A Class of Nonstationary Upper and Lower Triangular Splitting Iteration Methods for Ill-posed Inverse Problems

Jingjing Cui, Guohua Peng, Quan Lu, Zhengge Huang

Abstract—**In this study, by applying the minimize residual technique to a class of upper and lower triangular (ULT) methods, two types of nonstationary ULT iteration methods called the minimize residual ULT (MRULT) ones are established for solving ill-posed inverse problems. We provide the convergent analysis of the MRULT-type iteration methods, which shows that the proposed methods are convergent if the related parameter satisfies suitable restrictions. And the new methods further improve the convergence rates of the ULT ones. Finally, numerical experiments arising from a Fredholm integral equation of the first kind and image restoration are reported to further examine the feasibility and effectiveness of the proposed methods.**

Index Terms—**upper and lower triangular splitting, minimize residual, Tikhonov regularization, ill-posed problems, iteration method.**

I. Introduction

CONSIDER the problem of computing an approximate solution of large-scale least-squares problems of the form

$$\min_{f\in R^n} \|Af-g\|_2, \quad A\in R^{n\times n}, \quad f,g\in R^n, \tag{1}$$

where and throughout this paper $\|\cdot\|_2$ denotes the Euclidean vector norm or the associated induced matrix norm. The singular values of the matrix A gradually decay to zero without a significant gap. In particular, the Moore-Penrose pseudoinverse of A, denoted by A^+, is of very large norm. Hence, A is severely ill-conditioned and may be singular. Moveover, the data vector g represents available data and generally is contaminated by an error $e \in R^n$ that may stem from measurement inaccuracies, discretization error, and eletronic noise in the device used, i.e.,

$$g=\hat{g}+e, \tag{2}$$

where $\hat{g}\in R^n$ stands for the (unknown) error-free vector associated with g. Let the unavailable linear system of equation with the error-free right-hand side

$$Af=\hat{g} \tag{3}$$

Manuscript received August 5, 2019; revised October 26, 2019 and December 26, 2019. This work was supported by the National Natural Science Foundations of China (No.10802068), the National Science Foundation for Young Scientists of China (No. 11901123) and the Guangxi Natural Science Foundation (No. 2018JJB110062).

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be consistent and we denote its solution of minimal Euclidean norm by $\hat{f}$. It is our aim to determine an accurate approximation of $\hat{f}$ by computing an approximate solution of the available linear system of Equation (1).

Minimization problems (1) with a matrix of this kind are commonly referred to as linear discrete ill-posed problems. They arise from the suitably discretization of ill-posed problems in various scientific and engineering applications. For example, a large number of practical problems in image processing are the inverse problem of obtaining true data from observation data, and one of the basic problems is the linear inverse problem (1), such as image restoration [16], [22], [23], [25], image decomposition [7], image reconstruction [28] and so on. In addition, ill-posed problems arise from the discretization of a Fredholm integral equation of the first kind on a cube in two or more space-dimensions [14].

The solution of minimal Euclidean norm of (1), given by

$$A^+g=A^+\hat{g}+A^+e=\hat{f}+A^+e,$$

typically is not a meaningful approximate solution of the system (1), because typically $\|A^+e\|_2 \gg \|\hat{f}\|_2$. To compute a useful approximation of $\hat{f}$, the first step in our solution process is to replace (1) by a nearby problem, whose solution is less sensitive to the error e in g. This replacement is commonly referred to as regularization. One of the most popular regularization methods is due to Tikhonov [21], [24], [10]. Tikhonov regularization replaces the linear system (1) by the minimization problem

$$\min_{f} \|Af-g\|_2^2+\mu^2\|Lf\|_2^2, \tag{4}$$

where $\mu > 0$ is referred to as a regularization parameter (generally small, i.e., $0 < \mu < 1$) and the matrix L as a regularization matrix. The parameter μ balances the influence of the first term (the fidelity term) and the second term (the regularization term), which is determined by the regularization matrix. It is the purpose of the regularization term $\mu^2\|Lf\|_2^2$ to damp the propagated error in the computed approximation of $\hat{f}$. The solution of this system (4), less sensitive to the error e in g, is considered as an approximation of the solution of error-free linear system (3). When L is the identity matrix, the Tikhonov minimization problem (4) is said to be in standard form, otherwise it is said to be in general form. In this work, we limit our discussion to L being the identity matrix. The other cases can be obtained by using the similar technique. It is easy to see that the Tikhonov minimization problem is mathematically equivalent to solving the following normal equation

$$(A^TA+\mu^2 I)f=A^Tg. \tag{5}$$

Similar to [19], the problem (5) can be translated into an equivalent $2n \times 2n$ augmented system

$$\underbrace{\begin{pmatrix} I & A \\ -A^T & \mu^2 I \end{pmatrix}}_{\mathcal{K}} \underbrace{\begin{pmatrix} e \\ f \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} g \\ 0 \end{pmatrix}}_{b}, \tag{6}$$

where I and the variable e denote the identity matrix with proper dimension and additive noise $e = g - Af$, respectively. Note that the coefficient matrix $\mathcal{K}$ of the system (6) is non-Hermitian positive-definite. For non-Hermitian positive-definite systems, Bai et al. in [3] studied efficient iterative methods based on the Hermitian and skew-Hermitian splitting (HSS) of the coefficient matrix $\mathcal{K}$. Due to the high efficiency and the robustness of the HSS iteration method, it attracts many researchers attentions, and many HSS variants were proposed in recent years to solve different kinds of systems of linear equations [18], [11], [4]. Recently, for solving augmented system (6), Lv et al. in [19] established a special HSS (SHSS) iterative method by substituting $\alpha = 1$ into the second step of the HSS one, which makes the eigenvalues of the corresponding iteration matrix determined more conveniently. Numerical results included in [19] validate the SHSS iteration method is superior to the HSS one. Then Cheng et al. in [6] used the idea of [19] and proposed a new special HSS (NSHSS) iterative method by setting $\alpha = \mu^2$ in the second step of the HSS one. In [5], Benzi established a generalization of the HSS (GHSS) iteration method for solving positive definite, non-Hermitian linear systems. It is shown that the new scheme can outperform the standard HSS method in some situations. Subsequently, based on GHSS iteration method proposed in [5], Aghazadeh et al. [1] splitted the Hermitian part of $\mathcal{K}$ as sum of a Hermitian positive definite matrix and a Hermitian positive semidefinite matrix, and introduced a restricted version of the GHSS (RGHSS) iterative method for image restoration. Experimental results demonstrated that the RGHSS method is more effective and accurate than the SHSS method. After that, Aminikhah and Yousefi in [2] first constructed a new splitting of the Hermitian part of the coefficient matrix $\mathcal{K}$ for the GHSS method, and further presented a new special GHSS (SGHSS) method for solving ill-posed inverse problems. Lately, Fan et al. in [8] pointed out a class of upper and lower triangular (ULT) splitting iteration methods, the first type of the ULT (ULT-I) splitting iteration method and the second type of the ULT (ULT-II) splitting iteration method, for solving the augmented systems. The ULT-I and ULT-II iteration methods are the forms of

$$\begin{cases} \begin{pmatrix} I & 0 \\ -A^T & \mu^2 I + Q \end{pmatrix} x^{(k+\frac{1}{2})} = \begin{pmatrix} 0 & -A \\ 0 & Q \end{pmatrix} x^{(k)} + b \\ \begin{pmatrix} I & A \\ 0 & \mu^2 I + Q \end{pmatrix} x^{(k+1)} = \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix} x^{(k+\frac{1}{2})} + b \end{cases} \tag{7}$$

and

$$\begin{cases} \begin{pmatrix} I & 0 \\ -A^T & Q \end{pmatrix} x^{(k+\frac{1}{2})} = \begin{pmatrix} 0 & -A \\ 0 & Q - \mu^2 I \end{pmatrix} x^{(k)} + b \\ \begin{pmatrix} I & A \\ 0 & \mu^2 I + Q \end{pmatrix} x^{(k+1)} = \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix} x^{(k+\frac{1}{2})} + b \end{cases}, \tag{8}$$

respectively. The convergence rates and optimal iteration parameters of the ULT iteration methods were derived. And numerical experiments illustrated that the versions of the ULT methods considerably outperform the newly developed methods such as SHSS and RGHSS methods in terms of the numerical performance and image recovering quality.

In this paper, we continue to research on solving the $2n$-by-$2n$ linear system (6). In order to further improve the convergence rates of the above ULT methods, inspired by [27], [26], we introduce the control parameters into the ULT-I and ULT-II splitting iteration methods to construct a class of non-stationary ULT iteration methods. The control parameters involved in our methods are determined by minimizing the corresponding residual norms, thus the non-stationary ULT iteration methods are referred to as the minimize residual ULT (MRULT) iteration ones. It is expected that the MRULT-type methods converge faster than the ULT ones. In addition, the convergence properties of the MRULT-type methods are analyzed.

The arrangement of this paper is organized as follows. In Section II, we define the first type of the minimize residual ULT (MRULT-I) iteration method. The property of the parameters involved in the MRULT-I method is discussed and the corresponding convergence theory is established here. The second type of the minimize residual ULT (MRULT-II) iteration method is presented to solve the augmented systems (6) in Section III. Similarly, we investigate its convergence property for the ill-posed problems. Section IV is devoted to presenting numerical examples to examine the feasibility and effectiveness of the MRULT-I and MRULT-II methods. Finally in Section V, we give a brief conclusion for this paper.

II. THE FIRST TYPE OF THE MRULT METHOD AND ITS CONVERGENCE ANALYSIS

In [27], the authors proposed the minimize residual HSS (MRHSS) iteration method to solve non-Hermitian positive definite linear systems by applying the minimize residual technique to the HSS one. Two parameters involved in MRHSS method are adopted by minimizing the residual norms at each step of the HSS iteration scheme. Numerical experiments in [27] showed that the MRHSS method has advantages over the HSS one. Inspired by [27], [26], we introduce the control parameters into the ULT-I iteration scheme (7) and establish the MRULT-I iteration method to further accelerate the convergence rate of the ULT-I iteration one in this section.

Denote $r^{(k)} = b - \mathcal{K}x^{(k)}$, $r^{(k+\frac{1}{2})} = b - \mathcal{K}x^{(k+\frac{1}{2})}$, and

$$M_1 = \begin{pmatrix} I & 0 \\ -A^T & \mu^2 I + Q \end{pmatrix}, N_1 = \begin{pmatrix} 0 & -A \\ 0 & Q \end{pmatrix},$$
$$M_2 = \begin{pmatrix} I & A \\ 0 & \mu^2 I + Q \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix}, \tag{9}$$

then $\mathcal{K}_1 = M_1 - N_1 = M_2 - N_2$. And the ULT-I iteration scheme (7) can be equivalently written as

$$\begin{cases} x^{(k+\frac{1}{2})} = x^{(k)} + M_1^{-1} r^{(k)} \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + M_2^{-1} r^{(k+\frac{1}{2})} \end{cases}. \tag{10}$$

Enlightened by the idea of [26], [27], we introduce two arbitrary positive parameters β_k and γ_k into (10) to modify the ULT-I method, which leads to a new iteration scheme of the form

$$\begin{cases} x^{(k+\frac{1}{2})} = x^{(k)} + \beta_k M_1^{-1} r^{(k)} \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + \gamma_k M_2^{-1} r^{(k+\frac{1}{2})} \end{cases}. \tag{11}$$

As mentioned in [27], the parameters β_k and γ_k involved in the iteration scheme (11) are to control the step sizes. If they are determined by minimizing the corresponding residual norm, the convergence rate of (11) may be faster than that of (10). Thus, we adopt the parameters β_k and γ_k to minimize the residual norm at each step of the iteration scheme (11), and obtain the minimize residual ULT-I (MRULT-I) iteration method. Note that (1) is a real linear system, so it may be proper to adopt $\beta_k, \gamma_k > 0$ here. The residual form of the iteration scheme (11) can be expressed as

$$\begin{cases} r^{(k+\frac{1}{2})} = r^{(k)} - \beta_k \mathcal{K} M_1^{-1} r^{(k)} \\ r^{(k+1)} = r^{(k+\frac{1}{2})} - \gamma_k \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})} \end{cases}. \tag{12}$$

By simple calculation, the above two residual norms yield

$$\begin{aligned} &\|r^{(k+\frac{1}{2})}\|_2^2 \\ &= (r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}) \\ &= (r^{(k)} - \beta_k \mathcal{K} M_1^{-1} r^{(k)}, r^{(k)} - \beta_k \mathcal{K} M_1^{-1} r^{(k)}) \\ &= (r^{(k)}, r^{(k)}) - (r^{(k)}, \beta_k \mathcal{K} M_1^{-1} r^{(k)}) - (\beta_k \mathcal{K} M_1^{-1} r^{(k)}, r^{(k)}) \\ &\quad + \beta_k^2 \|\mathcal{K} M_1^{-1} r^{(k)}\|_2^2 \\ &= \|r^{(k)}\|_2^2 - 2\beta_k (r^{(k)}, \mathcal{K} M_1^{-1} r^{(k)}) + \beta_k^2 \|\mathcal{K} M_1^{-1} r^{(k)}\|_2^2 \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\|r^{(k+1)}\|_2^2 \\ &= (r^{(k+1)}, r^{(k+1)}) \\ &= (r^{(k+\frac{1}{2})} - \gamma_k \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})} - \gamma_k \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}) \\ &= (r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}) - (r^{(k+\frac{1}{2})}, \gamma_k \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}) \\ &\quad - (\gamma_k \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}, r^{(k+\frac{1}{2})}) + \gamma_k^2 \|\mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}\|_2^2 \\ &= \|r^{(k+\frac{1}{2})}\|_2^2 - 2\gamma_k (r^{(k+\frac{1}{2})}, \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}) \\ &\quad + \gamma_k^2 \|\mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}\|_2^2. \end{aligned} \tag{14}$$

Actually, $(r^{(k)}, \mathcal{K} M_1^{-1} r^{(k)}) = (\mathcal{K} M_1^{-1} r^{(k)}, r^{(k)})$ and $(r^{(k+\frac{1}{2})}, \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}) = (\mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}, r^{(k)})$ are due to the fact that $\mathcal{K} M_1^{-1}, \mathcal{K} M_2^{-1}$ and $r^{(k)}, r^{(k+\frac{1}{2})}$ are real matrices. By solving the equations $\frac{\partial \|r^{(k+\frac{1}{2})}\|_2^2}{\partial \beta_k} = 0$ and $\frac{\partial \|r^{(k+1)}\|_2^2}{\partial \gamma_k} = 0$, one then obtains

$$\beta_k = \frac{(r^{(k)}, \mathcal{K} M_1^{-1} r^{(k)})}{\|\mathcal{K} M_1^{-1} r^{(k)}\|_2^2}, \gamma_k = \frac{(r^{(k+\frac{1}{2})}, \mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})})}{\|\mathcal{K} M_2^{-1} r^{(k+\frac{1}{2})}\|_2^2}. \tag{15}$$

It is not difficult to check that $\frac{\partial^2 \|r^{(k+\frac{1}{2})}\|_2^2}{\partial \beta_k^2} > 0$ and $\frac{\partial^2 \|r^{(k+1)}\|_2^2}{\partial \gamma_k^2} > 0$, which reveals that β_k and γ_k in (15) are the minimum points for each of the functions $\|r^{(k+\frac{1}{2})}\|_2^2$ and $\|r^{(k+1)}\|_2^2$, respectively.

In summary, the detailed implementation of the MRULT-I iteration method can be listed as follows.

Algorithm 2.1:

1. Let $\beta, \gamma > 0$, and given an initial value $f^{(0)}$ and $e^{(0)} = g - Af^{(0)}$ with g being the available vector. Given $\tau > 0$ and M is the maximum prescribed number of outer iterations,
2. $r^{(0)} = b - \mathcal{K} x^{(0)}$, and divide $r^{(0)}$ into $(r_1^{(0)}; r_2^{(0)})$ with $r_1^{(0)}, r_2^{(0)} \in R^n$;
3. For $k = 0, 1, 2, \cdots$, until $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} > \tau$ or $k < M$,
4. compute $t_1 = A^T r_1^{(k)} + r_2^{(k)}$;
5. solve $(\mu^2 I + Q) t_2 = t_1$;
6. compute $t_3 = r_1^{(k)} + A t_2$ and $t_4 = -A^T r_1^{(k)} + \mu^2 t_2$;
7. compute the value of β_k: $\beta_k = \frac{(r_1^{(k)})^T t_3 + (r_2^{(k)})^T t_4}{\|t_3\|_2^2 + \|t_4\|_2^2}$;
8. compute $e^{(k+\frac{1}{2})} = e^{(k)} + \beta_k r_1^{(k)}$ and $f^{(k+\frac{1}{2})} = f^{(k)} + \beta_k t_2$;
9. compute $r_1^{(k+\frac{1}{2})} = r_1^{(k)} - \beta_k t_3$ and $r_2^{(k+\frac{1}{2})} = r_2^{(k)} - \beta_k t_4$;
10. solve $(\mu^2 I + Q) \bar{t}_2 = r_2^{(k+\frac{1}{2})}$;
11. compute $\bar{t}_3 = r_1^{(k+\frac{1}{2})}$ and $\bar{t}_4 = -A^T r_1^{(k)} + (\mu^2 I + A^T A) \bar{t}_2$;
12. compute the value of γ_k: $\gamma_k = \frac{(r_1^{(k+\frac{1}{2})})^T \bar{t}_3 + (r_2^{(k+\frac{1}{2})})^T \bar{t}_4}{\|\bar{t}_3\|_2^2 + \|\bar{t}_4\|_2^2}$;
13. compute $e^{(k+1)} = e^{(k+\frac{1}{2})} + \gamma_k (r_1^{(k+\frac{1}{2})} - A\bar{t}_2)$ and $f^{(k+1)} = f^{(k+\frac{1}{2})} + \gamma_k \bar{t}_2$;
14. compute $r_1^{(k+1)} = r_1^{(k+\frac{1}{2})} - \gamma_k \bar{t}_3$ and $r_2^{(k+1)} = r_2^{(k+\frac{1}{2})} - \gamma_k \bar{t}_4$;
15. end for

Remark 2.1: When $\beta_k = \gamma_k = 1$, the MRULT-I iteration method is exactly the ULT-I one presented in [8], which implies that the MRULT-I iteration method with proper parameters may be more efficient than the ULT-I one.

In the sequel, we first discuss the properties of the parameters β_k and γ_k, and then analyze the convergence of the MRULT-I iteration method. It follows from the derivations of β_k and γ_k that the parameters β_k and γ_k in (15) are the minimum points of the residual norms $\|r^{(k+\frac{1}{2})}\|$ and $\|r^{(k+1)}\|$, respectively. While, the residual norm $\|r^{(k+1)}\|$ can also be viewed as a real function of the real variable (β_k, γ_k). We will demonstrate that (β_k, γ_k) defined by (15) is the minimum point of $\|r^{(k+1)}\|$. To this end, we start with a lemma which is useful in our following proof.

Lemma 2.1: Let $\mathcal{K}$ and M_2 be defined as in (6) and (9), respectively, and symmetric positive definite $Q \in R^{n \times n}$ satisfy $(Q - A^T A)(\mu^2 I + Q)^{-1} = (\mu^2 I + Q)^{-1}(Q - A^T A)$. For any vector $z \in R^{2n}$, we have

$$\frac{d(z, \mathcal{K} M_2^{-1} z)}{d(\|z\|_2^2)} = \frac{1}{\|z\|_2^2}(z, \mathcal{K} M_2^{-1} z).$$

Proof. It follows from (9) that

$$\begin{aligned} \mathcal{K} M_2^{-1} &= I - N_2 M_2^{-1} \\ &= I - \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix} \begin{pmatrix} I & A \\ 0 & \mu^2 I + Q \end{pmatrix}^{-1} \\ &= I - \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix} \begin{pmatrix} I & -A(\mu^2 I + Q)^{-1} \\ 0 & (\mu^2 I + Q)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -A^T & I - (Q - A^T A)(\mu^2 I + Q)^{-1} \end{pmatrix}. \end{aligned}$$

We now divide the vector z into two parts $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ with $z_1, z_2 \in R^n$, then the inner product $(z, \mathcal{K} M_2^{-1} z)$ has the form of

$$\begin{aligned} &(z, \mathcal{K} M_2^{-1} z) \\ &= z_1^T z_1 - z_2^T A^T z_1 + z_2^T z_2 - z_2^T (Q - A^T A)(\mu^2 I + Q)^{-1} z_2. \end{aligned}$$

Inasmuch as $(Q - A^T A)(\mu^2 I + Q)^{-1} = (\mu^2 I + Q)^{-1}(Q - A^T A)$, the first-order derivative is as follows

$$\begin{aligned} &\frac{d(z, \mathcal{K} M_2^{-1} z)}{dz} \\ &= \begin{pmatrix} 2z_1 - A z_2 \\ -A^T z_1 + 2z_2 - 2(Q - A^T A)(\mu^2 I + Q)^{-1} z_2 \end{pmatrix}. \end{aligned} \tag{16}$$

From the proof of Lemma 1 in [27], one has

$$\frac{dz}{d(\|z\|_2^2)} = \frac{1}{2\|z\|_2^2} z,$$

which together with (16) leads to

$$\frac{d(z,\mathcal{K}M_2^{-1}z)}{d(\|z\|_2^2)} = (\frac{d(z,\mathcal{K}M_2^{-1}z)}{dz}, \frac{dz}{d(\|z\|_2^2)}) = \frac{1}{\|z\|_2^2}(z,\mathcal{K}M_2^{-1}z).$$

This completes the proof of this lemma.

By Lemma 2.1 and with the similar manner applied in the proof of Theorem 1 in [27], the fact that (β_k,γ_k) defined by (15) is the minimum point of $\|r^{(k+1)}\|$ is demonstrated in the following theorem.

Theorem 2.1: Let the symmetric positive definite matrix $Q \in R^{n\times n}$ satisfy $(Q - A^TA)(\mu^2 I + Q)^{-1} = (\mu^2 I + Q)^{-1}(Q-A^TA)$. Then the real variable pair (β_k,γ_k) defined by (15) is the minimum point of $\|r^{(k+1)}\|$ of the MRULT-I iteration scheme (11), which means the values of (β_k,γ_k) defined by (15) are optimal in the real field R.

Proof. Let $\hat{r}(\tau) = r^{(k)} - \tau\mathcal{K}M_1^{-1}r^{(k)}$ and

$$\phi(\hat{r}(\tau),\omega) = \|\hat{r}\|_2^2 - 2\omega(\hat{r},\mathcal{K}M_2^{-1}\hat{r}) + \omega^2\|\mathcal{K}M_2^{-1}\hat{r}\|_2^2.$$

Thus, we have

$$\hat{r}(\beta_k) = r^{(k+\frac{1}{2})}, \quad \phi(\hat{r}(\beta_k),\gamma_k) = \|r^{(k+1)}\|_2^2,$$

and

$$\|\hat{r}\|^2 = \|r^{(k)}\|_2^2 - 2\tau(r^{(k)},\mathcal{K}M_1^{-1}r^{(k)}) + \tau^2\|\mathcal{K}M_1^{-1}r^{(k)}\|_2^2.$$

By Lemma 2.1 and Lemma 1 in [27], one may deduce the following result

$$\frac{\partial\phi}{\partial(\|\hat{r}\|^2)} = \frac{\|\hat{r}\|^2}{\|\hat{r}\|^2} - \frac{2\omega}{\|\hat{r}\|^2}(\hat{r},\mathcal{K}M_2^{-1}\hat{r}) + \frac{\omega^2}{\|\hat{r}\|^2}\|\mathcal{K}M_2^{-1}\hat{r}\|_2^2 = \frac{\phi}{\|\hat{r}\|^2}.$$

Denote $\hat{\phi}(\tau,\omega) = \phi(\hat{r}(\tau),\omega)$, then the first-order partial derivatives of $\hat{\phi}(\tau,\omega)$ are as follows

$$\begin{aligned}\frac{\partial\hat{\phi}}{\partial\tau} &= \frac{\partial\phi}{\partial(\|\hat{r}\|^2)}\frac{d(\|\hat{r}\|^2)}{d\tau}\\ &= \frac{2\phi}{\|\hat{r}\|^2}\left(\tau\|\mathcal{K}M_1^{-1}r^{(k)}\|_2^2 - (r^{(k)},\mathcal{K}M_1^{-1}r^{(k)})\right),\\ \frac{\partial\hat{\phi}}{\partial\omega} &= 2\left(\omega\|\mathcal{K}M_2^{-1}\hat{r}\|_2^2 - (\hat{r},\mathcal{K}M_2^{-1}\hat{r})\right).\end{aligned}$$

It can be seen that the real number pair (β_k,γ_k) defined by (15) is the unique stationary point of the function $\hat{\phi}$. Moreover, (β_k,γ_k) is the minimum point of the function $\hat{\phi}$. Denote

$$\Phi_1(\tau) = \tau\|\mathcal{K}M_1^{-1}r^{(k)}\|_2^2 - (r^{(k)},\mathcal{K}M_1^{-1}r^{(k)})$$

and

$$\Phi_2(\tau,\omega) = \omega\|\mathcal{K}M_2^{-1}\hat{r}\|_2^2 - (\hat{r},\mathcal{K}M_2^{-1}\hat{r}).$$

Then, the second-order partial derivatives of $\hat{\phi}(\tau,\omega)$ are

$$\begin{aligned}\frac{\partial^2\hat{\phi}}{\partial\tau^2} &= \Phi_1(\tau)\frac{\partial}{\partial\tau}\left(\frac{2\phi}{\|\hat{r}\|^2}\right) + \frac{2\phi}{\|\hat{r}\|^2}\|\mathcal{K}M_1^{-1}r^{(k)}\|_2^2,\\ \frac{\partial^2\hat{\phi}}{\partial\tau\partial\omega} &= \Phi_1(\tau)\frac{\partial}{\partial\omega}\left(\frac{2\phi}{\|\hat{r}\|^2}\right),\\ \frac{\partial^2\hat{\phi}}{\partial\omega\partial\tau} &= 2\frac{\partial\Phi_2(\tau,\omega)}{\partial\|\hat{r}\|^2}\frac{d(\|\hat{r}\|^2)}{d\tau} = 2\frac{\Phi_2(\tau,\omega)}{\|\hat{r}\|^2}\frac{d(\|\hat{r}\|^2)}{d\tau},\\ \frac{\partial^2\hat{\phi}}{\partial\omega^2} &= 2\|\mathcal{K}M_2^{-1}\hat{r}\|_2^2.\end{aligned}$$

Keep in mind that $\Phi_1(\beta_k) = 0$ and $\Phi_2(\beta_k,\gamma_k) = 0$, then the Hessian matrix of $\hat{\phi}$ at this stationary point (β_k,γ_k) has the form of

$$\begin{pmatrix} \frac{\|r^{(k+1)}\|_2^2\|\mathcal{K}M_1^{-1}r^{(k)}\|_2^2}{\|r^{(k+\frac{1}{2})}\|_2^2} & 0\\ 0 & 2\|\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})}\|_2^2\end{pmatrix}.$$

It is easy to see that the Hessian matrix is Hermitian positive definite, which implies the stationary point (β_k,γ_k) defined by (15) is the unique minimum point of the function $\hat{\phi}$. Up to now, the proof has been completed.

Based on Theorem 2.1, we give the following theorem about the relation of $\|r^{(k+1)}\|$ and $\|r^{(k)}\|$.

Theorem 2.2: For the augmented system (6), if the parameter matrix $Q \in R^{n\times n}$ is symmetric positive definite and satisfies $(Q-A^TA)(\mu^2I+Q)^{-1} = (\mu^2I+Q)^{-1}(Q-A^TA)$, then the residual norm of the MRULT-I iteration method satisfies the following relation

$$\|r^{(k+1)}\|_2 \le \|\mathcal{A}\|_2\|r^{(k)}\|_2, \tag{17}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0\\ GA^T & G\end{pmatrix}$$

with $G = -A^TA(\mu^2I + Q)^{-1} + (Q - A^TA)(\mu^2I + Q)^{-1}Q(\mu^2I + Q)^{-1}$.

Proof. From (12), the residual $r^{(k+1)}$ of the MRULT-I method yields

$$\begin{aligned}&r^{(k+1)}\\ &= r^{(k)} - \beta_k\mathcal{K}M_1^{-1}r^{(k)} - \gamma_k\mathcal{K}M_2^{-1}\left(r^{(k)} - \beta_k\mathcal{K}M_1^{-1}r^{(k)}\right)\\ &= \left(I - \beta_k\mathcal{K}M_1^{-1} - \gamma_k\mathcal{K}M_2^{-1} + \beta_k\gamma_k\mathcal{K}M_2^{-1}\mathcal{K}M_1^{-1}\right)r^{(k)}\\ &= (I - \gamma_k\mathcal{K}M_2^{-1})(I - \beta_k\mathcal{K}M_1^{-1})r^{(k)},\end{aligned}$$

which together with (β_k,γ_k) being the minimum point of the function $\|r^{(k+1)}\|_2$ leads to

$$\begin{aligned}\|r^{(k+1)}\|_2 &= \|(I - \gamma_k\mathcal{K}M_2^{-1})(I - \beta_k\mathcal{K}M_1^{-1})r^{(k)}\|_2\\ &\le \|(I - \mathcal{K}M_2^{-1})(I - \mathcal{K}M_1^{-1})r^{(k)}\|_2\\ &\le \|(I - \mathcal{K}M_2^{-1})(I - \mathcal{K}M_1^{-1})\|_2\|r^{(k)}\|_2\\ &= \|N_2M_2^{-1}N_1M_1^{-1}\|_2\|r^{(k)}\|_2. \end{aligned}\tag{18}$$

The last equation is due to $I - \mathcal{K}M_2^{-1} = N_2M_2^{-1}$ and $I - \mathcal{K}M_1^{-1} = N_1M_1^{-1}$. Next, we analyze the structure of the matrix $N_2M_2^{-1}N_1M_1^{-1}$. By employing algebraic manipulations, we obtain

$$\begin{aligned}&N_2M_2^{-1}N_1M_1^{-1}\\ &= \begin{pmatrix}0 & 0\\ A^T & Q\end{pmatrix}\begin{pmatrix}I & A\\ 0 & \mu^2I+Q\end{pmatrix}^{-1}\\ &\times\begin{pmatrix}0 & -A\\ 0 & Q\end{pmatrix}\begin{pmatrix}I & 0\\ -A^T & \mu^2I+Q\end{pmatrix}^{-1}\\ &= \begin{pmatrix}0 & 0\\ A^T & Q\end{pmatrix}\begin{pmatrix}I & -A(\mu^2I+Q)^{-1}\\ 0 & (\mu^2I+Q)^{-1}\end{pmatrix}\\ &\times\begin{pmatrix}0 & -A\\ 0 & Q\end{pmatrix}\begin{pmatrix}I & 0\\ (\mu^2I+Q)^{-1}A^T & (\mu^2I+Q)^{-1}\end{pmatrix}\\ &= \begin{pmatrix}0 & 0\\ 0 & -A^TA + (Q-A^TA)(\mu^2I+Q)^{-1}Q\end{pmatrix}\\ &\times\begin{pmatrix}I & 0\\ (\mu^2I+Q)^{-1}A^T & (\mu^2I+Q)^{-1}\end{pmatrix}\\ &= \begin{pmatrix}0 & 0\\ GA^T & G\end{pmatrix},\end{aligned}$$

where $G = -A^TA(\mu^2I+Q)^{-1} + (Q - A^TA)(\mu^2I+Q)^{-1}Q(\mu^2I+Q)^{-1}$. It follows from the above proof that the conclusion (17) holds.

In particular, we study the convergence of the MRULT-I iteration method with $Q = sI$ and $Q = sI + A^TA$ $(s > 0)$ in detail, respectively. To this end, the following lemma is firstly presented.

Lemma 2.2: Let $X = diag(x_1, x_2, \cdots, x_n)$ and $Y = diag(y_1, y_2, \cdots, y_n)$ be two real diagonal matrices, then

$$\left\| \begin{pmatrix} 0 & 0 \\ XY & X \end{pmatrix} \right\|_2 = \max_{1\le i\le n} |x_i|\sqrt{1+y_i^2}.$$

Proof. The proof of this lemma is similar to that of Lemma 4.1 in [17], so we omit it here.

Theorem 2.3: For the augmented system (6), the MRULT-I iteration method with $Q = sI(s > 0)$ is convergent if the following conditions

$$\begin{cases} f_1(s) = (1+\cos\theta)s^2 + 2s(\mu^2\cos\theta - \sigma_1^2) \\ \qquad + \mu^2(\mu^2\cos\theta - \sigma_1^2) > 0 \\ f_2(s) = (1-\cos\theta)s^2 - 2s(\mu^2\cos\theta + \sigma_n^2) \\ \qquad - \mu^2(\mu^2\cos\theta + \sigma_n^2) < 0 \end{cases} \tag{19}$$

hold true, where $\tan\theta = \sigma_1$.

Proof. Since the matrix $Q = sI$ satisfies $(Q - A^TA)(\mu^2I + Q)^{-1} = (\mu^2I+Q)^{-1}(Q - A^TA)$, it follows from Theorem 2.2 that

$$\|r^{(k+1)}\|_2 \le \left\| \begin{pmatrix} 0 & 0 \\ HA^T & H \end{pmatrix} \right\|_2 \|r^{(k)}\|_2$$

with $H = \frac{-A^TA}{\mu^2+s} + \frac{s(sI-A^TA)}{(\mu^2+s)^2}$. The singular value decomposition (SVD) of A is defined as

$$A = U\Sigma V^T, \tag{20}$$

where $U = [u_1, u_2, \cdots, u_n], V = [v_1, v_2, \cdots, v_n] \in R^{n\times n}$ are orthogonal matrices and $\Sigma = \mathrm{diag}[\sigma_1, \sigma_2, \cdots, \sigma_n] \in R^{n\times n}$ has non-negative diagonal elements appearing in non-increasing order such that $\sigma_1 \ge \sigma_2 \ge \cdots \sigma_n \ge 0$. The numbers σ_i are the singular values of A. By (20), straightforward computation reveals that

$$\begin{aligned} &\|r^{(k+1)}\|_2 \\ \le &\left\| \begin{pmatrix} 0 & 0 \\ HA^T & H \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ = &\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \check{\Lambda}\Sigma^T & \check{\Lambda} \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ = &\left\| \begin{pmatrix} 0 & 0 \\ \check{\Lambda}\Sigma^T & \check{\Lambda} \end{pmatrix} \right\|_2 \|r^{(k)}\|_2, \end{aligned}$$

where $\check{\Lambda} = \frac{-\Sigma^T\Sigma}{\mu^2+s} + \frac{s(sI-\Sigma^T\Sigma)}{(\mu^2+s)^2}$. Therefore, by making use of Lemma 2.2, we can conduct the estimate

$$\begin{aligned} &\|r^{(k+1)}\|_2 \\ \le &\left\| \begin{pmatrix} 0 & 0 \\ \check{\Lambda}\Sigma^T & \check{\Lambda} \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ \le &\max_{1\le i\le n} \left\{ \left| \frac{-\sigma_i^2(\mu^2+s) + s(s-\sigma_i^2)}{(\mu^2+s)^2} \right| \sqrt{1+\sigma_i^2} \right\} \|r^{(k)}\|_2. \end{aligned} \tag{21}$$

To make the MRULT-I iteration method with $Q = sI$ convergent, it is enough to have

$$\left| \frac{-\sigma_i^2(\mu^2+s) + s(s-\sigma_i^2)}{(\mu^2+s)^2} \right|^2 (1+\sigma_i^2) < 1 \tag{22}$$

for all $\sigma_i (1 \le i \le n)$. For notational simplicity we denote by

$$c_i = \left| \frac{-\sigma_i^2(\mu^2+s) + s(s-\sigma_i^2)}{(\mu^2+s)^2} \right|.$$

If $c_i < \cos\theta$ and $\sigma_1 = \tan\theta$ hold for all $1 \le i \le n$, then $c_i^2(1+\sigma_i^2) < \cos^2\theta(1+\tan^2\theta) = 1$, which makes Inequality (22) valid. Thus, we only solve the inequality $c_i < \cos\theta$, i.e.,

$$-\cos\theta < \frac{-\sigma_i^2(\mu^2+s) + s(s-\sigma_i^2)}{(\mu^2+s)^2} < \cos\theta. \tag{23}$$

Due to the fact that $\frac{-\sigma_i^2(\mu^2+s)+s(s-\sigma_i^2)}{(\mu^2+s)^2} = \frac{-\sigma_i^2(\mu^2+2s)+s^2}{(\mu^2+s)^2}$ is a strictly monotone decreasing function with regard to σ_i^2, Inequality (23) is equivalent to

$$\frac{-\sigma_1^2(\mu^2+s) + s(s-\sigma_1^2)}{(\mu^2+s)^2} > -\cos\theta$$

and

$$\frac{-\sigma_n^2(\mu^2+s) + s(s-\sigma_n^2)}{(\mu^2+s)^2} < \cos\theta.$$

Directly solving the above inequalities yields the sufficient conditions (19), which guarantee the convergence of the MRULT-I iteration method with $Q = sI(s > 0)$.

Theorem 2.4: For the augmented system (6), the MRULT-I iteration method with $Q = sI + A^TA(s > 0)$ is convergent if the following conditions

$$\begin{cases} f_1(s) = (1+\cos\theta)s^2 + 2s(\mu^2+\sigma_1^2)\cos\theta \\ \qquad + (\mu^2+\sigma_1^2)[\cos\theta(\mu^2+\sigma_1^2) - \sigma_1^2] > 0 \\ f_2(s) = (1-\cos\theta)s^2 - 2s(\mu^2+\sigma_n^2)\cos\theta \\ \qquad - (\mu^2+\sigma_n^2)[\cos\theta(\mu^2+\sigma_n^2) + \sigma_n^2] < 0 \end{cases} \tag{24}$$

hold true, where $\tan\theta = \sigma_1$.

Proof. The matrix $Q = sI + A^TA$ satisfies $(Q - A^TA)(\mu^2I + Q)^{-1} = (\mu^2I+Q)^{-1}(Q - A^TA)$, thus, from Theorem 2.2, we have

$$\|r^{(k+1)}\|_2 \le \left\| \begin{pmatrix} 0 & 0 \\ WA^T & W \end{pmatrix} \right\|_2 \|r^{(k)}\|_2, \tag{25}$$

where $W = -A^TA[(\mu^2+s)I + A^TA]^{-1} + s[(\mu^2+s)I + A^TA]^{-1}(sI + A^TA)[(\mu^2+s)I + A^TA]^{-1}$. By means of (20), concrete computations give

$$W = V\Lambda V^T$$

and

$$WA^T = V\Lambda\Sigma^TU^T,$$

where $\Lambda = -\Sigma^T\Sigma\left[(\mu^2+s)I + \Sigma^T\Sigma\right]^{-1} + s\left[(\mu^2+s)I + \Sigma^T\Sigma\right]^{-1}(sI + \Sigma^T\Sigma)\left[(\mu^2+s)I + \Sigma^T\Sigma\right]^{-1}$ is a diagonal matrix. Submitting the above equations into (25) yields

$$\begin{aligned} &\|r^{(k+1)}\|_2 \\ \le &\left\| \begin{pmatrix} 0 & 0 \\ V\Lambda\Sigma^TU^T & V\Lambda V^T \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ = &\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \Lambda\Sigma^T & \Lambda \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ = &\left\| \begin{pmatrix} 0 & 0 \\ \Lambda\Sigma^T & \Lambda \end{pmatrix} \right\|_2 \|r^{(k)}\|_2. \end{aligned}$$

Taking into account that $\Lambda\Sigma^T$ and Λ are both diagonal matrices and using Lemma 2.2, we infer that

$$\begin{aligned}&\|r^{(k+1)}\|_2\\ \leq\ &\max_{1\leq i\leq n}\left\{\left|\frac{(s+\sigma_i^2)(s-\sigma_i^2)-\mu^2\sigma_i^2}{(\mu^2+s+\sigma_i^2)^2}\right|\sqrt{1+\sigma_i^2}\right\}\|r^{(k)}\|_2.\end{aligned}$$

The MRULT-I iteration method with $Q=sI+A^TA$ is convergent if and only if

$$\left|\frac{(s+\sigma_i^2)(s-\sigma_i^2)-\mu^2\sigma_i^2}{(\mu^2+s+\sigma_i^2)^2}\right|^2(1+\sigma_i^2)<1 \tag{26}$$

for all $\sigma_i(1\leq i\leq n)$. Let us denote

$$\hat{c}_i=\left|\frac{(s+\sigma_i^2)(s-\sigma_i^2)-\mu^2\sigma_i^2}{(\mu^2+s+\sigma_i^2)^2}\right|.$$

Similar to the proof of Theorem 2.3, if $\hat{c}_i<\cos\theta$ and $\sigma_1=\tan\theta$ hold for all $1\leq i\leq n$, then $\hat{c}_i^2(1+\sigma_i^2)<\cos^2\theta(1+\tan^2\theta)=1$, which is exactly (26). Moreover, $\hat{c}_i<\cos\theta$ is equivalent to

$$-\cos\theta<\frac{(s+\sigma_i^2)(s-\sigma_i^2)-\mu^2\sigma_i^2}{(\mu^2+s+\sigma_i^2)^2}<\cos\theta. \tag{27}$$

It is not difficult to verify that $\frac{(s+\sigma_i^2)(s-\sigma_i^2)-\mu^2\sigma_i^2}{(\mu^2+s+\sigma_i^2)^2}=\frac{s^2-\sigma_i^2(\mu^2+\sigma_i^2)}{(\mu^2+s+\sigma_i^2)^2}$ is a strictly monotone decreasing function with regard to σ_i^2, so Inequality (27) holds if and only if

$$\frac{(s+\sigma_1^2)(s-\sigma_1^2)-\mu^2\sigma_1^2}{(\mu^2+s+\sigma_1^2)^2}>-\cos\theta$$

and

$$\frac{(s+\sigma_n^2)(s-\sigma_n^2)-\mu^2\sigma_n^2}{(\mu^2+s+\sigma_n^2)^2}<\cos\theta.$$

The sufficient conditions (24) for the convergence of the MRULT-I iteration method with $Q=sI+A^TA$ follow by solving the above inequalities.

III. The Second Type of MRULT Method and Its Convergence Analysis

In this section, we discuss the second type of the MRULT (MRULT-II) iteration method for solving the augmented systems (6) by modifying the ULT-II iteration scheme (8). Denote

$$K_1=\begin{pmatrix}I&0\\-A^T&Q\end{pmatrix},L_1=\begin{pmatrix}0&-A\\0&Q-\mu^2I\end{pmatrix},$$

then an equivalent form of the ULT-II iteration scheme (8) is as follows

$$\begin{cases}x^{(k+\frac{1}{2})}=x^{(k)}+K_1^{-1}r^{(k)}\\ x^{(k+1)}=x^{(k+\frac{1}{2})}+M_2^{-1}r^{(k+\frac{1}{2})}\end{cases}, \tag{28}$$

where $r^{(k)}$ and $r^{(k+\frac{1}{2})}$ are defined as in Section II. Using the similar acceleration techniques of [27], [26], we propose the MRULT-II method incorporating two positive parameters $\bar{\beta}_k$ and $\bar{\gamma}_k$ to solve the augmented systems (6), which yields the following iteration scheme

$$\begin{cases}x^{(k+\frac{1}{2})}=x^{(k)}+\bar{\beta}_kK_1^{-1}r^{(k)}\\ x^{(k+1)}=x^{(k+\frac{1}{2})}+\bar{\gamma}_kM_2^{-1}r^{(k+\frac{1}{2})}\end{cases}. \tag{29}$$

The residual form of the iteration scheme (29) can be written as

$$\begin{cases}r^{(k+\frac{1}{2})}=r^{(k)}-\bar{\beta}_k\mathcal{K}K_1^{-1}r^{(k)}\\ r^{(k+1)}=r^{(k+\frac{1}{2})}-\bar{\gamma}_k\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})}\end{cases}. \tag{30}$$

By virtue of the same technique of the MRULT-I iteration method, the parameters $\bar{\beta}_k$ and $\bar{\gamma}_k$ in the MRULT-II iteration method are determined by minimizing the residual norms $\|r^{(k+\frac{1}{2})}\|$ and $\|r^{(k+1)}\|$, respectively. As a matter of fact, by direct computation, it has

$$\begin{aligned}&\|r^{(k+\frac{1}{2})}\|_2^2\\ =\ &(r^{(k+\frac{1}{2})},r^{(k+\frac{1}{2})})\\ =\ &(r^{(k)}-\bar{\beta}_k\mathcal{K}K_1^{-1}r^{(k)},r^{(k)}-\bar{\beta}_k\mathcal{K}K_1^{-1}r^{(k)})\\ =\ &(r^{(k)},r^{(k)})-(r^{(k)},\bar{\beta}_k\mathcal{K}K_1^{-1}r^{(k)})-(\bar{\beta}_k\mathcal{K}K_1^{-1}r^{(k)},r^{(k)})\\ &+\bar{\beta}_k^2\|\mathcal{K}K_1^{-1}r^{(k)}\|_2^2\\ =\ &\|r^{(k)}\|_2^2-2\bar{\beta}_k(r^{(k)},\mathcal{K}K_1^{-1}r^{(k)})+\bar{\beta}_k^2\|\mathcal{K}K_1^{-1}r^{(k)}\|_2^2\end{aligned}$$

and

$$\begin{aligned}&\|r^{(k+1)}\|_2^2\\ =\ &(r^{(k+1)},r^{(k+1)})\\ =\ &(r^{(k+\frac{1}{2})}-\bar{\gamma}_k\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})},r^{(k+\frac{1}{2})}-\bar{\gamma}_k\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})})\\ =\ &(r^{(k+\frac{1}{2})},r^{(k+\frac{1}{2})})-(r^{(k+\frac{1}{2})},\bar{\gamma}_k\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})})\\ &-(\bar{\gamma}_k\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})},r^{(k+\frac{1}{2})})+\bar{\gamma}_k^2\|\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})}\|_2^2\\ =\ &\|r^{(k+\frac{1}{2})}\|_2^2-2\bar{\gamma}_k(r^{(k+\frac{1}{2})},\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})})\\ &+\bar{\gamma}_k^2\|\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})}\|_2^2.\end{aligned}$$

As discussed in Section II, one then obtains

$$\bar{\beta}_k=\frac{(r^{(k)},\mathcal{K}K_1^{-1}r^{(k)})}{\|\mathcal{K}K_1^{-1}r^{(k)}\|_2^2},\bar{\gamma}_k=\frac{(r^{(k+\frac{1}{2})},\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})})}{\|\mathcal{K}M_2^{-1}r^{(k+\frac{1}{2})}\|_2^2}. \tag{31}$$

Therefore, we have the following specific description of the MRULT-II iteration method.

Algorithm 3.1:

1. Let $\bar{\beta},\bar{\gamma}>0$, and given an initial value $f^{(0)}$ and $e^{(0)}=g-Af^{(0)}$ with g being the available vector. Given $\tau>0$ and M is the maximum prescribed number of outer iterations,
2. $r^{(0)}=b-\mathcal{K}x^{(0)}$, and divide $r^{(0)}$ into $(r_1^{(0)};r_2^{(0)})$ with $r_1^{(0)},r_2^{(0)}\in R^n$;
3. For $k=0,1,2,\cdots$, until $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2}>\tau$ or $k<M$,
4. compute $t_1=A^Tr_1^{(k)}+r_2^{(k)}$;
5. solve $Qt_2=t_1$;
6. compute $t_3=r_1^{(k)}+At_2$ and $t_4=-A^Tr_1^{(k)}+\mu^2t_2$;
7. compute the value of $\bar{\beta}_k$: $\bar{\beta}_k=\frac{(r_1^{(k)})^Tt_3+(r_2^{(k)})^Tt_4}{\|t_3\|_2^2+\|t_4\|_2^2}$;
8. compute $e^{(k+\frac{1}{2})}=e^{(k)}+\bar{\beta}_kr_1^{(k)}$ and $f^{(k+\frac{1}{2})}=f^{(k)}+\bar{\beta}_kt_2$;
9. compute $r_1^{(k+\frac{1}{2})}=r_1^{(k)}-\bar{\beta}_kt_3$ and $r_2^{(k+\frac{1}{2})}=r_2^{(k)}-\bar{\beta}_kt_4$;
10. solve $(\mu^2I+Q)\bar{t}_2=r_2^{(k+\frac{1}{2})}$;
11. compute $\bar{t}_3=r_1^{(k+\frac{1}{2})}$ and $\bar{t}_4=-A^Tr_1^{(k)}+(\mu^2I+A^TA)\bar{t}_2$;
12. compute the value of $\bar{\gamma}_k$: $\bar{\gamma}_k=\frac{(r_1^{(k+\frac{1}{2})})^T\bar{t}_3+(r_2^{(k+\frac{1}{2})})^T\bar{t}_4}{\|\bar{t}_3\|_2^2+\|\bar{t}_4\|_2^2}$;
13. compute $e^{(k+1)}=e^{(k+\frac{1}{2})}+\bar{\gamma}_k(r_1^{(k+\frac{1}{2})}-A\bar{t}_2)$ and $f^{(k+1)}=f^{(k+\frac{1}{2})}+\bar{\gamma}_k\bar{t}_2$;
14. compute $r_1^{(k+1)}=r_1^{(k+\frac{1}{2})}-\bar{\gamma}_k\bar{t}_3$ and $r_2^{(k+1)}=r_2^{(k+\frac{1}{2})}-\bar{\gamma}_k\bar{t}_4$;
15. end for

Remark 3.1: If $\bar{\beta}_k = \bar{\gamma}_k = 1$, the MRULT-II iteration method automatically reduces to the ULT-II one, which means that the convergent rate of the MRULT-II iteration method is at least not slower than that of the ULT-II one.

By making using of Lemma 2.1 and the similar proof of Theorem 2.1 with only technical modifications, we can demonstrate that $\|r^{(k+1)}\|$ of the MRULT-II iteration scheme (29) can attain its minimum at the point $(\bar{\beta}_k, \bar{\gamma}_k)$ defined by (31).

Theorem 3.1: Let symmetric positive definite $Q \in R^{n\times n}$ satisfy $(Q - A^TA)(\mu^2 I + Q)^{-1} = (\mu^2 I + Q)^{-1}(Q - A^TA)$. Then the point $(\bar{\beta}_k, \bar{\gamma}_k)$ defined by (31) is the minimum point of $\|r^{(k+1)}\|$ of the MRULT-II iteration scheme (29), which reveals the values of $(\bar{\beta}_k, \bar{\gamma}_k)$ defined by (31) are optimal in the real field R.

What follows investigates the convergence properties of the proposed method. In particular, based on Theorem 3.1, we derive the following convergence properties of the MRULT-II iteration method with $Q = sI$ and $Q = sI + A^TA$ $(s > 0)$, respectively.

Theorem 3.2: For the augmented system (6), if the parameter matrix $Q \in R^{n\times n}$ is symmetric positive definite and satisfies $(Q - A^TA)(\mu^2 I + Q)^{-1} = (\mu^2 I + Q)^{-1}(Q - A^TA)$, then the residual norm of the MRULT-II iteration method satisfies the following relation

$$\|r^{(k+1)}\|_2 \le \|\bar{\mathcal{A}}\|_2 \|r^{(k)}\|_2, \tag{32}$$

where

$$\bar{\mathcal{A}} = \begin{pmatrix} 0 & 0 \\ \overline{G}A^T & \overline{G} \end{pmatrix}$$

with $\overline{G} = -A^TAQ^{-1} + (Q - A^TA)(\mu^2 I + Q)^{-1}(Q - \mu^2 I)Q^{-1}$.

Proof. Since $(\bar{\beta}_k, \bar{\gamma}_k)$ is the minimum point of the function $\|r^{(k+1)}\|_2$ and the equations $I - \mathcal{K}M_2^{-1} = N_2 M_2^{-1}$ and $I - \mathcal{K}K_1^{-1} = L_1 K_1^{-1}$ hold true, it immediately leads to the following result

$$\begin{aligned} &\|r^{(k+1)}\|_2 \\ &= \|(I - \bar{\gamma}_k \mathcal{K}M_2^{-1})(I - \bar{\beta}_k \mathcal{K}K_1^{-1})r^{(k)}\|_2 \\ &\le \|(I - \mathcal{K}M_2^{-1})(I - \mathcal{K}K_1^{-1})r^{(k)}\|_2 \\ &\le \|(I - \mathcal{K}M_2^{-1})(I - \mathcal{K}K_1^{-1})\|_2 \|r^{(k)}\|_2 \\ &= \|N_2 M_2^{-1} L_1 K_1^{-1}\|_2 \|r^{(k)}\|_2. \end{aligned} \tag{33}$$

Simple matrix calculations result in

$$\begin{aligned} & N_2 M_2^{-1} N_1 K_1^{-1} \\ &= \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix} \begin{pmatrix} I & A \\ 0 & \mu^2 I + Q \end{pmatrix}^{-1} \\ &\times \begin{pmatrix} 0 & -A \\ 0 & Q - \mu^2 I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^T & Q \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & 0 \\ A^T & Q \end{pmatrix} \begin{pmatrix} I & -A(\mu^2 I + Q)^{-1} \\ 0 & (\mu^2 I + Q)^{-1} \end{pmatrix} \\ &\times \begin{pmatrix} 0 & -A \\ 0 & Q - \mu^2 I \end{pmatrix} \begin{pmatrix} I & 0 \\ Q^{-1}A^T & Q^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -A^TA + (Q - A^TA)(\mu^2 I + Q)^{-1}(Q - \mu^2 I) \end{pmatrix} \\ &\times \begin{pmatrix} I & 0 \\ Q^{-1}A^T & Q^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \overline{G}A^T & \overline{G} \end{pmatrix}, \end{aligned}$$

where $\overline{G} = -A^TAQ^{-1} + (Q - A^TA)(\mu^2 I + Q)^{-1}(Q - \mu^2 I)Q^{-1}$. We then immediately get the conclusion (32).

Theorem 3.3: For the augmented system (6), the MRULT-II iteration method with $Q = sI (s > 0)$ is convergent if the parameter s satisfies

$$\frac{\mu^2(1 - \cos\theta) + 2\sigma_1^2}{1 + \cos\theta} < s < \frac{\mu^2(1 + \cos\theta) + 2\sigma_n^2}{1 - \cos\theta} \tag{34}$$

as $\mu^2 > \frac{\sigma_1^2 - \sigma_n^2 - \cos\theta(\sigma_1^2 + \sigma_n^2)}{2\cos\theta}$, where $\tan\theta = \sigma_1$.

Proof. By Theorem 3.2, it has

$$\|r^{(k+1)}\|_2 \le \left\| \begin{pmatrix} 0 & 0 \\ \overline{W}A^T & \overline{W} \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \tag{35}$$

with $\overline{W} = -\frac{A^TA}{s} + \frac{(s - \mu^2)(sI - A^TA)}{s(\mu^2 + s)}$. According to (20), we have

$$\overline{W} = V\bar{\Lambda}V^T$$

and

$$\overline{W}A^T = V\bar{\Lambda}\Sigma^T U^T,$$

where $\bar{\Lambda} = \frac{-(\mu^2 + s)\Sigma^T\Sigma + (s - \mu^2)(sI - \Sigma^T\Sigma)}{s(\mu^2 + s)}$ is a diagonal matrix. Taking the above equations into (35) leads to

$$\begin{aligned} &\|r^{(k+1)}\|_2 \\ &\le \left\| \begin{pmatrix} 0 & 0 \\ V\bar{\Lambda}\Sigma^T U^T & V\bar{\Lambda}V^T \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ &= \left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \bar{\Lambda}\Sigma^T & \bar{\Lambda} \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ &= \left\| \begin{pmatrix} 0 & 0 \\ \bar{\Lambda}\Sigma^T & \bar{\Lambda} \end{pmatrix} \right\|_2 \|r^{(k)}\|_2, \end{aligned}$$

which gives

$$\|r^{(k+1)}\|_2 \le \max_{1\le i\le n} \left\{ \left| \frac{-2\sigma_i^2 + s - \mu^2}{\mu^2 + s} \right| \sqrt{1 + \sigma_i^2} \right\} \|r^{(k)}\|_2$$

in view of Lemma 2.2. It is well known that the MRULT-II iteration method with $Q = sI$ is convergent if and only if

$$\left| \frac{-2\sigma_i^2 + s - \mu^2}{\mu^2 + s} \right|^2 (1 + \sigma_i^2) < 1 \tag{36}$$

for all $\sigma_i (1 \le i \le n)$. Denote by

$$\bar{c}_i = \left| \frac{-2\sigma_i^2 + s - \mu^2}{\mu^2 + s} \right|.$$

With a quite similar strategy utilized in Theorem 2.3, if $\bar{c}_i < \cos\theta$ and $\sigma_1 = \tan\theta$ hold for all $1 \le i \le n$, then $\bar{c}_i^2(1 + \sigma_i^2) < \cos^2\theta(1 + \tan^2\theta) = 1$, i.e., Inequality (36) holds true. Solving the inequality $\bar{c}_i < \cos\theta$ is working out

$$-\cos\theta < \frac{-2\sigma_i^2 + s - \mu^2}{\mu^2 + s} < \cos\theta. \tag{37}$$

It then follows from the decreasing property of $\frac{-2\sigma_i^2 + s - \mu^2}{\mu^2 + s}$ with regard to σ_i^2 that Inequality (37) is equivalent to

$$\frac{-2\sigma_1^2 + s - \mu^2}{\mu^2 + s} > -\cos\theta \text{ and } \frac{-2\sigma_n^2 + s - \mu^2}{\mu^2 + s} < \cos\theta,$$

from which one may deduce the sufficient condition (34) for the convergence of the MRULT-I iteration method with $Q = sI$.

Theorem 3.4: For the augmented system (6), the MRULT-II iteration method with $Q = sI + A^TA (s > 0)$ is convergent if the following conditions

$$\begin{cases} f_1(s) = (1+\cos\theta)s^2 + [(\cos\theta - 1)\mu^2 + 2\sigma_1^2\cos\theta]s \\ \qquad + \sigma_1^2(\mu^2+\sigma_1^2)(\cos\theta - 1) > 0 \\ f_2(s) = (1-\cos\theta)s^2 - [(\cos\theta + 1)\mu^2 + 2\sigma_n^2\cos\theta]s \\ \qquad - \sigma_n^2(\mu^2+\sigma_n^2)(\cos\theta + 1) < 0 \end{cases} \tag{38}$$

hold true, where $\tan\theta = \sigma_1$.

Proof. By Theorem 3.2 and the SVD of A in (20), we get

$$\begin{aligned} &\|r^{(k+1)}\|_2 \\ \leq &\left\| \begin{pmatrix} 0 & 0 \\ V\tilde{\Lambda}\Sigma^T U^T & V\tilde{\Lambda}V^T \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ = &\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \tilde{\Lambda}\Sigma^T & \tilde{\Lambda} \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix} \right\|_2 \|r^{(k)}\|_2 \\ = &\left\| \begin{pmatrix} 0 & 0 \\ \tilde{\Lambda}\Sigma^T & \tilde{\Lambda} \end{pmatrix} \right\|_2 \|r^{(k)}\|_2, \end{aligned}$$

where $\tilde{\Lambda} = -\Sigma^T\Sigma(sI + \Sigma^T\Sigma)^{-1} + s[(\mu^2+s)I + \Sigma^T\Sigma]^{-1}[(s-\mu^2)I + \Sigma^T\Sigma](sI+\Sigma^T\Sigma)^{-1}$. Based on Lemma 2.2, one has

$$\begin{aligned} &\|r^{(k+1)}\|_2 \\ \leq &\max_{1\leq i\leq n}\left\{ \left| \frac{-(\mu^2+\sigma_i^2)\sigma_i^2 + s(s-\mu^2)}{(\mu^2+s+\sigma_i^2)(s+\sigma_i^2)} \right| \sqrt{1+\sigma_i^2} \right\} \|r^{(k)}\|_2. \end{aligned}$$

To guarantee the MRULT-II iteration method with $Q = sI + A^TA$ convergent is to make

$$\left| \frac{-(\mu^2+\sigma_i^2)\sigma_i^2 + s(s-\mu^2)}{(\mu^2+s+\sigma_i^2)(s+\sigma_i^2)} \right|^2 (1+\sigma_i^2) < 1 \tag{39}$$

for all $\sigma_i (1 \leq i \leq n)$. Denote by

$$\tilde{c}_i = \left| \frac{-(\mu^2+\sigma_i^2)\sigma_i^2 + s(s-\mu^2)}{(\mu^2+s+\sigma_i^2)(s+\sigma_i^2)} \right|.$$

By making use of the technique applied in Theorem 2.3, if $\tilde{c}_i < \cos\theta$ and $\sigma_1 = \tan\theta$ hold for all $1 \leq i \leq n$, then $\tilde{c}_i^2(1+\sigma_i^2) < \cos^2\theta(1+\tan^2\theta) = 1$. The inequality $\tilde{c}_i < \cos\theta$ can be equivalently transformed into the following inequality

$$-\cos\theta < \frac{-(\mu^2+\sigma_i^2)\sigma_i^2 + s(s-\mu^2)}{(\mu^2+s+\sigma_i^2)(s+\sigma_i^2)} < \cos\theta. \tag{40}$$

Taking a derivative of $\frac{-(\mu^2+\sigma_i^2)\sigma_i^2 + s(s-\mu^2)}{(\mu^2+s+\sigma_i^2)(s+\sigma_i^2)}$ with respect to σ_i^2 reveals that it is strictly monotone decreasing about σ_i^2, thus we can rewrite Inequality (40) into the following equivalent form

$$\frac{-(\mu^2+\sigma_1^2)\sigma_1^2 + s(s-\mu^2)}{(\mu^2+s+\sigma_1^2)(s+\sigma_1^2)} > -\cos\theta$$

and

$$\frac{-(\mu^2+\sigma_n^2)\sigma_n^2 + s(s-\mu^2)}{(\mu^2+s+\sigma_n^2)(s+\sigma_n^2)} < \cos\theta.$$

The conclusions (38) follow by directly solving the above inequalities.

IV. Numerical examples

In this section, two examples arising from a Fredholm integral equation of the first kind and image restoration are presented to examine the effectiveness of the proposed methods. The numerical results of the MRULT-type methods including the number of iteration steps (denoted by 'IT') and the total computing times in seconds (denoted by 'CPU') are compared with those of the special HSS (SHSS), new special HSS (NSHSS), restricted version of the generalized HSS (RGHSS), special GHSS (SGHSS) and the ULT-type ones mentioned in Section I. The versions of the MRULT-I iteration method with the parameter matrices $Q = sI$ and $Q = sI + A^TA$ are denoted by $\mathrm{MRULT-I}_{Q_1}$ and $\mathrm{MRULT-I}_{Q_2}$ ones, respectively, and the MRULT-II iteration method with the parameter matrices $Q = sI$ and $Q = sI + A^TA$ are abbreviated as $\mathrm{MRULT-II}_{Q_1}$ and $\mathrm{MRULT-II}_{Q_2}$ ones, respectively. All experiments are implemented in MATLAB (version R2016b) on a personal computer with Intel (R) Pentium (R) CPU G3240T 2.870GHz, 16.0 GB memory and Windows 10 system.

The error vector e in g has normally distributed entries with zero mean and unit variance and is scaled so that the contaminated g, defined by (2), has a specified noise level

$$\epsilon = \|e\|/\|\hat{g}\|,$$

where the error-free right-hand side $\hat{g}$ is defined as in (3). The initial approximate solution $f^{(0)} = 0$ is used for all the iterative methods in Example 4.1, while for image restoration problems in Example 4.2 the initial approximate solution $f^{(0)} = g$ is adopted. The parameter ϵ is set to 0.001 in two examples.

In examples, the optimal values of unknown parameters for the SHSS, NSHSS, RGHSS, SGHSS and ULT-type methods have been presented in [19], [6], [1], [2], [8], and the optimal parameters of the $\mathrm{MRULT-I}$ and $\mathrm{MRULT-II}$ methods are chosen as the experimentally found optimal ones which lead to the least number of iteration steps. The optimal value of the regularization parameter has been investigated [12], [20], [9]. In our computation, the GCV scheme is adopted to determine a suitable value for regularization parameter μ [9]. The best regularization parameter value μ minimizes the following GCV functional

$$G(\mu) = \frac{\|A(A^TA+\mu^2I)^{-1}A^Tg - g\|_2^2}{(trace(I - A(A^TA+\mu^2I)^{-1}A^T))^2}.$$

A quantity, the relative error (RES), is commonly applied to measure the accuracy of these methods for ill-posed problems and image restoration. The proposed quantity is defined as follows:

$$\mathrm{RES} = \frac{\|f_{numerical} - f_{exact}\|_2}{\|f_{exact}\|_2},$$

where $f_{numerical}$ and f_{exact} are the numerical solutions (or restored images) and exact solutions (or the original images), respectively. In image restoration problem, a smaller RES-value usually implies that the restoration is of higher quality. Somehow this may be not consistent with visual judgment. Therefore, the restored images also are displayed in the following examples.

Example 4.1: (Example from Hansen Tools [13]) Consider the test example $\mathbf{gravity(500, 1)}$ from Hansen Tools

TABLE I: Numerical results of methods of Example 4.1 with $\mu = 0.0068$.

Method	Parameters	IT	CPU	RES
SHSS	$\alpha = 0.9543$	500	0.2322	0.8130
NSHSS	$\alpha = 4.5749e-6$	500	0.2347	0.9554
RGHSS	$(\alpha, \beta) = (0.001, 0.001)$	19	0.0108	0.0010
SGHSS	$(\omega, \tau) = (0.6, 1.0274e-3)$	217	0.1078	0.0010
$\mathrm{ULT-I}_{Q_1}$	$s = 0.4172$	500	1.3257	4.3841
$\mathrm{ULT-I}_{Q_2}$	$s = 0.8$	500	0.0972	0.0221
$\mathrm{ULT-II}_{Q_1}$	$s = 0.4172$	500	0.4823	4.7534
$\mathrm{ULT-II}_{Q_2}$	$s = 0.0437$	3	0.0159	0.0187
$\mathrm{MRULT-I}_{Q_1}$	$s = 1.9$	121	0.2770	0.0233
$\mathrm{MRULT-I}_{Q_2}$	$s = 0.01$	2	0.0065	0.0158
$\mathrm{MRULT-II}_{Q_1}$	$s = 1.6$	121	0.2901	0.0240
$\mathrm{MRULT-I}_{Q_2}$	$s = 0.01$	2	0.0071	0.0147

TABLE II: Numerical results of methods of Example 4.2 with $\mu = 0.0042$.

Method	Parameters	IT	CPU	RES
SHSS	$\alpha = 0.3491$	500	1.6402	0.0717
NSHSS	$\alpha = 1.9858e-3$	500	2.9766	0.1965
RGHSS	$(\alpha, \beta) = (0.0361, 0.001)$	8	0.0267	0.0548
SGHSS	$(\omega, \tau) = (0.6, 2.0622e-3)$	9	0.0302	0.0473
$\mathrm{ULT-I}_{Q_1}$	$s = 1.02$	500	1.7532	0.0851
$\mathrm{ULT-I}_{Q_2}$	$s = 12$	500	2.0577	0.1953
$\mathrm{ULT-II}_{Q_1}$	$s = 1.0179$	500	1.2897	0.0851
$\mathrm{ULT-II}_{Q_2}$	$s = 0.1338$	423	1.7649	0.0565
$\mathrm{MRULT-I}_{Q_1}$	$s = 0.99$	366	1.6182	0.0568
$\mathrm{MRULT-I}_{Q_2}$	$s = 0.001$	6	2.0308	0.0544
$\mathrm{MRULT-II}_{Q_1}$	$s = 0.98$	476	0.0427	0.0569
$\mathrm{MRULT-II}_{Q_2}$	$s = 0.001$	6	0.0429	0.0543

[13]. The associated perturbed data vector g is obtained by using Matlab code $g = \hat{g} + 0.001 \times rand(size(\hat{g}))$ in the test.

We apply Algorithms 2.1 and 3.1 with $Q = sI$ and $Q = sI + A^TA$ to Example 4.1 and compare the IT, CPU times and RES of the MRULT-type methods with those of the SHSS, NSHSS, RGHSS, SGHSS and ULT-type ones. For the test problem, the regularization parameter μ determined by GCV is 0.0068. All iteration processes are terminated once the current residual satisfies $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-5}$ or the number of iterations exceeds the largest prescribed iteration step $M = 500$, where $r^{(k)} = b - \mathcal{K}x^{(k)}$ is the residual at the kth iteration. In our implementations, the linear sub-systems are solved by the sparse Cholesky factorization when the coefficient matrix is symmetric positive definite.

The parameters, IT, CPU times and RES of the tested iteration methods are listed in Table I, and the exact and numerical solutions are draw in Figure 1 for $n = 500$. From Table 1, it can be seen that that the MRULT-type iteration methods can successfully compute approximate solution of high quality. Moreover, the MRULT-type iteration methods can achieve smaller relative error with least IT and CPU times compared with the SHSS and NSHSS ones. Although the relative errors of the MRULT-type iteration methods are bigger than that of the RGHSS and SGHSS ones, the $\mathrm{MRULT-I}_{Q_2}$ and $\mathrm{MRULT-II}_{Q_2}$ iteration methods need less iteration steps. Besides, from these numerical results, we can observe that the MRULT-I iteration methods with $Q = sI$ and $Q = sI + A^TA$ are superior to the ULT-I ones with $Q = sI$ and $Q = sI + A^TA$, respectively, and the conclusion also holds true for the MRULT-II iteration methods. In order to show more clearly that the convergence rates of the MRULT-type methods are faster than that of the ULT-type ones, we draw the plots of their relative errors with respect to iterations k in Figure 2. From Figure 2, it can be seen that the MRULT-type methods further improve the convergence rates of the ULT-type ones. Moreover, the plots of relative errors with respect to iterations k are also drawn in Figure 3 to compare the convergence behaviour of the MRULT-I and MRULT-II methods with different parameter matrix Q. And Figure 3 indicates that the MRULT-I and MRULT-II methods are comparable. As these results, our proposed algorithms surpass some other ones and are more effective for solving the ill-posed problems.

Example 4.2: (Image restoration) In this example, we let the exact image be the 'brain' image from MATLAB. It is represented by an array of 128×128 pixels and shown on the left-hand side of Figure 4. Choose $PSF = psfDefocus([7, 7], 3)$ in [15] to blur the example, and a 'noisy' available vector g is generated by using MATLAB code $g = \hat{g} + 0.001 \times rand(size(\hat{g}))$. The PSF and blurred image in the example are shown on the middle side and the right-hand side of Figure 4, respectively. The purpose of the experiment is to illustrate that the MRULT-type iterative methods perform better than the SHSS, NSHSS, RGHSS, SGHSS and ULT-type ones for image restoration. Furthermore, the relative error of the blurred image is 0.2772.

In this test, we apply the periodic BCs to construct the blurring matrix A. Here the matrix A is block circulant with circulant blocks, which can perform matrix-vector multiplications and provide the spectral factorization of the blurring matrix A via two-dimensional fast Fourier transformations (FFTs). Therefore the regularization parameter μ computed by the GCV method and the optimal values of the unknown parameters for all iterative methods involving the singular values of A can be easily obtained via FFTs, see [15] for more details. Moreover, we can also use the FFTs to solve the linear systems with the symmetric positive definite coefficient matrices in Algorithms 2.1 and 3.1. All computations for the tested iteration methods are terminated once the current residual satisfies $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-4}$ or the number of iterations exceeds the largest prescribed iteration step $M = 600$, where $r^{(k)} = b - \mathcal{K}x^{(k)}$ is the residual at the kth iteration.

IT, CPU times and RES for the tested methods and the restored images by the twelve iterative methods are exhibited in Table II and in Figure 5, respectively. From Table II, we can see that the $\mathrm{MRULT-I}_{Q_2}$ and $\mathrm{MRULT-II}_{Q_2}$ iteration methods just need 6 iterations and less CPU as they are terminated. Moreover, the restorations with the MRULT-type methods are of higher quality, because the relative errors of the MRULT-type methods are smaller than some others. Figure 5 also demonstrates the higher visual quality of the restored images with the MRULT-type methods. As a whole, the proposed methods in this paper perform better than some tested ones and are effective for solving image restoration problems.

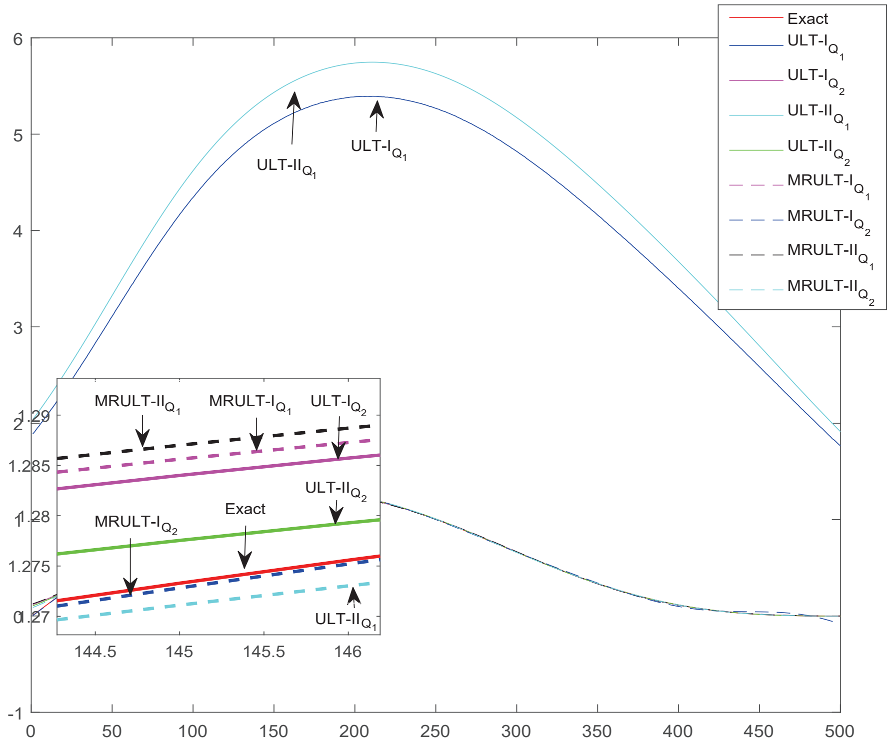

Fig. 1: Example 4.1-$gravity(500, 1)$ test case: the exact solution and its numerical solution.

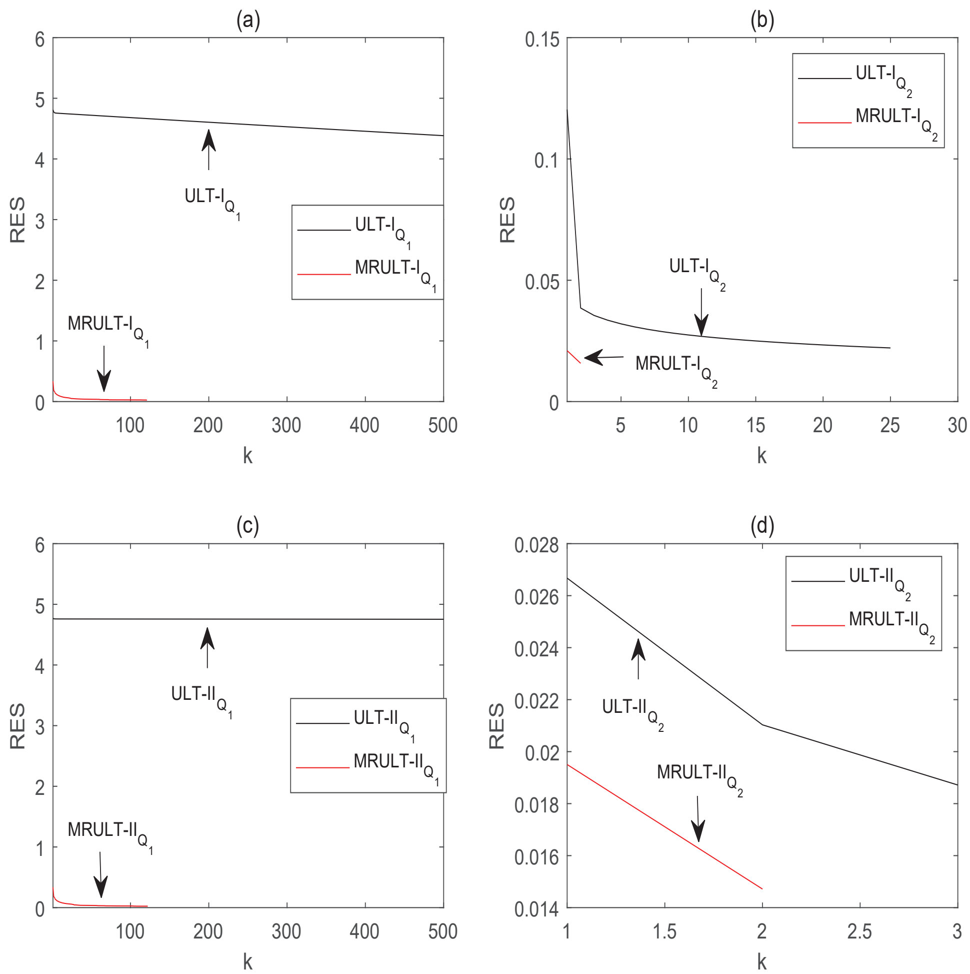

Fig. 2: Example 4.1-$gravity(500, 1)$ test case: the relative error versus iteration k for tested iterative methods.

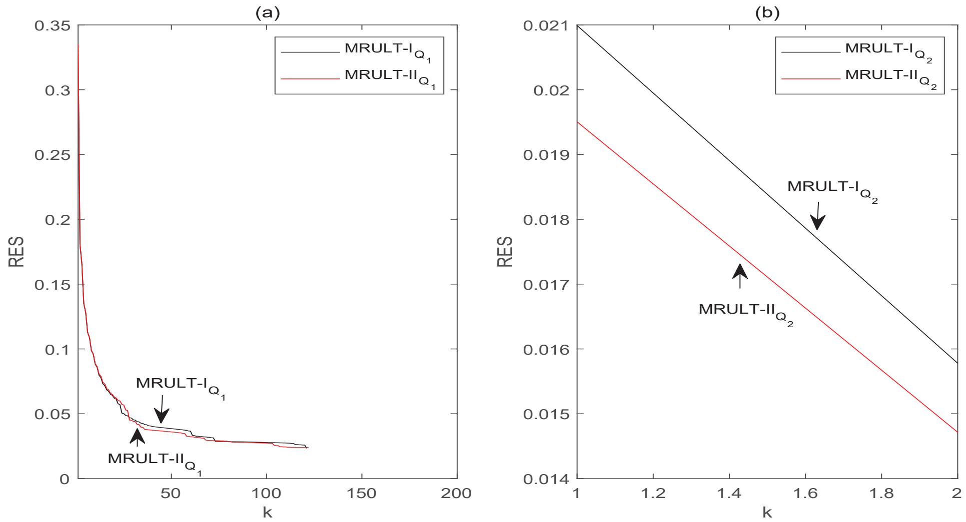

Fig. 3: Example 4.1-$gravity(500, 1)$ test case: the relative error versus iteration k for tested iterative methods.

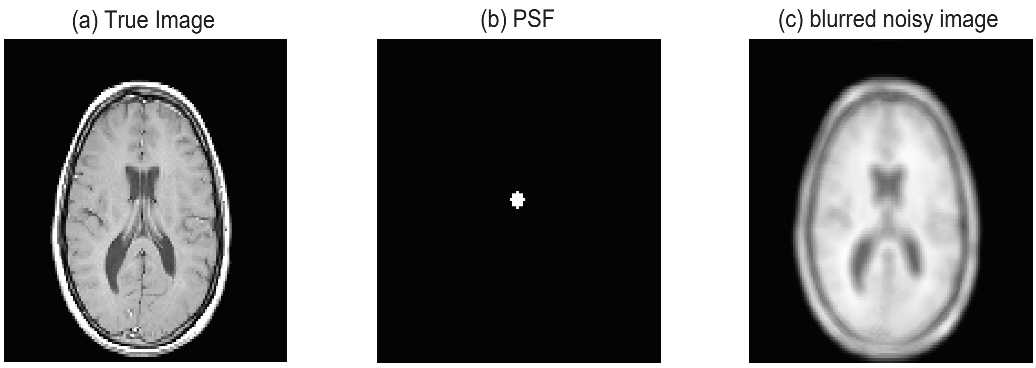

Fig. 4: True image, PSF and blurred image of Example 4.2.

V. Conclusion

Inspired by [27], this paper puts forward two types of non-stationary upper and lower triangular iteration methods called the minimize residual ULT (MRULT) ones to give the approximate solution of for ill-posed inverse problems. By introducing the control parameters into the ULT-I and ULT-II splitting iteration methods, we establish the iterative sequences (11) and (29) to accelerate the convergence rates of the ULT-I and ULT-II ones, respectively. Two parameters involved in the MRULT methods are adopted by minimizing the residual norms at each step of the MRULT ones. And when the parameters are adopted as specific values, the MRULT-I and MRULT-II methods reduce to the ULT-I and ULT-II ones, respectively. Thus, the convergent rates of the MRULT methods are at least not slower than those of the ULT ones. Moreover, we investigate analytically the convergence properties of the MRULT methods. And the presented numerical examples verify that our methods are superior to some existing ones and effective for solving ill-posed problems.

Acknowledgment

This research is supported by the National Natural Science Foundations of China (No.10802068), the National Science Foundation for Young Scientists of China (No.11901123) and the Guangxi Natural Science Foundation (No.2018JJB110062).

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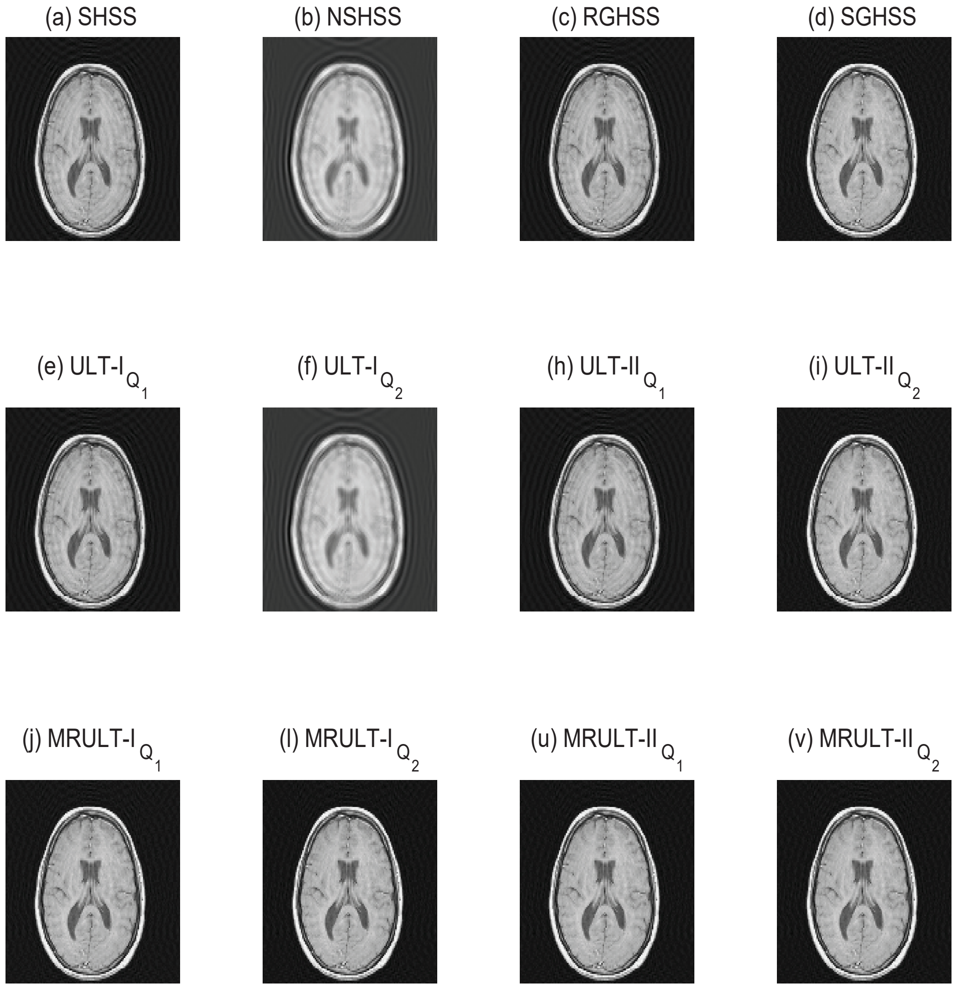

Fig. 5: Restored images with the tested methods of Example 4.2.

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