \xrightarrow{P} 0. \quad (30)$$

Moreover, when $\varepsilon \to 0$ and $n \to \infty$, it is easy to check that

$$\frac{\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} dZ_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \xrightarrow{P} 0. \quad (31)$$

Therefore, by (13), (16), (19) and (20), When $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$, we have

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta.$$

Using the same methods, it can be easily to check that When $\varepsilon \to 0$, $n \to \infty$ and $\varepsilon n^{1-\frac{1}{\alpha}} \to 0$, we have

$$\widehat{\gamma}_{n,\varepsilon} \xrightarrow{P} \gamma.$$

The proof is complete. ∎

Theorem 3: When $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$,

$$\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon} - \theta) \xrightarrow{d} \frac{((\int_0^1 (X_t^0)^\alpha dt)^{\frac{1}{\alpha}}\int_0^1 X_t^0 dt - \int_0^1 (X_t^0)^2 dt)}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt} S_\alpha(1,0,0),$$

$$\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon} - \gamma) \xrightarrow{d} \frac{((\int_0^1 (X_t^0)^\alpha dt)^{\frac{1}{\alpha}} - \int_0^1 X_t^0 dt)}{(\int_0^1 X_t^0 dt)^2 - \int_0^1 (X_t^0)^2 dt} S_\alpha(1,0,0).$$

Proof: According to the explicit decomposition for $\widehat{\theta}_{n,\varepsilon}$, it is obvious that

$$\begin{aligned}
&\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon} - \theta) \\
&= \frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \\
&- \frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i} X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \\
&+ \frac{\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i} dZ_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \\
&- \frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} dZ_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}.
\end{aligned}$$

When $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$,

$$\begin{aligned}&|\varepsilon^{-1}\gamma\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds| \\ &\le \varepsilon^{-1}\gamma\sum_{i=1}^{n}|X_{t_{i-1}}||\int_{t_{i-1}}^{t_i}X_s ds| \\ &\le \varepsilon^{-1}n^{-1}\theta_2\sum_{i=1}^{n}(|X_{t_{i-1}}-X^0_{t_{i-1}}|+|X^0_{t_{i-1}}|) \\ &\sup_{t_{i-1}\le t\le t_i}|X_t| \\ &\xrightarrow{P} 0.\end{aligned}$$

Then, it is easy to check that

$$\varepsilon^{-1}\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds \xrightarrow{P} 0.$$

Hence

$$\frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}} \xrightarrow{P} 0, \tag{32}$$

and

$$\frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}} \xrightarrow{P} 0. \tag{33}$$

Since

$$\begin{aligned}&\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dZ_s \\ &=\sum_{i=1}^{n}(X_{t_{i-1}}-X^0_{t_{i-1}}+X^0_{t_{i-1}})\int_{t_{i-1}}^{t_i}dZ_s \\ &=\sum_{i=1}^{n}(X_{t_{i-1}}-X^0_{t_{i-1}})\int_{t_{i-1}}^{t_i}dZ_s \\ &+\sum_{i=1}^{n}X^0_{t_{i-1}}\int_{t_{i-1}}^{t_i}dZ_s.\end{aligned}$$

According to Theorem 1, we have

$$\sum_{i=1}^{n}(X_{t_{i-1}}-X^0_{t_{i-1}})\int_{t_{i-1}}^{t_i}dZ_s \xrightarrow{P} 0. \tag{34}$$

Moreover,

$$\begin{aligned}&\sum_{i=1}^{n}X^0_{t_{i-1}}\int_{t_{i-1}}^{t_i}dZ_s \\ &=\int_0^1\sum_{i=1}^{n}X^0_{t_{i-1}}1_{(t_{i-1},t_i]}(s)dZ_s \\ &=Z'\circ\int_0^1\sum_{i=1}^{n}(X^0_{t_{i-1}}1_{(t_{i-1},t_i]}(s))^\alpha ds,\end{aligned}$$

where $Z' \stackrel{d}{=} Z$.

Since

$$\int_0^1\sum_{i=1}^{n}(X^0_{t_{i-1}}1_{(t_{i-1},t_i]}(s))^\alpha ds \to \int_0^1(X^0_s)^\alpha ds, \tag{35}$$

it is clear that

$$Z'\circ\int_0^1\sum_{i=1}^{n}(X^0_{t_{i-1}}1_{(t_{i-1},t_i]}(s))^\alpha ds \xrightarrow{a.s.} Z'\circ\int_0^1(X^0_s)^\alpha ds. \tag{36}$$

It immediately follows that

$$\sum_{i=1}^{n}X^0_{t_{i-1}}\int_{t_{i-1}}^{t_i}dZ_s \xrightarrow{d} (\int_0^1(X^0_t)^\alpha dt)^{\frac{1}{\alpha}}S_\alpha(1,0,0). \tag{37}$$

Then, we have

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\theta}_{n,\varepsilon}-\theta) \xrightarrow{d} \\ &\frac{((\int_0^1(X^0_t)^\alpha dt)^{\frac{1}{\alpha}}\int_0^1 X^0_t dt-\int_0^1(X^0_t)^2dt)}{(\int_0^1 X^0_t dt)^2-\int_0^1(X^0_t)^2dt}S_\alpha(1,0,0).\end{aligned}$$

As

$$\begin{aligned}&\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon}-\gamma) \\ &=\frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}} \\ &-\frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}}-\varepsilon^{-1}\gamma \\ &+\frac{\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dZ_s}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}} \\ &-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}dZ_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}}.\end{aligned}$$

It is obvious that

$$\begin{aligned}&\frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s ds\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}} \\ &-\frac{\varepsilon^{-1}\gamma\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}X_s ds}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}}-\varepsilon^{-1}\gamma \\ &\xrightarrow{P} 0,\end{aligned}$$

and

$$\begin{aligned}&\frac{\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}dZ_s}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}} \\ &-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}dZ_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}}{(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}})^2-\frac{1}{n}\sum_{i=1}^{n}X^2_{t_{i-1}}} \\ &\xrightarrow{d} \frac{((\int_0^1(X^0_t)^\alpha dt)^{\frac{1}{\alpha}}-\int_0^1 X^0_t dt)}{(\int_0^1 X^0_t dt)^2-\int_0^1(X^0_t)^2dt}S_\alpha(1,0,0).\end{aligned}$$

Then, we have

$$\varepsilon^{-1}(\widehat{\gamma}_{n,\varepsilon}-\gamma) \xrightarrow{d} \frac{((\int_0^1(X^0_t)^\alpha dt)^{\frac{1}{\alpha}}-\int_0^1 X^0_t dt)}{(\int_0^1 X^0_t dt)^2-\int_0^1(X^0_t)^2dt}S_\alpha(1,0,0). \tag{38}$$

The proof is complete. ■

IV. Simulations

In this experiment, we generate a discrete sample $(X_{t_i})_{i=0,1,\dots,n}$ and compute $\widehat{\theta}_n$ and $\widehat{\gamma}_n$ from the sample. We let $x_0 = 1$ and $\alpha = 1.8$. For every given true value of the parameters-θ and γ, the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1, $h = 0.1$, the size is increasing from 1000 to 5000. In Table 2, $h = 0.01$, the size is increasing from 10000 to 50000. The tables list the value of "$\theta - LSE$" and "$\gamma - LSE$", and the absolute errors (AE) of LSE, LSE means least squares estimator.

Two tables illustrate that when n is large enough and h is small enough, the obtained estimators are very close to the true parameter value. Therefore, the methods used in this paper are effective and the obtained estimators are good.

TABLE I
LSE SIMULATION RESULTS OF θ AND γ

True		Aver		AE	
(θ, γ)	Size n	$\theta - LSE$	$\gamma - LSE$	θ	γ
(1,-2)	1000	1.3526	-2.4132	0.3526	0.4132
	2000	1.2350	-2.2915	0.2350	0.2915
	3000	1.1706	-2.1852	0.1706	0.1852
	5000	1.0217	-2.0471	0.0217	0.0471
(2,-3)	1000	2.4016	-3.3218	0.4016	0.3218
	2000	2.2908	-3.2651	0.2908	0.2651
	3000	2.1683	-3.1710	0.1683	0.1710
	5000	2.0462	-3.0235	0.0462	0.0235

V. CONCLUSION

The aim of this paper is to estimate the parameters for Vasicek model driven by α-stable noises from discrete observation. The contrast function has been introduced to obtain the least squares estimators. The strong consistency of the estimators have been discussed by using ergodic theorem, Hölder inequality and Markov inequality. Some numerical calculus and simulations have been given to verify the effectiveness of estimators. Further research tops will include parameter estimation for partially observed stochastic differential equation driven by α-stable noises.

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TABLE II
LSE SIMULATION RESULTS OF θ AND γ

True		Aver		AE	
(θ, γ)	Size n	$\theta - LSE$	$\gamma - LSE$	θ	γ
(1,-2)	10000	1.2315	-2.3041	0.2315	0.3041
	20000	1.1426	-2.1986	0.1426	0.1986
	30000	1.0610	-2.0973	0.0610	0.0973
	50000	1.0012	-2.0027	0.0012	0.0027
(2,-3)	10000	2.3107	-3.2145	0.3107	0.2145
	20000	2.2046	-3.1308	0.2046	0.1308
	30000	2.1154	-3.0641	0.1154	0.0641
	50000	2.0063	-3.0017	0.0063	0.0017

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