The Hamiltonicity, Hamiltonian Connectivity, and Longest (s, t)-path of L-shaped Supergrid Graphs

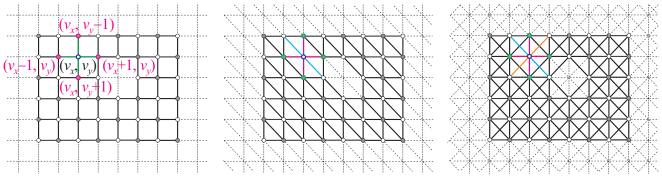
Fatemeh Keshavarz-Kohjerdi[1] and Ruo-Wei Hung[2,*]

***Abstract*—Supergrid (or called strong grid) graphs contain grid graphs and triangular grid graphs as their subgraphs. The Hamiltonian (*s, t*)-path of a graph is a Hamiltonian path between any two distinct vertices *s* and *t* in the graph, and the longest (*s, t*)-path is a simple path with the maximum number of vertices from *s* to *t* in the graph. A graph is called Hamiltonian if it contains a Hamiltonian cycle, and is said to be Hamiltonian connected if there exists a Hamiltonian (*s, t*)-path in it. These problems are known to be NP-complete for general supergrid graphs. As far as we know, their complexities are still unknown for solid supergrid graphs which are supergrid graphs without any hole. In this paper, we will study these problems on *L*-shaped supergrid graphs which form a subclass of solid supergrid graphs. First, we prove *L*-shaped supergrid graphs to be Hamiltonian except one trivial condition. We then verify the Hamiltonian connectivity of *L*-shaped supergrid graphs except few conditions. The Hamiltonicity and Hamiltonian connectivity of *L*-shaped supergrid graphs can be applied to compute the minimum trace of computerized embroidery machine and 3D printer when a *L*-like object is printed. Finally, we present a linear-time algorithm to compute the longest (*s, t*)-paths of *L*-shaped supergrid graphs. This study can be regarded as the first attempt for solving the Hamiltonian and longest (*s, t*)-path problems on solid supergrid graphs.**

***Index Terms*—Hamiltonicity, Hamiltonian connectivity, longest (*s, t*)-path, solid supergrid graphs, *L*-shaped supergrid graphs, computer embroidery machines, 3D printers.**

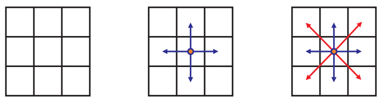
I. Introduction

The studied graphs, namely *supergrid* (or called *strong grid*) *graphs*, derive from our industry-university cooperative research project. They can be applied to computerized sewing machines. The process flow of a computerized sewing machine is as follows: The computerized sewing software is given by a colour image. First, it uses the image processing technique to produce some blocks of different colors. Then, the software computes the stitching trace for each block of colors. Finally, the software transmits its computed stitching trace to computerized sewing machine, and the machine then performs the sewing action along the received stitching trace. Since each stitch position of a sewing machine can be moved to its eight neighbor positions (left, right, up, down, up-left, up-right, down-left, and down-right), we define *supergrid graphs* in [13] as follows: Each lattice of a block of color will be represented by a vertex and each vertex v is coordinated as (v_x, v_y), denoted by $v = (v_x, v_y)$, where v_x and v_y are integers, and v_x and v_y represent the x and y coordinates of node v, respectively. Two vertices u and v are adjacent if and only if $|u_x - v_x| \leqslant 1$ and $|u_y - v_y| \leqslant 1$. Thus, the possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph contain $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, $(v_x + 1, v_y + 1)$, $(v_x + 1, v_y - 1)$, and $(v_x - 1, v_y + 1)$. Supergrid graphs contain grid [19] and triangular grid [32] graphs as their subgraphs. For instance, Figs. 1(a)–(c) depict a grid, a triangular grid, and a supergrid graph, respectively. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (nodes) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are bipartite [19] but triangular grid graphs and supergrid graphs are not bipartite.



(a) (b) (c)

Fig. 1. (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, where circles represent the vertices and solid lines indicate the edges in the graphs.



(a) (b) (c)

Fig. 2. (a) A set of lattices, (b) the neighbors of one lattice in a grid graph, and (c) the neighbors of one lattice in a supergrid graph, where each lattice is denoted by a vertex in a graph and arrow lines indicate the adjacent neighbors of one lattice.

Another intuitive motivation of proposing supergrid graphs is given below. Consider a set of lattices, shown in Fig. 2(a), where each lattice is denoted as a vertex in a graph. For a grid graph, the neighbors of a lattice include its up, down, left, and right lattices, see Fig. 2(b). However, in the real word and other applications, the neighbors of a lattice may also contain its up-left, up-right, down-left, and down-right adjacent lattices. Thus, supergrid graphs can be used in these applications.

A *Hamiltonian path* (resp., *cycle*) in a graph is a spanning path (resp., cycle) of the graph. The *Hamiltonian path* (resp., *cycle*) *problem* involves determining whether a graph contains a Hamiltonian path (resp., cycle). A graph is called *Hamiltonian* if it contains a Hamiltonian cycle. A simple path

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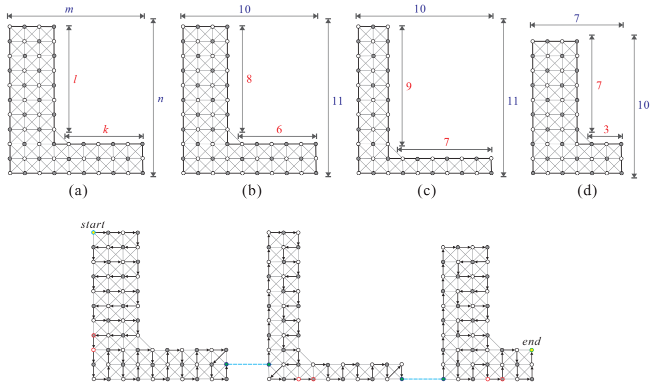
from vertex s to t is denoted by (s,t)-path. A graph G is said to be *Hamiltonian connected* if it contains a Hamiltonian (s,t)-path for any two vertices s and t of G. The longest (s,t)-path of a graph is a simple path with the maximum number of vertices from s to t in the graph. The longest (s,t)-path problem is to compute the longest (s,t)-path of a graph given any two distinct vertices s and t. It is well known that the Hamiltonian and longest (s,t)-path problems are NP-complete for general graphs [7], [20]. The same holds true for bipartite graphs [28], split graphs [8], circle graphs [6], undirected path graphs [1], grid graphs [19], triangular grid graphs [9], supergrid graphs [13], and so on. In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks, see [3], [5], [10]–[12], [30], [31].

In [13], we proved the Hamiltonian problems on general supergrid graphs to be NP-complete. A solid supergrid graph is a supergrid graph without any hole. For example, the graph in Fig. 1(c) is a supergrid graph but it is not a solid supergrid graph. The Hamiltonian problems on solid supergrid graphs are still open. In this paper, we will solve the Hamiltonian and longest (s,t)-path problems on L-shaped supergrid graphs, which form a subclass of solid supergrid graphs, in linear time. Let $R(m,n)$ be a supergrid graph such that its vertex set $V(R(m,n)) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n\}$. A *rectangular supergrid graph* is a supergrid graph which is isomorphic to $R(m,n)$. Let $L(m,n;k,l)$ be a supergrid graph obtained from a rectangular supergrid graph $R(m,n)$ by removing its subgraph $R(k,l)$ from the upper-right corner. A L-shaped supergrid graph is isomorphic to $L(m,n;k,l)$. In this paper, we only consider $L(m,n;k,l)$. Note that the number of vertices in $L(m,n;k,l)$ equals to $mn - kl$. In the figures, we will assume that $(1,1)$ are coordinates of the vertex located at the upper-left corner of a supergrid graph. For example, Fig. 3(a) indicates the structure of $L(m,n;k,l)$, and Figs. 3(b)–(d) indicate $L(10,11;6,8)$, $L(10,11;7,9)$, and $L(7,10;3,7)$, respectively. The width and height of L-shaped supergrid graph $L(m,n;k,l)$ can be adjusted according to the parameters m, n, k, and l. The main idea of our strategy is presented as follows. It first separates the input graph into many parts. Then, we verify whether there exists a Hamiltonian (s,t)-path in the input graph by checking these separated parts. When there exists no Hamiltonian (s,t)-path in the input graph, we then combine the longest paths of these separated parts to get a longest (s,t)-path of the input graph. Although this idea does seem to be simple, there are still many issues to be solved. That is, the precise partitions is important. If the separation is done in a wrong way then the result may be wrong. In this paper, we will verify our partition can separate the graph in a correct way.

The possible application of the Hamiltonian connectivity of L-shaped supergrid graphs is given below. Consider a computerized embroidery machine for sewing a varied-sized letter L into the object, e.g. clothes. First, we produce a set of lattices to represent the letter. Then, a path is computed to visit the lattices of the set such that each lattice is visited exactly once. Finally, the software transmits the stitching trace of the computed path to the computerized embroidery machine, and the machine then performs the sewing work along the trace on the object. Since each stitch position of an embroidery machine can be moved to its eight neighboring



(a) (b) (c) (d)
(e)

Fig. 3. (a) The structure of L-shaped supergrid graph $L(m,n;k,l)$, (b) $L(10,11;6,8)$, (c) $L(10,11;7,9)$, (d) $L(7,10;3,7)$, and (e) a minimum sewing trace for the sets of lattices in (b)–(d), where each lattice is represented by a node, solid arrow lines indicate the computed trace and dashed arrow lines indicate the jump lines connecting two continuous letters.

positions, one set of neighboring lattices forms a L-shaped supergrid graph. In this case, each lattice will be represented by a vertex of a supergrid graph. The desired sewing trace of the set of adjacent lattices is the Hamiltonian path of the corresponding L-shaped supergrid graph. Given a string with varied-sized L letters. By the Hamiltonian connectivity of L-shaped supergrid graphs, we can compute the end nodes of Hamiltonian paths in the corresponding L-shaped supergrid graphs so that the total length of jump lines connecting two L-shaped supergrid graphs is minimum. For instance, given three L-shaped supergrid graphs in Figs. 3(b)–(d), in which each L-shaped supergrid graph represents a set of lattices, Fig. 3(e) shows such a minimum sewing trace for the sets of lattices.

Another possible application of Hamiltonian connectivity of L-shaped supergrid graphs is to compute the minimum printing trace of 3D printers. Consider a 3D printer with a L-type object being printed. The software produces a series of thin layers, designs a path for each layer, combines these paths of produced layers, and transmits the above paths to 3D printer. Because 3D printing is performed layer by layer (see Fig. 4(a)), each layer can be considered as a L-shaped supergrid graph. Suppose that there are k layers under the above 3D printing. If the Hamiltonian connectivity of L-shaped supergrid graphs holds, then we can find a Hamiltonian (s_i, t_i)-path of an L-shaped supergrid graph L_i, where L_i, $1 \leqslant i \leqslant k$, represents a layer under 3D printing. Thus, we can design an optimal trace for the above 3D printing, where t_i is adjacent to s_{i+1} for $1 \leqslant i \leqslant k-1$. In this application, we restrict the 3D printer nozzle to be located at integer coordinates. For example, Fig. 4(a) shows 4 layers L_1–L_4 of a 3D printing for a L-type object, Fig. 4(b) depicts the Hamiltonian (s_i, t_i)-paths of L_i for $1 \leqslant i \leqslant 4$, and the result of this 3D printing is shown in Fig. 4(c).

Previous related works on supergrid graphs are summarized as follows. The supergrid graphs were first introduced in [13], in which we proved that the Hamiltonian cycle and path problems on supergrid graphs are NP-complete, and every rectangular supergrid graph is Hamiltonian. Since the Hamiltonian problems on general supergrid graphs are NP-complete, an important investgated direction is to discover

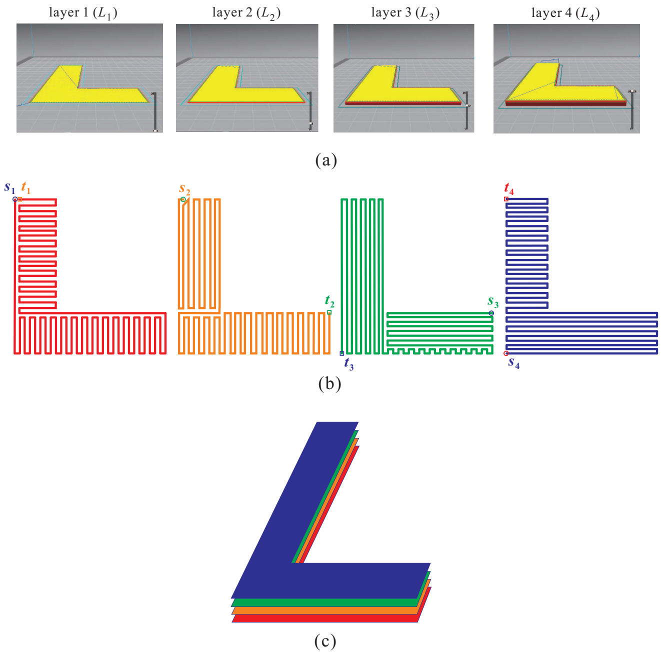


Fig. 4. (a) The four layers L_1–L_4 of a 3D printing model while printing a L-type object, (b) the computing Hamiltonian (s_i, t_i)-path of each layer L_i in (a), and (c) the final result while performing the 4-layered 3D printing.

the complexities of special subclasses of supergrid graphs. In [14], we proved that linear-convex supergrid graphs always contain Hamiltonian cycles. In [15], we proved that rectangular supergrid graphs (with one trivial exception) are always Hamiltonian connected. Recently, we verified the Hamiltonicity and Hamiltonian connectivity of some shaped supergrid graphs, including triangular, parallelogram, and trapezoid [16]. Very recently, we verified the Hamiltonicity and Hamiltonian connectivity of alphabet supergrid graphs [18]. A preliminary version of this paper has appeared in [17]. For the related works about grid and triangular grid graphs, we refer the readers to [4], [9], [19], [21]–[27], [29], [32]–[34].

The rest of the paper is organized as follows. In Section II, some notations and observations are given. Previous results are also introduced. In Section III, we prove two Hamiltonian connected properties of rectangular supergrid graphs. These two properties will be used in proving the Hamiltonian connectivity of L-shaped supergrid graphs. Section IV shows that L-shaped supergrid graphs are Hamiltonian and Hamiltonian connected except one or three conditions. In Section V, we propose a linear-time algorithm to solve the longest (s, t)-path problem on L-shaped supergrid graphs. Finally, we make some concluding remarks in Section VI.

II. Notations and Previous Results

In this section, we will introduce some notations and previously established results. For graph-theoretic terminology not defined here, the reader is referred to [2].

Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let $S \subseteq V(G)$ and let $u, v \in V(G)$. The subgraph of G *induced* by S is represented as $G[S]$, and $G - S$ is used to denote $G[V - S]$ for convenience. In general, we write $G - v$ instead of $G - \{v\}$. We denote by (u, v) an edge in G, where u is *adjacent* to v, and u is called a neighbor of v. The notation $u \sim v$ (resp., $u \nsim v$) means that vertices u and v are adjacent (resp., non-adjacent). Two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are said to be *parallel* if $u_1 \sim v_1$ and $u_2 \sim v_2$, denote this by $e_1 \approx e_2$. We use $N_G(v)$ to denote the set of neighbors of v in G, and let $N_G[v] = N_G(v) \cup \{v\}$. The number of vertices adjacent to vertex v in G is called the *degree* of v in G and is denoted by $deg(v)$.

A path $P = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P|-1} \rightarrow v_{|P|}$ in G is a sequence $(v_1, v_2, \cdots, v_{|P|-1}, v_{|P|})$ of vertices such that $(v_i, v_{i+1}) \in E(G)$ for $1 \leqslant i < |P|$, and all vertices except $v_1, v_{|P|}$ are distinct. If $v_1 = v_{|P|}$ and $|P| \geqslant 4$, then P is called a cycle of G. The vertices v_1 and $v_{|P|}$ are called the *path-start* and *path-end* of P, denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote "P visits vertex v_i" and use $(v_i, v_{i+1}) \in P$ to denote "P visits edge (v_i, v_{i+1})". A path from v_1 to v_k is denoted by (v_1, v_k)-path. For convenience, we will use P to refer to $V(P)$ if no ambiguity occurs. Let P_1 and P_2 be two paths (or cycles) in G. If $V(P_1) \cap V(P_2) = \emptyset$, then they are called *vertex-disjoint*. When P_1 and P_2 are vertex-disjoint and $end(P_1) \sim start(P_2)$, then they can be concatenated into a path, denoted by $P_1 \Rightarrow P_2$.

Rectangular supergrid graphs first appeared in [13], in which we solved the Hamiltonian cycle problem in linear time. A rectangular supergrid graph $R(m, n)$ is a supergrid graph with $V(R(m, n)) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n\}$, and it is called n-rectangle. In this paper, without loss of generality we will assume that $m \geqslant n$. Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. The vertex v is called the *upper-left* (resp., *upper-right*, *down-left*, *down-right*) *corner* of $R(m, n)$ if for any vertex $w = (w_x, w_y) \in R(m, n)$, $w_x \geqslant v_x$ and $w_y \geqslant v_y$ (resp., $w_x \leqslant v_x$ and $w_y \geqslant v_y$, $w_x \geqslant v_x$ and $w_y \leqslant v_y$, $w_x \leqslant v_x$ and $w_y \leqslant v_y$). Notice that in the figures we will assume that $(1, 1)$ are coordinates of the upper-left corner of $R(m, n)$, except we explicitly change this assumption. The edge (u, v) is called *horizontal* (resp., *vertical*) if $u_y = v_y$ (resp., $u_x = v_x$), and is said to be *crossed* if it is neither a horizontal nor a vertical edge. There are four boundaries in a rectangular supergrid graph $R(m, n)$ with $m, n \geqslant 2$. The edge in the boundary of $R(m, n)$ is called *boundary edge*. A path is called *boundary* of $R(m, n)$ if it visits all vertices of the same boundary in $R(m, n)$ and its length equals to the number of vertices in the visited boundary. For example, Fig. 5 shows a rectangular supergrid graph $R(10, 8)$ which is called 8-rectangle and contains $2 \times (9 + 7) = 32$ boundary edges. Fig. 5 also indicates the types of edges and corners.

A *L-shaped supergrid graph*, denoted by $L(m, n; k, l)$, is a supergrid graph obtained from a rectangular supergrid graph $R(m, n)$ by removing its subgraph $R(k, l)$ from the upper-right corner, where $m, n > 1$ and $k, l \geqslant 1$. Then, $m - k \geqslant 1$, $n - l \geqslant 1$, and $|V(L(m, n; k, l))| = mn - kl$. The structure of $L(m, n; k, l)$ is depicted in Fig. 3(a).

In proving our results, we need to partition a rectangular or L-shaped supergrid graph into two disjoint parts. The partition is defined as follows:

Definition 1. Let G be a L-shaped supergrid graph $L(m, n; k, l)$ or a rectangular supergrid graph $R(m, n)$. A *separation operation* of G is a partition of G into two

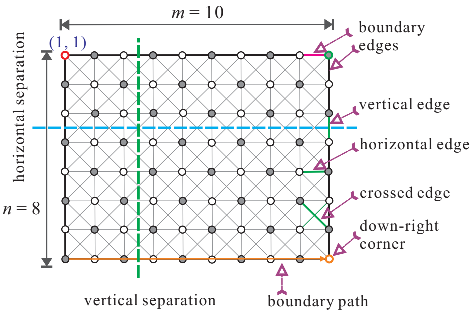


Fig. 5. A rectangular supergrid graph $R(m, n)$, where $m = 10$, $n = 8$, and bold dashed lines indicate vertical and horizontal separations.

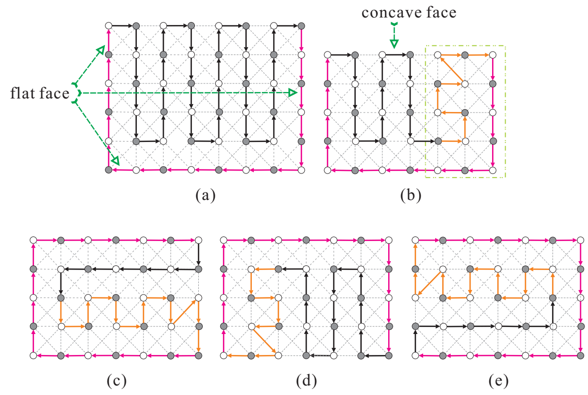


Fig. 6. A canonical Hamiltonian cycle containing three flat faces and one concave face for (a) $R(8, 6)$ and (b)–(e) $R(7, 5)$, where solid arrow lines indicate the edges in the cycles and $R(7, 5)$ includes four distinct canonical Hamiltonian cycles in (b)–(e) such that their concave faces are located on different boundaries.

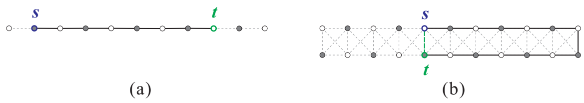


Fig. 7. Rectangular supergrid graphs in which Hamiltonian (s, t)-path does not exist for (a) 1-rectangle $R(m, 1)$, and (b) 2-rectangle $R(m, 2)$, where solid lines indicate the longest (s, t)-path.

vertex-disjoint supergrid subgraphs G_1 and G_2, i.e., $V(G) = V(G_1) \cup V(G_2)$ and $V(G_1) \cap V(G_2) = \emptyset$. A separation is called *vertical* if it consists of a set of horizontal edges, and is called *horizontal* if it contains a set of vertical edges. For instance, the bold dashed vertical (resp., horizontal) line in Fig. 5 indicates a vertical (resp., horizontal) separation of $R(10, 8)$ which is partitioned into $R(3, 8)$ and $R(7, 8)$ (resp., $R(10, 3)$ and $R(10, 5)$).

Let $R(m, n)$ with $m \geqslant n$ be a rectangular supergrid graph. In [13], we proved $R(m, n)$ to be Hamiltonian except $n = 1$. Let $\mathcal{C}$ be a cycle of $R(m, n)$, and let B be a boundary of $R(m, n)$. The restriction of $\mathcal{C}$ to B is denoted by $\mathcal{C}_{|B}$. If $|\mathcal{C}_{|B}| = 1$, i.e., $\mathcal{C}_{|B}$ is a boundary path on B, then $\mathcal{C}_{|B}$ is called *flat face* on B. If $|\mathcal{C}_{|B}| > 1$ and $\mathcal{C}_{|B}$ contains at least one boundary edge of B, then $\mathcal{C}_{|B}$ is called *concave face* on B. A Hamiltonian cycle $\mathcal{HC}$ of $R(m, n)$ with $m \geqslant n \geqslant 2$ is called *canonical* if
(1) $n = 3$, $\mathcal{HC}$ contains three flat faces on two shorter boundaries and one longer boundary, and $\mathcal{HC}$ contains one concave face on the other boundary; or
(2) $n = 2$ or $n \geqslant 4$, $\mathcal{HC}$ contains three flat faces on three boundaries, and $\mathcal{HC}$ contains one concave face on the other boundary.
The following lemma shows the Hamiltonicity of rectangular supergrid graphs and appears in [13].

Lemma 1. *(See [13].) Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. Then, $R(m, n)$ contains a canonical Hamiltonian cycle. Moreover, $R(m, n)$ contains four canonical Hamiltonian cycles with concave faces being located on different boundaries when $n \neq 3$.*

Fig. 6 shows canonical Hamiltonian cycles for rectangular supergrid graphs found in Lemma 1. Each Hamiltonian cycle found by this lemma contains all the boundary edges on any three sides of the rectangular supergrid graph. This shows that for any rectangular supergrid graph $R(m, n)$ with $m \geqslant n \geqslant 4$, we can always construct four canonical Hamiltonian cycles such that their concave faces are placed on different boundaries. For an example, the four distinct canonical Hamiltonian cycles of $R(7, 5)$ are depicted in Figs. 6(b)–(e).

Let (G, s, t) denote the supergrid graph G with two distinct vertices s and t. We will assume, without loss of generality, that $s_x \leqslant t_x$ except we explicitly change this assumption. A Hamiltonian path between s and t in G is denoted by $HP(G, s, t)$. From Lemma 1, $HP(R(m, n), s, t)$ does exist if $m, n \geqslant 2$ and (s, t) is an edge in the constructed Hamiltonian cycle of $R(m, n)$. In addition, we will use $\hat{L}(G, s, t)$ to denote the length of longest (s, t)-path in (G, s, t). Note that we denote the length of a path by the number of vertices in the path.

Recently, the Hamiltonian connectivity of rectangular supergrid graphs except one condition has been verified in [15]. The forbidden condition for $HP(R(m, n), s, t)$ is satisfied only for 1-rectangle or 2-rectangle. To describe the exception condition, we define the *cut vertex* and *vertex cut* of a graph as follows:

Definition 2. Let G be a connected graph and let $V_1 \subseteq V(G)$. The set V_1 is a *vertex cut* of G if $G - V_1$ is disconnected. A vertex v of G is a *cut vertex* of G if $\{v\}$ is a vertex cut of G. For instance, in Fig. 7(a) s or t is a cut vertex and in Fig. 7(b) $\{s, t\}$ is a vertex cut.

Then, the following condition implies $HP(R(m, 1), s, t)$ and $HP(R(m, 2), s, t)$ do not exist.

(F1) s or t is a cut vertex of $R(m, 1)$, or $\{s, t\}$ is a vertex cut of $R(m, 2)$ (see Figs. 7(a)–(b)).

The following lemma can be easily verified by the same arguments in [24].

Lemma 2. *Let $R(m, n)$ be a rectangular supergrid graph, and let s and t be its two vertices. If $(R(m, n), s, t)$ satisfies condition* (F1)*, then $HP(R(m, n), s, t)$ does not exist.*

In [15], we obtained the following lemma to show the Hamiltonian connectivity of rectangular supergrid graphs.

Lemma 3. *(See [15].) Let $R(m, n)$ be a rectangular supergrid graph, and let s and t be its two vertices. If $(R(m, n), s, t)$ does not satisfy condition* (F1)*, then*

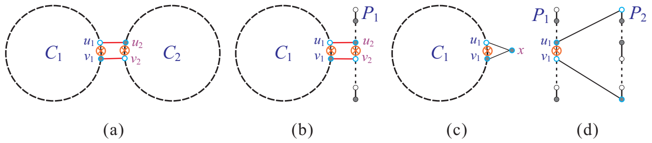


Fig. 8. A schematic diagram for (a) Statement (1), (b) Statement (2) , (c) Statement (3), and (d) Statement (4) of Proposition 5, where $\otimes$ represents the destruction of an edge while constructing a merged cycle or path.

$HP(R(m,n),s,t)$ does exist.

Combining with the above two lemmas, we have the following theorem.

Theorem 4. *Let $R(m,n)$ be a rectangular supergrid graph, and let s and t be its two vertices. Then, $HP(R(m,n),s,t)$ does exist if and only if $(R(m,n),s,t)$ does not satisfy condition* (F1).

The Hamiltonian (s,t)-path of $R(m,n)$ constructed in [15] is to contain at least one boundary edge of each boundary, and is called *canonical*.

We next give some propositions on the relations among cycle, path, and vertex. These observations will be used in proving our results and are given in [13], [14], [15].

Proposition 5. *(See [13], [14], [15].) Let G be a connected graph, C_1 and C_2 be two vertex-disjoint cycles of G, C_1 and P_1 be a cycle and a path, respectively, of G such that $V(C_1) \cap V(P_1) = \emptyset$, and let x be a vertex in $G - V(C_1)$ or $G - V(P_1)$. Then, the following statements hold:*
(1) If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be merged into a cycle of G (see Fig. 8(a)).
(2) If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be merged into a path of G (see Fig. 8(b)).
(3) If vertex x adjoins one edge (u_1,v_1) of C_1 (resp., P_1), then$x$ and C_1 (resp., P_1) can be combined into a cycle (resp., path) of G (see Fig. 8(c)).
(4) If there exists one edge $(u_1,v_1) \in C_1$ such that $u_1 \sim start(P_1)$ and $v_1 \sim end(P_1)$, then C_1 and P_1 can be combined into a cycle C of G (see Fig. 8(d)).

In [15], Hung *et al.* gave the following formula to compute the length of a longest (s,t)-path in $R(m,n)$:

$$\hat{L}(R(m,n),s,t) = \begin{cases} t_x - s_x + 1 & \text{, if } n = 1; \\ \max\{2s_x, \\ 2(m - s_x + 1)\} \text{ or } 2m & \text{, if } n = 2; \\ mn & \text{, if } n \geqslant 3. \end{cases}$$

Theorem 6. *(See [15].) Given a rectangular supergrid graph $R(m,n)$ with $mn > 2$, and two distinct vertices s and t in $R(m,n)$, a longest (s,t)-path can be computed in $O(mn)$-linear time.*

In this paper, we will show that a longest (s,t)-path of $(L(m,n;k,l),s,t)$ can be computed in $O(mn-kl)$-linear time.

III. Two Hamiltonian connected properties of rectangular supergrid graphs

By Theorem 4, rectangular supergrid graph $R(m,n)$ contains a Hamiltonian (s,t)-path if and only if $(R(m,n),s,t)$ does not satisfy condition (F1). The Hamiltonian (s,t)-path of $R(m,n)$ constructed in [15] contains at least one boundary edge of each boundary. In this section, we will prove two additional Hamiltonian connected properties of rectangular supergrid graphs under some conditions. These two properties will be used to prove the Hamiltonian connectivity of L-shaped supergrid graphs. Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant 3$ and $n \geqslant 2$, and let $w = (1,1)$, $z = (2,1)$, and $f = (3,1)$ be three vertices in $R(m,n)$. We will prove the following two Hamiltonian connected properties of $R(m,n)$:

(P1) If $s = w = (1,1)$ and $t = z = (2,1)$, then there exists a Hamiltonian (s,t)-path P of $R(m,n)$ such that edge $(z,f) \in P$.
(P2) If ($n = 2$ and $\{s,t\} \notin \{\{w,z\}, \{(1,1),(2,2)\}, \{(2,1),(1,2)\}\}$) or ($n \geqslant 3$ and $\{s,t\} \neq \{w,z\}$), then there exists a Hamiltonian (s,t)-path Q of $R(m,n)$ such that edge $(w,z) \in Q$, where $(R(m,n),s,t)$ does not satisfy condition (F1).

First, we verify the first property (P1) as follows:

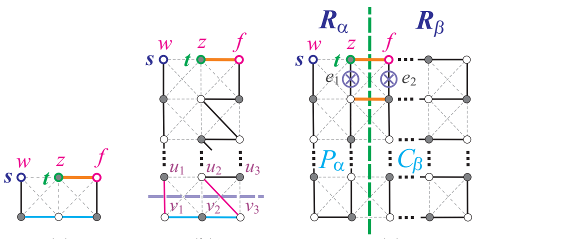
Lemma 7. *Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant 3$ and $n \geqslant 2$, and let $s = w = (1,1)$, $t = z = (2,1)$, and $f = (3,1)$. Then, there exists a Hamiltonian (s,t)-path P of $R(m,n)$ such that edge $(z,f) \in P$.*

Proof: Depending on whether $m = 3$, we consider the following two cases:

Case 1: $m = 3$. In this case, we claim that there exists a Hamiltonian (s,t)-path P of $R(m,n)$ such that $(z,f) \in P$ and a boundary path connecting down-left corner and down-right corner is a subpath of P.

We will prove the above claim by induction on n. Initially, let $n = 2$. The desired Hamiltonian (s,t)-path P of $R(3,2)$ can be easily constructed and is depicted in Fig. 9(a). Assume that the claim holds true when $n = k \geqslant 2$. Let $u_1 = (1,k)$, $u_2 = (2,k)$, and $u_3 = (3,k)$. By induction hypothesis, there exists a Hamiltonian (s,t)-path P_k of $R(m,k)$ such that $(z,f) \in P_k$ and P_k contains the boundary path $P' = u_1 \to u_2 \to u_3$ as a subpath. Let $P_k = P_1 \Rightarrow P' \Rightarrow P_2$. Consider $n = k+1$. Let $v_1 = (1,k+1)$, $v_2 = (2,k+1)$, $v_3 = (3,k+1)$, and let $\tilde{P} = v_1 \to v_2 \to v_3$. Then, $P_1 \Rightarrow u_1 \Rightarrow \tilde{P} \Rightarrow u_2 \to u_3 \Rightarrow P_2$ is the desired Hamiltonian (s,t)-path of $R(3,k+1)$. The constructed Hamiltonian (s,t)-path of $R(3,k+1)$ is shown in Fig. 9(b). By induction, the claim holds and hence, the lemma holds true in the case of $m = 3$.

Case 2: $m > 3$. In this case, we first make a vertical separation on $R(m,n)$ to partition it into two disjoint rectangular supergrid subgraphs $R_\alpha = R(2,n)$ and $R_\beta = R(m-2,n)$, as depicted in Fig. 9(c). We can easily construct a Hamiltonian (s,t)-path P_α of R_α such that P_α contains a boundary path placed to face R_β, as shown in Fig. 9(c). By Lemma 1, R_β contains a canonical Hamiltonian cycle C_β. We can place one flat face of C_β to face R_α. Then, there exist two edges



(a) (b) (c)

Fig. 9. The Hamiltonian (s,t)-path of rectangular supergrid graph $R(m,n)$ containing edge (z,f), where $s = w = (1,1)$, $t = z = (2,1)$, and $f = (3,1)$, for (a) $m = 3$ and $n = 2$, (b) $m = 3$ and $n = k+1 \geqslant 3$, and (c) $m \geqslant 4$ and $n \geqslant 2$, where solid lines indicate the Hamiltonian path between s and t and $\otimes$ represents the destruction of an edge while constructing such a Hamiltonian path.

$e_1 \in P_\alpha$ and $e_2 \in C_\beta$ such that $t(=z)$ is a vertex of e_1, f is a vertex of e_2, and $e_1 \approx e_2$. By Statement (2) of Proposition 5, P_α and C_β can be combined into a Hamiltonian (s,t)-path P of $R(m,n)$ such that edge $(z,f) \in P$. The constructed Hamiltonian (s,t)-path of $R(m,n)$ is depicted in Fig. 9(c). Thus, the lemma holds true when $m \geqslant 4$.

It immediately follows from the above cases that the lemma holds true. ■

Next, we will verify the second Hamiltonian connected property (P2) of $R(m,n)$, where $m \geqslant 3$ and $n \geqslant 2$. We first consider the following forbidden condition such that there exists no Hamiltonian (s,t)-path Q of $R(m,n)$ with edge $(w,z) \in Q$:

(F2) $n = 2$ and $\{s,t\} \in \{\{w,z\}, \{(1,1),(2,2)\}, \{(2,1),(1,2)\}\}$, or $n \geqslant 3$ and $\{s,t\} = \{w,z\}$.

The above condition states that $R(m,n)$ has no Hamiltonian (s,t)-path containing edge (w,z) if $(R(m,n),s,t)$ satisfies condition (F2). We will prove property (P2) by constructing a Hamiltonian (s,t)-path of $R(m,n)$ visiting edge (w,z) when $(R(m,n),s,t)$ does not satisfy conditions (F1) and (F2). To verify property (P2), we first consider the special case, in Lemma 8, that $m = 3$, $n \geqslant 2$, and either $s = z$ or $t = z$. This lemma can be proved by similar arguments in proving Case 1 of Lemma 7.

Lemma 8. *Let $R(m,n)$ be a rectangular supergrid graph with $m = 3$ and $n \geqslant 2$, s and t be its two distinct vertices, and let $w = (1,1)$ and $z = (2,1)$. If $(R(m,n),s,t)$ does not satisfy conditions* (F1) *and* (F2), *and either $s = z$ or $t = z$, then there exists a Hamiltonian (s,t)-path Q of $R(m,n)$ such that edge $(w,z) \in Q$.*

Proof: Without loss of generality, assume that $s = z$. Then, $t_x \leqslant s_x$ or $t_x \geqslant s_x$. That is, t may be to the left of s. Let $x = (1,n)$, $y = (2,n)$, and $r = (3,n)$ be three vertices of $R(m,n)$. We claim that
there exists a Hamiltonian (s,t)-path Q of $R(m,n)$ such that edge $(w,z) \in Q$, and $(x,y) \in Q$ if $t = r$; and $(y,r) \in Q$ otherwise.

We will prove the above claim by induction on n. Initially, let $n = 2$. Since $(R(m,n),s,t)$ does not satisfy conditions (F1) and (F2), $t \notin \{(1,1),(1,2),(2,2)\}$. Thus, $t \in \{(3,1),(3,2)\}$. Then, the desired Hamiltonian (s,t)-path Q of $R(3,2)$ can be easily constructed and is depicted in Fig. 10(a). Assume that the claim holds true when $n = k \geqslant 2$. Let $x_1 = (1,k)$, $y_1 = (2,k)$, and $r_1 = (3,k)$. By induction hypothesis, there exists Hamiltonian (s,p)-path Q_k of $R(3,k)$ such that edge $(w,z) \in Q_k$, and $(x_1,y_1) \in Q_k$ or $(y_1,r_1) \in Q_k$ depending on whether or not $p = r_1$. Consider that $n = k+1$. We first make a horizontal separation on $R(3,k+1)$ to obtain two disjoint parts $R_1 = R(3,k)$ and $R_2 = R(3,1)$, as shown in Fig. 10(b). Let $x_2 = (1,k+1)$, $y_2 = (2,k+1)$, and $r_2 = (3,k+1)$ be the three vertices of R_2. We will construct a Hamiltonian (s,t)-path Q_{k+1} of $R(3,k+1)$ such that $(w,z) \in Q_{k+1}$, and $(x_2,y_2) \in Q_{k+1}$ or $(y_2,r_2) \in Q_{k+1}$ as follows. Depending on the location of t, there are the following two cases:

Case 1: $t \in R_1$. Let $P_2 = x_2 \rightarrow y_2 \rightarrow r_2$. By induction hypothesis, there exists Hamiltonian (s,t)-path Q_k of $R(m,k)$ such that edge $(w,z) \in Q_k$, and $(x_1,y_1) \in Q_k$ if $t = r_1$; and $(y_1,r_1) \in Q_k$ otherwise. Thus, there exists an edge (u_k,v_k) in Q_k such that $start(P_2) \sim u_k$ and $end(P_2) \sim v_k$, where $(u_k,v_k) = (x_1,y_1)$ or (y_1,r_1). By Statement (4) of Proposition 5, Q_k and P_2 can be combined into a Hamiltonian (s,t)-path Q_{k+1} of $R(3,k+1)$ such that edges $(w,z),(x_2,y_2),(y_2,r_2) \in Q_{k+1}$. The construction of such a Hamiltonian path is depicted in Fig. 10(b).

Case 2: $t \in R_2$. In this case, $t \in \{x_2,y_2,r_2\}$. Then, there are the following three subcases:

Case 2.1: $t = x_2$. Let $p = r_1 \in R_1$ and $q = r_2 \in R_2$. Then, $p \sim q$. Let $P_2 = r_2(=q) \rightarrow y_2 \rightarrow x_2(=t)$. By induction hypothesis, there exists Hamiltonian (s,p)-path Q_k of $R(m,k)$ such that edges $(w,z),(x_1,y_1) \in Q_k$. Then, $Q_{k+1} = Q_k \Rightarrow P_2$ forms a Hamiltonian (s,t)-path of $R(m,k+1)$ with $(w,z),(x_2,y_2),(y_2,r_2) \in Q_{k+1}$. Fig. 10(c) shows the construction of such a Hamiltonian (s,t)-path.

Case 2.2: $t = r_2$. Let $p = x_1 \in R_1$ and $q = x_2 \in R_2$. Let $P_2 = x_2(=q) \rightarrow y_2 \rightarrow r_2(=t)$. By induction hypothesis, there exists Hamiltonian (s,p)-path Q_k of $R(m,k)$ such that edges $(w,z),(y_1,r_1) \in Q_k$. Then, $Q_{k+1} = Q_k \Rightarrow P_2$ forms a Hamiltonian (s,t)-path of $R(m,k+1)$ with $(w,z),(x_2,y_2),(y_2,r_2) \in Q_{k+1}$. Fig. 10(d) shows the construction of such a Hamiltonian (s,t)-path.

Case 2.3: $t = y_2$. Let $p = r_1 \in R_1$. Let $P_2 = r_2 \rightarrow y_2(=t)$. By induction hypothesis, there exists Hamiltonian (s,p)-path Q_k of $R(m,k)$ such that edges $(w,z),(x_1,y_1) \in Q_k$. Then, $Q'_k = Q_k \Rightarrow P_2$ is a Hamiltonian (s,t)-path of $R(m,k+1) - x_2$ such that edges $(w,z),(x_1,y_1),(y_2,r_2) \in Q'_k$. Since $x_2 \sim x_1$, $x_2 \sim y_1$, and edge $(x_1,y_1) \in Q'_k$, by Statement (3) of Proposition 5 Q'_k and x_2 can be combined into a Hamiltonian (s,t)-path Q_{k+1} of $R(3,k+1)$ such that edges $(w,z),(y_2,r_2) \in Q_{k+1}$. Fig. 10(e) depicts such a construction of Hamiltonian (s,t)-path.

It immediately follows from the above cases that the claim holds true when $n = k+1$. By induction, the claim holds true and, hence, the lemma is true. ■

We next verify property (P2) in the following lemma.

Lemma 9. *Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant 3$ and $n \geqslant 2$, s and t be its two distinct vertices, and let $w = (1,1)$ and $z = (2,1)$. If $(R(m,n),s,t)$ does not satisfy conditions* (F1) *and* (F2), *then there exists a Hamiltonian (s,t)-path Q of $R(m,n)$ such that edge $(w,z) \in Q$.*

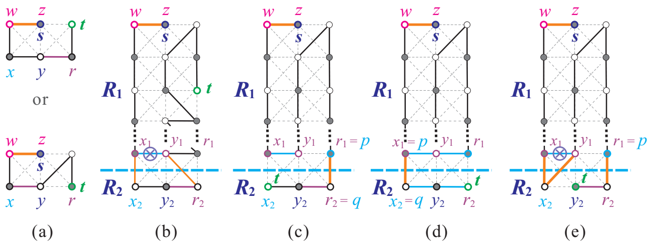


Fig. 10. The Hamiltonian (s, t)-path of 3-rectangle $R(3, n)$ containing edge (w, z), where $s = z = (1, 2)$ and $w = (1, 1)$, for (a) $n = 2$, (b) $n = k + 1 \geqslant 3$ and $t \in R_1(= R(3, k))$, and (c)–(e) $n = k + 1 \geqslant 3$ and $t \in R_2(= R(3, 1))$, where solid lines indicate the constructed Hamiltonian (s, t)-path and $\otimes$ represents the destruction of an edge while constructing such a Hamiltonian path.

Proof: We will provide a constructive method to prove this lemma. By assumption of this lemma, $\{s, t\} \neq \{w, z\}$ and, hence, $0 \leqslant |\{s, t\} \cap \{w, z\}| \leqslant 1$. Then, there are three cases:

Case 1: $\{s, t\} \cap \{w, z\} = \emptyset$. In this case, $s, t \notin \{w, z\}$. By Lemma 3, $R(m, n)$ contains a Hamiltonian (s, t)-path $\tilde{Q}$. If edge $(w, z) \in \tilde{Q}$, then $\tilde{Q}$ is the desired Hamiltonian (s, t)-path of $R(m, n)$. Suppose that edge $(w, z) \notin \tilde{Q}$. Let $x = (1, 2)$ and $y = (2, 2)$. Then, $N(w) - \{z\} = \{x, y\}$. Let $\tilde{Q} = Q_1^w \Rightarrow w \Rightarrow Q_2^w$. Since $N(w) - \{z\} = \{x, y\}$, $\{end(Q_1^w), start(Q_2^w)\} = \{x, y\}$ and, hence, $end(Q_1^w) \sim start(Q_2^w)$. Then, $\tilde{Q}' = Q_1^w \Rightarrow Q_2^w$ is a Hamiltonian (s, t)-path of $R(m, n) - w$, where edge $(end(Q_1^w), start(Q_2^w)) = (x, y)$ is visited by $\tilde{Q}'$. Let $\tilde{Q}' = Q_1^z \Rightarrow z \Rightarrow Q_2^z$. Depending on whether $end(Q_1^z) \sim start(Q_2^z)$, we consider the following two subcases:

Case 1.1: $end(Q_1^z) \sim start(Q_2^z)$. In this subcase, $Q^z = Q_1^z \Rightarrow Q_2^z$ is a Hamiltonian (s, t)-path of $R(m, n) - \{w, z\}$, where edge (x, y) is visited by Q^z. Let $P' = w \to z$. Then, there exist one edge $(x, y) \in Q^z$ such that $start(P') \sim x$ and $end(P') \sim y$. By Statement (4) of Proposition 5, Q^z and P' can be combined into a Hamiltonian (s, t)-path Q of $R(m, n)$ such that edge $(w, z) \in Q$. The construction of such a Hamiltonian (s, t)-path is depicted in Fig. 11(a).

Case 1.2: $end(Q_1^z) \nsim start(Q_2^z)$. Since $N(z) - \{w, x\}$ forms a clique, $x \in \{end(Q_1^z), start(Q_2^z)\}$. Then, $z \to x \to y$ is a subpath of $\tilde{Q}'$. Let $\tilde{Q}' = Q_1^x \Rightarrow x \Rightarrow Q_2^x$. Then, $\{end(Q_1^x), start(Q_2^x)\} = \{y, z\}$. Thus, $Q^x = Q_1^x \Rightarrow Q_2^x$ is a Hamiltonian (s, t)-path of $R(m, n) - \{w, x\}$, where edge (y, z) is visited by Q^x. Let $P' = w \to x$. Then, there exist one edge $(y, z) \in Q^x$ such that $start(P') \sim z$ and $end(P') \sim y$. By Statement (4) of Proposition 5, Q^x and P' can be combined into a Hamiltonian (s, t)-path Q of $R(m, n)$ such that edge $(w, z) \in Q$. The construction of such a Hamiltonian (s, t)-path is shown in Fig. 11(b).

Case 2: $s = w$ or $t = w$. Without loss of generality, assume that $s = w$. First, consider that $n = 2$. Then, $R(m, n)$ is a 2-rectangle. By assumption of the lemma, $(R(m, n), s, t)$ does not satisfy condition (F2), and, hence, $t \notin \{(2, 1), (2, 2)\}$. If $t = (1, 2)$, then a Hamiltonian (s, t)-path Q of $R(m, n)$ can be easily constructed by visiting each boundary edge of $R(m, n)$ except boundary edge (s, t), and, hence, $(w, z) \in Q$. Let $t = (t_x, t_y)$ satisfy that $t_x \geqslant 3$. We first make a vertical separation on $R(m, n)$ to obtain two disjoint parts R_α and R_β, as depicted in Fig. 11(c). Let $p = (t_x - 1, 2) \in R_\alpha$ and $q = (t_x, t_y - 1)$ or $(t_x, t_y + 1)$ in

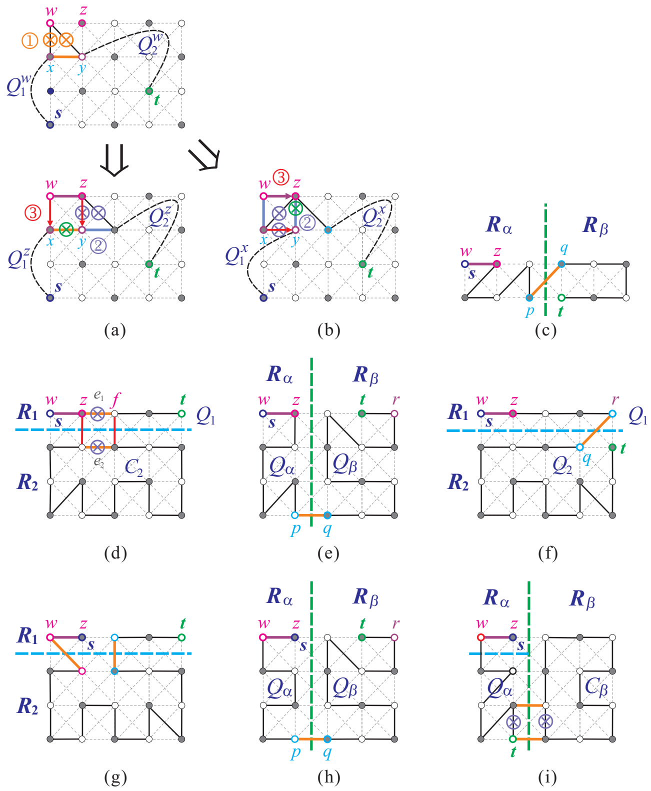


Fig. 11. The construction of Hamiltonian (s, t)-path Q in $R(m, n)$ with edge $(w, z) \in Q$ for (a)–(b) $s, t \notin \{w, z\}$, (c) $s = w$ and $n = 2$, (d)–(f) $s = w$ and $n \geqslant 3$, and (g)–(i) $s = z$, $m \geqslant 4$, and $n \geqslant 3$, where bold dashed lines indicate the subpaths of the constructed Hamiltonian (s, t)-path, solid (arrow) lines indicate the edges in the constructed Hamiltonian path, and $\otimes$ represents the destruction of an edge while constructing such a Hamiltonian path.

R_β, where $q \neq t$ and $q_x = t_x$. Then, $p \sim q$ and we can easily construct Hamiltonian (s, p)-path Q_α and (q, t)-path Q_β of R_α and R_β, respectively, such that edge $(w, z) \in Q_\alpha$. Thus, $Q = Q_\alpha \Rightarrow Q_\beta$ is a Hamiltonian (s, t)-path of $R(m, n)$ with $(w, z) \in Q$. The construction of such a Hamiltonian (s, t)-path is depicted in Fig. 11(c). Next, consider that $n \geqslant 3$. Let $t = (t_x, t_y)$. Depending on the location of t, we consider the following three subcases:

Case 2.1: $t_y = 1$ and $t_x = m$. In this subcase, t is located at the up-right corner of $R(m, n)$. We first make a horizontal separation on $R(m, n)$ to obtain two disjoint parts $R_1 = R(m, 1)$ and $R_2 = R(m, n - 1)$, as shown in Fig. 11(d). Note that $m \geqslant 3$ and $n - 1 \geqslant 2$. By visiting all boundary edges of R_1 from s to t, we get a Hamiltonian (s, t)-path Q_1 of R_1 with edge $(w, z) \in Q_1$. By Lemma 1, we can construct a canonical Hamiltonian cycle C_2 of R_2 such that its one flat face is placed to face R_1. Then, there exist two edges $e_1(= (z, f)) \in Q_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, where $z = (2, 1)$ and $f = (3, 1)$. By Statement (2) of Proposition 5, P_1 and C_2 can be merged into a Hamiltonian (s, t)-path Q of $R(m, n)$ such that edge $(w, z) \in Q$. The construction of such a Hamiltonian (s, t)-path is shown in Fig. 11(d).

Case 2.2: $t_y = 1$ and $t_x < m$. Let $r = (m, 1)$ be the up-right corner of $R(m, n)$. Then, $z_x < t_x < r_x$, i.e., $2 < t_x < m$, and, hence, $m \geqslant 4$. We first make a vertical separation on $R(m, n)$ to get two disjoint parts $R_\alpha = R(2, n)$ and $R_\beta = R(m - 2, n)$, as depicted in Fig.

11(e), where $n \geqslant 3$ and $m - 2 \geqslant 2$. Let $p = (2, n)$ be the down-right corner of R_α and let $q = (3, n)$ be the down-left corner of R_β. Then, $p \sim q$ and, (R_α, s, p) and (R_β, q, t) do not satisfy condition (F1). Since R_α is a 2-rectangle, we can easily construct a a Hamiltonian (s, p)-path Q_α of R_α such that edge $(w, z) \in Q_\alpha$, as shown in Fig. 11(e). By Lemma 3, there exists a Hamiltonian (q, t)-path Q_β of R_β. Then, $Q = Q_\alpha \Rightarrow Q_\beta$ forms a Hamiltonian (s, t)-path of $R(m, n)$ such that edge $(w, z) \in Q$. Such a Hamiltonian (s, t)-path is depicted in Fig. 11(e).

Case 2.3: $t_y > 1$. In this subcase, we first make a horizontal separation on $R(m, n)$ to obtain two disjoint parts $R_1 = R(m, 1)$ and $R_2 = R(m, n-1)$, as shown in Fig. 11(f), where $m \geqslant 3$ and $n - 1 \geqslant 2$. Let $r = (m, 1)$, then $r \in R_1$. Let $q = (m, 2)$ if $t \neq (m, 2)$; otherwise $q = (m - 1, 2)$. A simple check shows that (R_2, q, t) does not satisfy condition (F1). By visiting every vertex of R_1 from s to r, we get a Hamiltonian (s, t)-path Q_1 of R_1 with edge $(w, z) \in Q_1$. By Lemma 3, there exists a Hamiltonian (q, t)-path Q_2 of R_2. Then, $Q = Q_1 \Rightarrow Q_2$ is a Hamiltonian (s, t)-path of $R(m, n)$ with $(w, z) \in Q$. The constructed Hamiltonian (s, t)-path in this subcase can be found in Fig. 11(f).

Case 3: $s = z$ or $t = z$. By symmetry, assume that $s = z$. Then, t may be to the left of s, i.e., $t_x < s_x$. When $n = 2$, a Hamiltonian (s, t)-path Q of $R(m, n)$ with $(w, z) \in Q$ can be constructed by similar arguments in Fig. 11(c). By Lemma 8, the desired Hamiltonian (s, t)-path of $R(m, n)$ can be constructed when $m = 3$. In the following, suppose that $m \geqslant 4$ and $n \geqslant 3$. We then make a horizontal separation on $R(m, n)$ to obtain two disjoint parts $R_1 = R(m, 1)$ and $R_2 = R(m, n - 1)$, as shown in Fig. 11(g), where $m \geqslant 4$ and $n - 1 \geqslant 2$. Then, $s \in R_1$. Depending on whether $t \in R_1$, we consider the following two subcases:

Case 3.1: $t \in R_1$. A Hamiltonian (s, t)-path Q of $R(m, n)$ with $(w, z) \in Q$ can be constructed by similar arguments in proving Case 2.1 and Case 2.2. Figs. 11(g)–(h) show such constructions of the desired Hamiltonian (s, t)-paths of $R(m, n)$.

Case 3.2: $t \in R_2$. In this subcase, we make a vertical separation on $R(m, n)$ to obtain two disjoint parts $R_\alpha = R(2, n)$ and $R_\beta = R(m-2, n)$, where $m-2 \geqslant 2$ and $n \geqslant 3$, as shown in Fig. 11(i). Suppose that $t \in R_\alpha$. By similar technique in Fig. 11(c) and Lemma 3, we can easily construct a Hamiltonian (s, t)-path Q_α of R_α such that $(w, z) \in Q_\alpha$ and Q_α contains one boundary edge e_α that is placed to face R_β, as depicted in Fig. 11(i). By Lemma 1, there exists a canonical Hamiltonian cycle C_β of R_β such that its one flat face is placed to face R_α. Then, there exist two edges $e_\alpha \in Q_\alpha$ and $e_\beta \in C_\beta$ such that $e_\alpha \approx e_\beta$. By Statement (2) of Proposition 5, Q_α and C_β can be combined into a Hamiltonian (s, t)-path Q of $R(m, n)$ with edge $(w, z) \in Q$. The construction of such a Hamiltonian (s, t)-path is shown in Fig. 11(i). On the other hand, suppose that $t \in R_\beta$. Let $p \in R_\alpha$ and $q \in R_\beta$ such that $p \sim q$ and, (R_α, s, p) and (R_β, q, t) do not satisfy condition (F1). By Lemma 3, there exist Hamiltonian (s, p)-path Q_α and Hamiltonian (q, t)-path Q_β of R_α and R_β, respectively. Since R_α is a 2-rectangle, we can easily construct Q_α to satisfy $(w, z) \in Q_\alpha$. Then, $Q = Q_\alpha \Rightarrow Q_\beta$ is a Hamiltonian (s, t)-path of $R(m, n)$ with edge $(w, z) \in Q$.

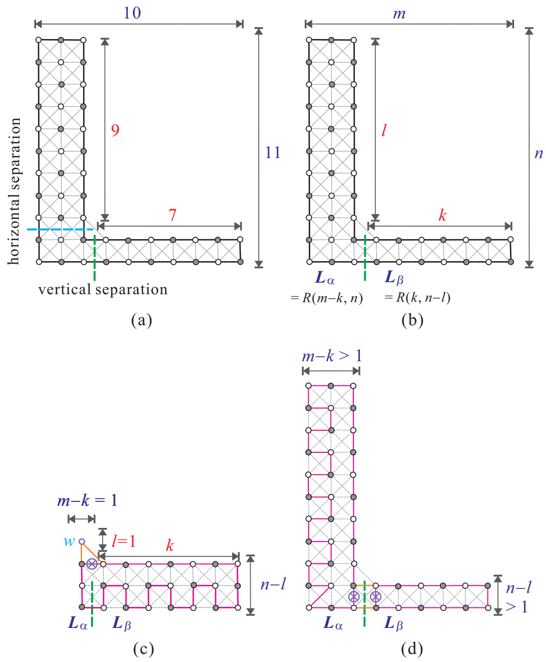


Fig. 12. (a) Separations on $L(10, 11; 7, 9)$, (b) a vertical separation on $L(m, n; k, l)$ to obtain $L_\alpha = R(m - k, n)$ and $L_\beta = R(k, l)$, (c) a Hamiltonian cycle of $L(m, n; k, l)$ when $m - k = 1$ and $n - l \geqslant 2$, and (d) a Hamiltonian cycle of $L(m, n; k, l)$ when $m - k \geqslant 2$, $n - l \geqslant 2$, and $k \geqslant 2$, where bold dashed vertical (resp., horizontal) line in (a) indicates a vertical (resp., horizontal) separation on $L(10, 11; 7, 9)$, and $\otimes$ represents the destruction of an edge while constructing a Hamiltonian cycle of $L(m, n; k, l)$.

We have considered any case to construct a Hamiltonian (s, t)-path Q of $R(m, n)$ with edge $(w, z) \in Q$. This completes the proof of the lemma. ■

IV. The Hamiltonian and Hamiltonian connected properties of L-shaped supergrid graphs

In this section, we will verify the Hamiltonicity and Hamiltonian connectivity of L-shaped supergrid graphs. Let $L(m, n; k, l)$ be a L-shaped supergrid graph. We will make a vertical or horizontal separation on $L(m, n; k, l)$ to obtain two disjoint rectangular supergrid graphs. For an example, the bold dashed vertical (resp., horizontal) line in Fig. 12(a) indicates a vertical (resp., horizontal) separation on $L(10, 11; 7, 9)$ that is to partition it into $R(3, 11)$ and $R(7, 2)$ (resp., $R(3, 9)$ and $R(10, 2)$). The following two subsections will prove the Hamiltonicity and Hamiltonian connectivity of $L(m, n; k, l)$ respectively.

A. The Hamiltonian property of L-shaped supergrid graphs

In this subsection, we will prove the Hamiltonicity of L-shaped supergrid graphs. Obviously, $L(m, n; k, l)$ contains no Hamiltonian cycle if there exists a vertex w in $L(m, n; k, l)$ such that $deg(w) = 1$. Thus, $L(m, n; k, l)$ is not Hamiltonian when the following condition is satisfied.

(F3) there exists a vertex w in $L(m, n; k, l)$ such that $deg(w) = 1$.

When the above condition is satisfied, we get that ($m-k = 1$ and $l > 1$) or ($n - l = 1$ and $k > 1$). We then show the Hamiltonicity of L-shaped supergrid graphs as follows:

Theorem 10. *Let $L(m,n;k,l)$ be a L-shaped supergrid graph. Then, $L(m,n;k,l)$ contains a Hamiltonian cycle if and only if it does not satisfy condition* (F3).

Proof: Obviously, $L(m,n;k,l)$ contains no Hamiltonian cycle if it satisfies condition (F3). In the following, we will prove that $L(m,n;k,l)$ contains a Hamiltonian cycle if it does not satisfy condition (F3). Assume that $L(m,n;k,l)$ does not satisfy condition (F3). We prove it by constructing a Hamiltonian cycle of $L(m,n;k,l)$. First, we make a vertical separation on $L(m,n;k,l)$ to obtain two disjoint rectangular supergrid subgraphs $L_\alpha = R(m-k,n)$ and $L_\beta = R(k,n-l)$, as depicted in Fig. 12(b). Depending on the sizes of L_α and L_β, there are the following two cases:

Case 1: $m-k=1$ or $n-l=1$. By symmetry, we assume that $m-k=1$. Since there exists no vertex w in $L(m,n;k,l)$ such that $deg(w)=1$, we get that $l=1$ (see Fig. 12(c)). Consider that $n-l=1$. Then, $k=1$. Thus, $L(m,n;k,l)$ consists of only three vertices which forms a cycle. On the other hand, consider that $n-l \geqslant 2$. Let w be a vertex of L_α with $deg(w)=2$, $L^*_\alpha = L_\alpha - \{w\}$, and let $L^* = L^*_\alpha \cup L_\beta$. Then, $L^* = R(k+1,n-l) = R(m,n-1)$, where $k+1 \geqslant 2$ and $n-l \geqslant 2$. By Lemma 1, L^* contains a canonical Hamiltonian cycle HC^*. We can place one flat face of HC^* to face w. Thus, there exists an edge (u,v) in HC^* such that $w \sim u$ and $w \sim v$. By Statement (3) of Proposition 5, w and HC^* can be combined into a Hamiltonian cycle of $L(m,n;k,l)$. For example, Fig. 12(c) depicts a such construction of Hamiltonian cycle of $L(m,n;k,l)$ when $m-k=1$ and $n-l \geqslant 2$. Thus, $L(m,n;k,l)$ is Hamiltonian if $m-k=1$ or $n-l=1$.

Case 2: $m-k \geqslant 2$ and $n-l \geqslant 2$. In this case, $L_\alpha = R(m-k,n)$ and $L_\beta = R(k,n-l)$ satisfy that $m-k \geqslant 2$ and $n-l \geqslant 2$. Since $n-l \geqslant 2$ and $l \geqslant 1$, we get that $n \geqslant l+2 \geqslant 3$. Thus, $L_\alpha = R(m-k,n)$ satisfies that $m-k \geqslant 2$ and $n \geqslant 3$. By Lemma 1, L_α contains a canonical Hamiltonian cycle HC_α whose one flat face is placed to face L_β. Consider that $k=1$. Then, $L_\beta = R(k,n-l)$ is a 1-rectangle. Let $V(L_\beta) = \{v_1, v_2, \cdots, v_{n-l}\}$, where $v_{i+1_y} = v_{i_y}+1$ for $1 \leqslant i \leqslant n-l-1$. Since HC_α contains a flat face that is placed to face L_β, there exists an edge (u,v) in HC_α such that $u \sim v_1$ and $v \sim v_1$. By Statement (3) of Proposition 5, v_1 and HC_α can be combined into a cycle HC^1_α. By the same argument, v_2, v_3, $\cdots$, v_{n-l} can be merged into the cycle to form a Hamiltonian cycle of $L(m,n;k,l)$. On the other hand, consider that $k \geqslant 2$. Then, $L_\beta = R(k,n-l)$ satisfies that $k \geqslant 2$ and $n-l \geqslant 2$. By Lemma 1, L_β contains a canonical Hamiltonian cycle HC_β such that its one flat face is placed to face L_α. Then, there exist two edges $e_1 = (u_1,v_1) \in HC_\alpha$ and $e_2 = (u_2,v_2) \in HC_\beta$ such that $e_1 \approx e_2$. By Statement (1) of Proposition 5, HC_α and HC_β can be combined into a Hamiltonian cycle of $L(m,n;k,l)$. For instance, Fig. 12(d) shows a Hamiltonian cycle of $L(m,n;k,l)$ when $m-k \geqslant 2$, $n-l \geqslant 2$, and $k \geqslant 2$. Thus, $L(m,n;k,l)$ contains a Hamiltonian cycle in this case.

It immediately follows from the above cases that $L(m,n;k,l)$ contains a Hamiltonian cycle if it does not satisfy condition (F3). Thus, the theorem holds true. ■

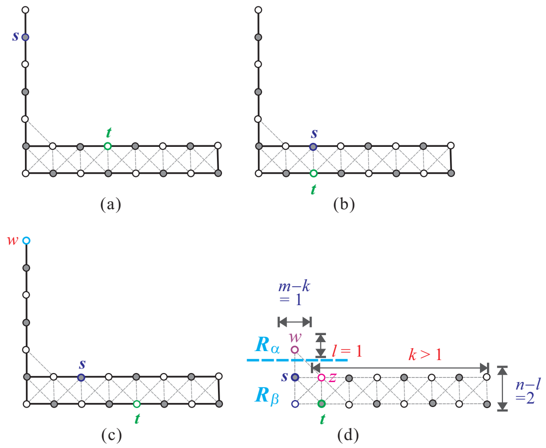


Fig. 13. L-shaped supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) s is a cut vertex, (b) $\{s,t\}$ is a vertex cut, (c) there exists a vertex w such that $deg(w)=1$, $w \neq s$, and $w \neq t$, and (d) $m-k=1$, $n-l=2$, $l=1$, $k \geqslant 2$, $s=(1,2)$, and $t=(2,3)$.

B. The Hamiltonian connected property of L-shaped supergrid graphs

In this subsection, we will verify the Hamiltonian connectivity of L-shaped supergrid graphs. Besides condition (F1) (as depicted in Fig. 13(a) and Fig. 13(b)), whenever one of the following conditions is satisfied then $HP(L(m,n;k,l),s,t)$ does not exist.

- (F4) there exists a vertex w in $L(m,n;k,l)$ such that $deg(w)=1$, $w \neq s$, and $w \neq t$ (see Fig. 13(c)).
- (F5) $m-k=1$, $n-l=2$, $l=1$, $k \geqslant 2$, and $\{s,t\} = \{(1,2),(2,3)\}$ or $\{(1,3),(2,2)\}$ (see Fig. 13(d)).

The following lemma shows the necessary condition for that $HP(L(m,n;k,l),s,t)$ does exist.

Lemma 11. *Let $L(m,n;k,l)$ be a L-shaped supergrid graph with two vertices s and t. If $HP(L(m,n;k,l),s,t)$ does exist, then $(L(m,n;k,l),s,t)$ does not satisfy conditions* (F1), (F4), *and* (F5).

Proof: Assume that $(L(m,n;k,l),s,t)$ satisfies one of conditions (F1), F(4), and (F5). For condition (F1), the proof is the same as that of Lemma 2. For condition (F4), it is easy to see that $HP(L(m,n;k,l),s,t)$ does not exist (see Fig. 13(c)). For (F5), we make a horizontal separation on it to obtain two disjoint rectangular supergrid subgraphs $R_\alpha = R(m-k,l)$ and $R_\beta = R(m,n-l)$, as depicted in Fig. 13(d). Suppose that $m-k=1$, $n-l=2$, $l=1$, and $k \geqslant 2$. Then, R_α contains only one vertex w. Let $s=(1,2)$, $t=(2,3)$, and $z=(2,2)$. Then, there exists no Hamiltonian (s,t)-path of R_α such that it contains edge (s,z). Thus, w can not be combined into the Hamiltonian (s,t)-path of R_α and hence $HP(L(m,n;k,l),s,t)$ does not exist. ■

We then prove that $HP(L(m,n;k,l),s,t)$ does exist when $(L(m,n;k,l),s,t)$ does not satisfy conditions (F1), (F4), and (F5). First, we consider the case that $m-k=1$ or $n-l=1$ in the following lemma.

Lemma 12. *Let $L(m,n;k,l)$ be a L-shaped supergrid graph, and let s and t be its two distinct vertices such*

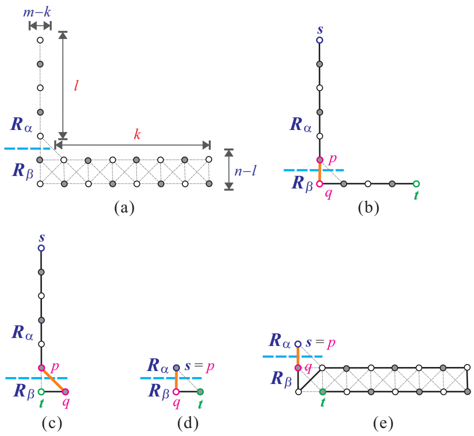


Fig. 14. (a) The horizontal separation on $L(m, n; k, l)$ to obtain $R_\alpha = R(m-k, l)$ and $R_\beta = R(m, n-l)$ under that $m-k=1$, and (b)–(e) a Hamiltonian (s,t)-path of $L(m, n; k, l)$ for $m-k=1$, $s \in R_\alpha$, and $t \in R_\beta$, where bold solid lines indicate the constructed Hamiltonian (s,t)-path.

that $(L(m, n; k, l), s, t)$ *does not satisfy conditions* (F1), (F4), *and* (F5). *Assume that* $m-k=1$ *or* $n-l=1$. *Then,* $L(m, n; k, l)$ *contains a Hamiltonian* (s,t)*-path, i.e.,* $HP(L(m, n; k, l), s, t)$ *does exist if* $m-k=1$ *or* $n-l=1$.

Proof: We prove this lemma by showing how to construct a Hamiltonian (s,t)-path of $L(m, n; k, l)$ when $m-k=1$ or $n-l=1$. By symmetry, we assume that $m-k=1$. We make a horizontal separation on $L(m, n; k, l)$ to obtain two disjoint rectangular supergrid graphs $R_\alpha = R(m-k, l)$ and $R_\beta = R(m, n-l)$ (see Fig 14(a)). Consider the following cases:

Case 1: s_y(resp., t_y) $\leqslant l$ and t_y(resp., s_y) $> l$. Without loss of generality, assume that $s_y \leqslant l$ and $t_y > l$. Let $p \in V(R_\alpha)$ and $q \in V(R_\beta)$ such that $p \sim q$, $p = (1, l)$, and $q = (1, l+1)$ if $t \neq (1, l+1)$; otherwise $q = (2, l+1)$. Notice that, in this case, if $|V(R_\alpha)| = 1$, then $p = s$. Clearly, $s = (1,1)$. If $l > 1$ and $s_y > 1$, then $(L(m, n; k, l), s, t)$ satisfies condition (F1), a contradiction. Consider (R_α, s, p). Since $s = (1,1)$ and $p = (1, l)$, (R_α, s, p) does not condition (F1). Consider (R_β, q, t). Condition (F1) holds, if

(i) $k > 1$, $n-l = 1$, and $t \neq (m, n)$. If this case holds, then $(L(m, n; k, l), s, t)$ satisfies (F1), a contradiction.
(ii) $n-l = 2$ and $q_x = t_x > m-k(=1)$. Since $(q_x = 1$ and $t_x \geqslant 1)$ or $(q_x = 2$ and $t = (1, l+1))$, clearly $q_x \neq t_x$ or $t_x = q_x = 1$.

Therefore, (R_β, q, t) does not satisfy condition (F1). Since (R_α, s, p) and (R_β, q, t) do not satisfy condition (F1), by Lemma 3 there exist Hamiltonian (s,p)-path P_α and Hamiltonian (q,t)-path P_β of R_α and R_β, respectively. Then, $P_\alpha \Rightarrow P_\beta$ is a Hamiltonian (s,t)-path of $L(m, n; k, l)$. The construction of a such Hamiltonian (s,t)-path is depicted in Figs. 14(b)–(e).

Case 2: $s_y, t_y > l$. In this case, $l = 1$ and $|V(R_\alpha)| = 1$. Otherwise, it satisfies condition (F4). Let $r \in V(R_\alpha)$, $w = (1, l+1)$, and $z = (2, l+1)$. Consider (R_β, s, t). If (R_β, s, t) satisfies condition (F1), then $(L(m, n; k, l), s, t)$ satisfies (F1), a contradiction. Also, (R_β, s, t) does not satisfy condition (F2). Otherwise, $(L(m, n; k, l), s, t)$ satisfies (F1) or (F5), a contradiction. Since (R_β, s, t) does not satisfy conditions (F1) and (F2), by Lemma 3, where $n-l=1$, or Lemma 9, where $n-l \geqslant 2$, there exists a Hamiltonian (s,t)-path P_β of R_β such that $(w, z) \in P_\beta$. By Statement (3) of Proposition 5, vertex r can be combined into path P_β to form a Hamiltonian (s,t)-path of $L(m, n; k, l)$. The construction of a such Hamiltonian (s,t)-path of $L(m, n; k, l)$ is depicted in Fig. 15. Notice that, in this subcase, we have constructed a Hamiltonian (s,t)-path P such that an edge $(r, w) \in P$. ∎

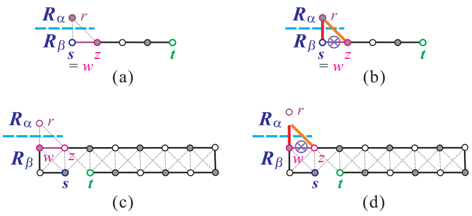


Fig. 15. (a) and (c) The Hamiltonian (s,t)-path of R_β containing edge (w,z) under that $m-k=1$ and $s,t \in R_\beta$, and (b) and (d) the Hamiltonian (s,t)-path of $L(m, n; k, l)$ for (a) and (c) respectively, where bold solid lines indicate the constructed Hamiltonian (s,t)-path and ⊗ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path of $L(m, n; k, l)$.

Next, we consider the case that $m-k \geqslant 2$ and $n-l \geqslant 2$. Notice that in this case $(L(m, n; k, l), s, t)$ does not satisfy conditions (F4) and (F5).

Lemma 13. *Let* $L(m, n; k, l)$ *be a L-shaped supergrid graph with* $m-k \geqslant 2$ *and* $n-l \geqslant 2$*, and let* s *and* t *be its two distinct vertices such that* $(L(m, n; k, l), s, t)$ *does not satisfy condition* (F1). *Then,* $L(m, n; k, l)$ *contains a Hamiltonian* (s,t)*-path, i.e.,* $HP(L(m, n; k, l), s, t)$ *does exist.*

Proof: We will provide a constructive method to prove this lemma. That is, a Hamiltonian (s,t)-path of $L(m, n; k, l)$ will be constructed. Since $m-k \geqslant 2$, $n-l \geqslant 2$, and $k, l \geqslant 1$, we get that $m \geqslant 3$ and $n \geqslant 3$. Note that $L(m, n; k, l)$ is obtained from $R(m, n)$ by removing $R(k, l)$ from its upper-right corner. Based on the sizes of k and l, there are the following two cases:

Case 1: $k = 1$ and $l = 1$. Let z be the only vertex in $V(R(m, n) - L(m, n; k, l))$. Then, $z = (m, 1)$ is the upper-right corner of $R(m, n)$. By Lemma 3, there exists a Hamiltonian (s,t)-path P of $R(m, n)$. Let $P = P_1 \Rightarrow z \Rightarrow P_2$. Since $N(z)$ forms a clique, $end(P_1) \sim start(P_2)$. Thus, $P_1 \Rightarrow P_2$ forms a Hamiltonian (s,t)-path of $L(m, n; k, l)$. The construction of a such Hamiltonian (s,t)-path is depicted in Fig. 16(a).

Case 2: $k \geqslant 2$ or $l \geqslant 2$. By symmetry, we can only consider that $k \geqslant 2$. Depending on the locations of s and t, we consider the following three subcases:

Case 2.1: $s_x, t_x \leqslant m-k$. Let $\tilde{R}$ be the graph with $V(\tilde{R}) = \{v \in V(L(m, n; k, l)) | v_x \leqslant m-k\}$. Then, $\tilde{R} = R(m-k, n)$ and $s, t \in \tilde{R}$. Depending on whether $\{s, t\}$ is a vertex cut of $\tilde{R}$, there are the following two subcases:

Case 2.1.1: $(m-k \geqslant 3)$ or $(m-k = 2$ and $[(s_y \neq t_y), (s_y = t_y = 1)$, or $(s_y = t_y = n)])$. In this subcase, $\{s, t\}$ is not a vertex cut of $\tilde{R}$. We make a vertical separation on $L(m, n; k, l)$ to obtain two disjoint rectangular supergrid graphs $R_\alpha = R(m-k, n)$ and $R_\beta = R(k, n-l)$. Consider (R_α, s, t). Condition (F1) holds only if $m-k = 2$ and $2 \leqslant s_y = t_y \leqslant n-1$. Since $s_y \neq t_y$, $s_y = t_y = 1$, or $s_y = t_y = n$, it is clear that (R_α, s, t) does not satisfy

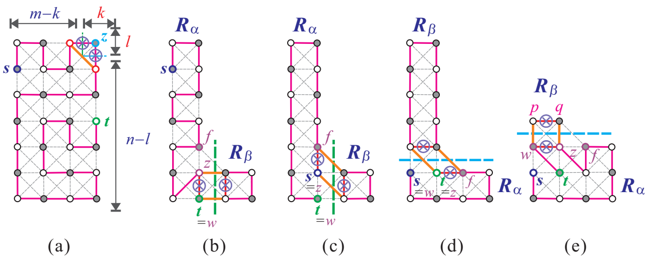


Fig. 16. The construction of Hamiltonian (s,t)-path of $L(m,n;k,l)$ under that $m-k \geqslant 2$ and $n-l \geqslant 2$ for (a) $k=1$ and $l=1$, (b)–(c) $k \geqslant 2$, $s_x, t_x \leqslant m-k$ and $\{s,t\}$ is not a vertex cut of $\tilde{R}$ with vertex set $\{v \in V(L(m,n;k,l)) | v_x \leqslant m-k\}$, and (d)–(e) $k \geqslant 2$, $s_x, t_x \leqslant m-k$ and $\{s,t\}$ is a vertex cut of $\tilde{R}$, where bold lines indicate the constructed Hamiltonian (s,t)-path and $\otimes$ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path of $L(m,n;k,l)$.

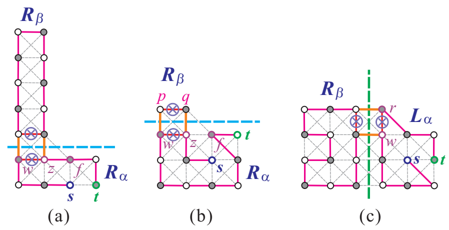


Fig. 17. The construction of Hamiltonian (s,t)-path of $L(m,n;k,l)$ under that $m-k \geqslant 2$, $n-l \geqslant 2$, $k \geqslant 2$, and $s_x, t_x > m-k$ for (a)–(b) $(l > 1)$ or ($l=1$ and $m-k=2$)), and (c) $l=1$ and $m-k>2$, where bold lines indicate the constructed Hamiltonian (s,t)-path and $\otimes$ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path of $L(m,n;k,l)$.

condition (F1). Let $w=(m-k,n)$, $z=(m-k,n-1)$, and $f=(m-k,n-2)$. Also, assume $(1,1)$ is the down-right corner of R_α. Since (R_α,s,t) does not satisfy condition (F1), by Lemma 3 (when (R_α,s,t) satisfies condition (F2)), Lemma 7, and Lemma 9, we can construct a Hamiltonian (s,t)-path P_α of R_α such that edge (w,z) or (z,f) is in P_α. By Lemma 1, there exists a Hamiltonian cycle C_β of R_β such that its one flat face is placed to face R_α. Then, there exist two edges $e_1 \in C_\beta$ and (w,z) (or (z,f)) $\in P_\alpha$ such that $e_1 \approx (w,z)$ or $e_1 \approx (z,f)$. By Statement (2) of Proposition 5, P_α and C_β can be combined into a Hamiltonian (s,t)-path of $L(m,n;k,l)$. The construction of a such Hamiltonian path is shown in Figs. 16(b)–(c).

Case 2.1.2: $m-k=2$ and $2 \leqslant s_y = t_y \leqslant n-1$. In this subcase, $\{s,t\}$ is a vertex cut of $\tilde{R}$. If $s_y = t_y \leqslant l$, then $(L(m,n;k,l),s,t)$ satisfies condition (F1), a contradiction. Thus, $s_y = t_y > l$. Let $w=(1,l+1)$, $z=(2,l+1)$, and $f=(3,l+1)$. We make a horizontal separation on $L(m,n;k,l)$ to obtain two disjoint rectangular supergrid graphs $R_\beta = R(m-k,l)$ and $R_\alpha = R(m,n-l)$. A simple check shows that (R_α,s,t) does not satisfy condition (F1). Since (R_α,s,t) does not satisfy conditions (F1), by Lemma 7 and Lemma 9, we can construct a Hamiltonian (s,t)-path P_α of R_α such that edge (w,z) or (z,f) is in P_α depending on whether $\{s,t\} = \{(1,l+1),(2,l+1)\}$. First, let $l>1$. By Lemma 1, there exists a Hamiltonian cycle C_β of R_β such that its one flat face is placed to face R_α. Then, there exist two edges $e_1 \in C_\beta$ and (w,z) (or (z,f)) $\in P_\alpha$ such that $e_1 \approx (w,z)$ or $e_1 \approx (z,f)$. By Statement (2) of Proposition 5, P_α and C_β can be combined into a Hamiltonian (s,t)-path of $L(m,n;k,l)$. The construction of a such Hamiltonian path is depicted in Fig. 16(d). Next, let $l=1$. Then, $|V(R_\beta)|=2$ and R_β consists of only two vertices p and q with $p_x < q_x$. Since $(p,q) \approx (w,z)$ or $(p,q) \approx (z,f)$. By Statement (4) of Proposition 5, edge (p,q) in R_β can be combined into path P_α to form a Hamiltonian (s,t)-path of $L(m,n;k,l)$. The construction of a such Hamiltonian (s,t)-path is shown in Fig. 16(e).

Case 2.2: $s_x, t_x > m-k$. Based on the size of l, we consider the following two subcases:

Case 2.2.1: $(l>1)$ or ($l=1$ and $m-k=2$). A Hamiltonian (s,t)-path of $L(m,n;k,l)$ can be constructed by similar arguments in proving Case 2.1.2. Figs. 17(a)–(b) depict the construction of a such Hamiltonian (s,t)-path of $L(m,n;k,l)$ in this subcase.

Case 2.2.2: $l=1$ and $m-k>2$. Let $r=(m-k,1)$ and $w=(m-k,2)$ be two vertices in $L(m,n;k,l)$. We make a vertical separation on $L(m,n;k,l)$ to obtain two disjoint supergrid subgraphs $R_\beta = R(m',n)$ and $L_\alpha = L(m-m',n;k,l)$, where $m'=m-k-1$; as depicted in Fig. 17(c). Clearly, $m-m'=1$ and (L_α,s,t) lies on Case 2 of Lemma 12. By Lemma 12, we can construct a Hamiltonian (s,t)-path P_α of L_α such that edge $(r,w) \in P_\alpha$. By Lemma 1, there exists a Hamiltonian cycle C_β of R_β such that its one flat face is placed to face L_α. Then, there exist two edges $e_1 \in C_\beta$ and $(r,w) \in P_\alpha$ such that $e_1 \approx (r,w)$. By Statement (2) of Proposition 5, P_α and C_β can be combined into a Hamiltonian (s,t)-path of $L(m,n;k,l)$. The construction of a such Hamiltonian path is shown in Fig. 17(c).

Case 2.3: $s_x \leqslant m-k$ and $t_x > m-k$. We make a vertical separation on $L(m,n;k,l)$ to obtain two disjoint rectangles $R_\alpha = R(m',n)$ and $R_\beta = R(k,n-l)$, where $m'=m-k$. Let $p \in V(R_\alpha)$, $q \in V(R_\beta)$, $p \sim q$, and

$$\begin{cases} p=(m',n) \text{ and} & \\ q=(m'+1,n), & \text{if } s \neq (m',n) \text{ and } t \neq (m'+1,n); \\ p=(m',n-1) \text{ and} & \\ q=(m'+1,n-1), & \text{if } s=(m',n) \text{ and } t=(m'+1,n); \\ p=(m',n) \text{ and} & \\ q=(m'+1,n-1), & \text{if } s \neq (m',n) \text{ and } t=(m'+1,n); \\ p=(m',n-1) \text{ and} & \\ q=(m'+1,n), & \text{if } s=(m',n) \text{ and } t \neq (m'+1,n). \end{cases}$$

Consider (R_α,s,p) and (R_β,q,t). Condition (F1) holds, if ($m-k=2$ and $s_y=p_y=n-1$) or ($k=2$ and $q_y=t_y=n-1$). This is impossible, because if $p_y=q_y=n-1$, then $s_y=n$ and $t_y=n$. Therefore, (R_α,s,p) and (R_β,q,t) do not satisfy condition (F1). By Lemma 3, there exist Hamiltonian (s,p)-path P_α and Hamiltonian (q,t)-path P_β of R_α and R_β, respectively. Then, $P_\alpha \Rightarrow P_\beta$ forms a Hamiltonian (s,t)-path of $L(m,n;k,l)$.

We have considered any case to construct a Hamiltonian (s,t)-path of $L(m,n;k,l)$ when $m-k \geqslant 2$, $n-l \geqslant 2$, and $(L(m,n;k,l),s,t)$ does not satisfy condition (F1). Thus, the lemma holds true. ■

It immediately follows from Lemmas 11–13 that we get the following theorem.

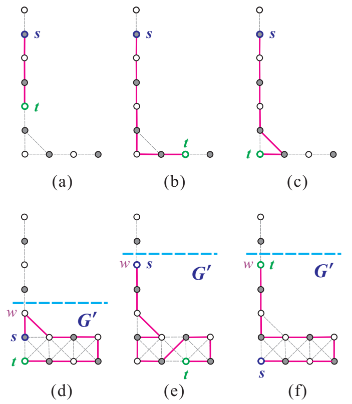


Fig. 18. The longest (s, t)-path for (a) (UB1), (b) (UB2), (c) (UB3), and (d)–(f) (UB4), where bold lines indicate the constructed longest (s, t)-path.

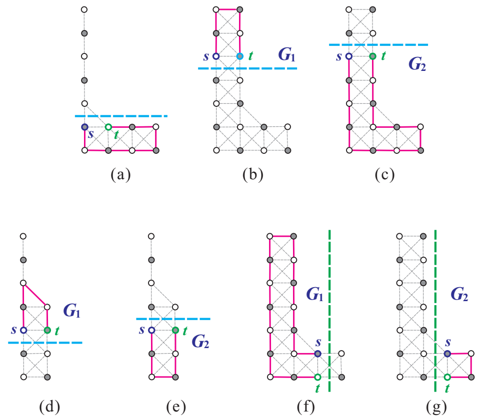


Fig. 19. The longest (s, t)-path for (a) (UB5), and (b)–(g) (UB6), where bold lines indicate the constructed longest (s, t)-path.

Theorem 14. *Let $L(m, n; k, l)$ be a L-shaped supergrid graph with vertices s and t. Then, $L(m, n; k, l)$ contains a Hamiltonian (s, t)-path if and only if $(L(m, n; k, l), s, t)$ does not satisfy conditions* (F1), (F4), *and* (F5).

V. The longest (s, t)-path algorithm

It follows from Theorem 14 that if $(L(m, n; k, l), s, t)$ satisfies one of conditions (F1), (F4), and (F5), then $(L(m, n; k, l), s, t)$ contains no Hamiltonian (s, t)-path. So in this section, first for these cases we give upper bounds on the lengths of longest paths between s and t. Then, we show that these upper bounds equal to the lengths of longest paths between s and t. Recall that $\hat{L}(G, s, t)$ denotes the length of longest (s, t)-path in G, and the length of a path is the number of vertices in the path. In the following, we will use $\hat{U}(G, s, t)$ to indicate the upper bound on the length of longest (s, t)-paths in G, where G is a rectangular or L-shaped supergrid graph. Notice that the isomorphic cases are omitted. Depending on the sizes of $m - k$ and $n - l$, we provide the following two lemmas to compute the upper bounds when $(L(m, n; k, l), s, t)$ satisfies either condition (F1) or (F4).

Lemma 15. *Let $m - k = n - l = 1$ and $l > 1$. Then, the following implications (conditions) hold:*

- (UB1) *If $s_y, t_y \leqslant l$, then the length of any path between s and t cannot exceed $|t_y - s_y| + 1$ (see Fig. 18(a)).*
- (UB2) *If $s_y < l$ and $t_x > 1$, then the length of any path between s and t cannot exceed $n - s_y + t_x$ (see Fig. 18(b)).*
- (UB3) *If $s_x = t_x = 1$, $\max\{s_y, t_y\} = n$, and $[(k > 1)$ or $(k = 1$ and $\min\{s_y, t_y\} > 1)]$, then the length of any path between s and t cannot exceed $|t_y - s_y| + 2$ (see Fig. 18(c)).*

Proof: Since $n - l = m - k = 1$, there is only one single path between s and t that has the specified. ∎

Lemma 16. *Let $n - l > 1$. Then, the following implications (conditions) hold:*

- (UB4) *If $m - k = 1$, $l > 1$, and $[(s_y, t_y > l$ and $\{s, t\}$ is not a vertex cut$)$, $(s_y \leqslant l$ and $t_y > l)$, or $(t_y \leqslant l$ and $s_y > l)]$, then the length of any path between s and t cannot exceed $\hat{L}(G', s, t)$; where $G' = L(m, n - n'; k, l')$ and $l' = l - n'$, and $n' = l - 1$ if $s_y, t_y \geqslant l$; otherwise $n' = \min\{s_y, t_y\} - 1$ (see Figs. 18(d)–(f)).*
- (UB5) *If $m - k = 1$, $k > 1$ $(m > 2)$, $s = (1, l + 1)$, and $t = (2, l + 1)$, then the length of any path between s and t cannot exceed $\hat{L}(G', s, t)$, where $G' = R(m, n - l)$ (see Fig. 19(a)).*
- (UB6) *If $(m - k = 2$, $l > 1$, and $2 \leqslant s_y = t_y \leqslant n - 1)$, $(m = 2$, $n - l > 2$, and $l + 1 \leqslant s_y = t_y \leqslant n - 1)$, or $(n - l = 2$, $k > 1$, and $m - k + 1 \leqslant s_x = t_x \leqslant m - 1)$, then the length of any path between s and t cannot exceed $\max\{\hat{L}(G_1, s, t), \hat{L}(G_2, s, t)\}$, where G_1 and G_2 are defined in Figs. 19(b)–(g).*

Proof: For (UB4), let $w = (1, l)$ if $s_y, t_y \geqslant l$; otherwise $w = \min\{s_y, t_y\}$. Since w is a cut vertex, hence removing w clearly disconnects $L(m, n; k, l)$ into two components, and a simple path between s and t can only go through a component that contains s and t, let this component be G'. Therefore, its length cannot exceed $\hat{L}(G', s, t)$. For (UB5), consider Fig. 19(a). Since $\{s, t\}$ is a vertex cut of $L(m, n; k, l)$, the length of any path between s and t cannot exceed $\max\{3, \hat{L}(G', s, t)\}$. Since $n - l > 1$ and $m > 2$, it follows that $|V(G')| > 3$. Moreover, since $\hat{L}(G', s, t)| \leqslant |V(G')|$, its length cannot exceed $\hat{L}(G', s, t)$. For (UB6), removing s and t clearly disconnects $L(m, n; k, l)$ into two components G_1 and G_2. Thus, a simple path between s and t can only go through one of these components. Therefore, its length cannot exceed the size of the largest component. ∎

We have computed the upper bounds of the longest (s, t)-paths when $(L(m, n; k, l), s, t)$ satisfies condition (F1) or (F4). The following lemma shows the upper bound when $(L(m, n; k, l), s, t)$ satisfies condition (F5).

Lemma 17. *If $(L(m, n; k, l), s, t)$ satisfies condition* (F5), *then the length of any path between s and t cannot exceed $mn - kl - 1$.*

Proof: Consider Fig. 20. We can easily check that the length of any path between s and t cannot exceed $\hat{L}(G_1, s, p) + \hat{L}(G_2, q, t) = mn - kl - 1$. ∎

It is easy to show that any $(L(m, n; k, l), s, t)$ must satisfy one of conditions (L0), (UB1), (UB2), (UB3), (UB4), (UB5), (UB6), and (F5), where (L0) is defined as follows:

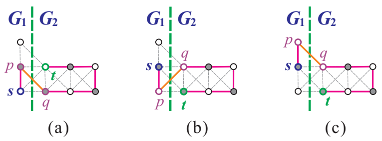


Fig. 20. The longest (s,t)-path when condition (F5) holds, where bold lines indicate the longest (s,t)-path.

(L0) $(L(m,n;k,l),s,t)$ does not satisfy any of conditions (F1), (F4), and (F5).

If $(L(m,n;k,l),s,t)$ satisfies (L0), then $\hat{U}(L(m,n;k,l),s,t)$ is $mn-kl$. Otherwise, $\hat{U}(L(m,n;k,l),s,t)$ can be computed by using Lemmas 15–17. So, we have the following formula of upper bounds:

$$\hat{U}(L(m,n;k,l),s,t) = \begin{cases} |t_y-s_y|+1, & \text{if (UB1) holds;} \\ n-s_y+t_x, & \text{if (UB2) holds;} \\ |t_y-s_y|+2, & \text{if (UB3) holds;} \\ \hat{L}(G',s,t), & \text{if (UB4) or (UB5) holds;} \\ \max\{\hat{L}(G_1,s,t),\hat{L}(G_2,s,t)\}, & \text{if (UB6) holds;} \\ mn-kl-1, & \text{if (F5) holds;} \\ mn-kl, & \text{if (L0) holds.} \end{cases}$$

Now, we show how to obtain a longest (s,t)-path for L-shaped supergrid graphs. Notice that if $(L(m,n;k,l),s,t)$ satisfies (L0), then by Theorem 14, it contains a Hamiltonian (s,t)-path.

Lemma 18. *If* $(L(m,n;k,l),s,t)$ *satisfies one of the conditions* (UB1), (UB2), (UB3), (UB4), (UB5), (UB6), *and* (F5), *then* $\hat{L}(L(m,n;k,l),s,t)=\hat{U}(L(m,n;k,l),s,t)$.

Proof: Consider the following cases:

Case 1: conditions (UB1), (UB2), and (UB3) hold. Clearly the lemma holds for the single possible path between s and t (see Figs. 18(a)–(c)).

Case 2: condition (UB4) holds. Then, by Lemma 16, $\hat{U}(L(m,n;k,l),s,t)=\hat{L}(G',s,t)$. In this case, G' is a L-shaped supergrid graph. There are two subcases:

Case 2.1: (s_y(resp., t_y) $\leqslant l$ and t_y(resp., s_y) $>l$) or ($s_y,t_y>l$ and $[(n-l>2)$ or $(n-l=2$ and $\{s,t\}\neq\{(1,n-1),(2,n)\}$ or $\{(1,n),(2,n-1)\})]$). First, let s_y(resp., t_y) $\leqslant l$ and t_y(resp., s_y) $>l$. Without loss of generality, assume that $s_y\leqslant l$ and $t_y>l$. Consider (G',s,t) and see Fig. 18(e). Then, $G'=L(m,n-n';k,l')$, where $n'=s_y-1$ and $l'=l-n'$. Since $s_y=1$ in G', $t_y>l'$, and $n-n'\geq 2$, it is obvious that (G',s,t) does not satisfies conditions (F1), (F4), and (F5). Now, let $s_y,t_y>l$. Then, $G'=L(m,n-n';k,l')$ satisfies that $n'=l-1$ and $l'=1$. Consider Fig. 18(d). Since $n-n'-l'\geqslant 2$, $l'=1$, $\{s,t\}$ is not a vertex cut, and $\{s,t\}\neq\{(1,n-1),(2,n)\}$ or $\{(1,n),(2,n-1)\}$, (G',s,t) does not satisfy conditions (F1), (F4), and (F5). Thus, by Theorem 14 (G',s,t) contains a Hamiltonian (s,t)-path.

Case 2.2: $s_y,t_y>l$, $n-l=2$, and $\{s,t\}=\{(1,n-1),(2,n)\}$ or $\{(1,n),(2,n-1)\}$. In this subcase, (G',s,t) satisfies condition (F5). Hence, (G',s,t) lies on Case 5 (see later).

Case 3: condition (UB5) holds. In this case, $\{s,t\}$ is a vertex cut of $L(m,n;k,l)$ (see Fig. 19(a)). By Lemma 16, $\hat{U}(L(m,n;k,l),s,t)=\hat{L}(G',s,t)$, where $G'=R(m,n-l)$ is a rectangular supergrid graph. Since $n-l>1$, $s=(1,l+1)$, and $t=(2,l+1)$, (G',s,t) does not satisfy condition (F1). Thus, by Lemma 3, (G',s,t) contains a Hamiltonian (s,t)-path.

Case 4: condition (UB6) holds. In this case, $\{s,t\}$ is a vertex cut of $L(m,n;k,l)$ (see Figs. 19(b)–(g)). Then, removing s and t splits $L(m,n;k,l)$ into two components G_1' and G_2'. Let $G_1=G_1'\cup\{s,t\}$ and $G_2=G_2'\cup\{s,t\}$. Thus,

- if $m-k=2$ and $s_y=t_y$, then $G_1=R(m-k,s_y)$ and $G_2=L(m,n-s_y+1;k,l-s_y+1)$ (see Figs. 19(b) and 19(c)).
- if $m-k=1$ and $m=2$, then $G_1=L(m,s_y;k,l)$ and $G_2=R(m,n-s_y+1)$ (see Figs. 19(d) and 19(e)).
- if $n-l=2$ and $s_x=t_x$, then $G_1=L(s_x,n;s_x-(m-k),l)$ and $G_2=R(m-s_x+1,n-l)$ (see Figs. 19(f) and 19(g)).

Then the path going through vertices of the larger subgraph between G_1 and G_2 has the length equal to $\hat{U}(L(m,n;k,l),s,t)$. The longest (s,t)-path in each subgraph computed by Lemma 3, 12, 13, or Case 5; as depicted in Figs. 19(b)–(g).

Case 5: condition (F5) holds. In this case, $m-k=1$, $n-l=2$, $l=1$, $k\geqslant 2$, and $\{s,t\}=\{(1,2),(2,3)\}$ or $\{(1,3),(2,2)\}$ (see Fig. 13(d)). Consider Fig. 20. By Lemma 17, $\hat{U}(L(m,n;k,l),s,t)=\hat{L}(G_1,s,p)+\hat{L}(G_2,q,t)$. By Theorem 6, there exist a longest (s,p)-path P_1 and longest (q,t)-path P_2 of G_1 and G_2, respectively. Then, $P_1\Rightarrow P_2$ forms a Hamiltonian (s,t)-path of $L(m,n;k,l)$. ∎

It follows from Theorem 14 and Lemmas 15–18 that the following theorem concludes our result.

Theorem 19. *Given a L-shaped supergrid $L(m,n;k,l)$ and two distinct vertices s and t in $L(m,n;k,l)$, a longest (s,t)-path can be computed in $O(mn-kl)$-linear time.*

The linear-time algorithm is formally presented as Algorithm 1.

VI. Concluding Remarks

Based on the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs, we first obtain two Hamiltonian connected properties of rectangular supergrid graphs. Using the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs, we prove L-shaped supergrid graphs to be Hamiltonian and Hamiltonian connected except one or three conditions. Furthermore, we present a linear-time algorithm to compute the longest (s,t)-path of a L-shaped supergrid graph. The Hamiltonian cycle problem on solid grid graphs was known to be polynomial solvable. However, it remains open for solid supergrid graphs in which there exists no hole. This result can be regarded as the first attempt for solving the Hamiltonian and longest (s,t)-path problems on solid supergrid graphs, where L-shaped supergrid graphs form a subclass of solid supergrid graphs.

Algorithm 1: The longest (s,t)-path algorithm

Input: A L-shaped supergrid graph $L(m,n;k,l)$ with $mn \geqslant 2$, and two distinct vertices s and t in $L(m,n;k,l)$.

Output: The longest (s,t)-path.

Method:

1. **if** ($m-k=1$ or $n-l=1$) and (($L(m,n;k,l),s,t$) does not satisfy conditions (F1), (F4), and (F5)) **then output** $HP(L(m,n;k,l),s,t))$ constructed from Lemma 12;
   // construct Hamiltonian (s,t)-path when $m-k=1$ or $n-l=1$
2. **if** ($m-k \geqslant 2$ and $n-l \geqslant 2$) and (($L(m,n;k,l),s,t$) does not satisfy conditions (F1), (F4), and (F5)) **then output** $HP(L(m,n;k,l),s,t))$ constructed from Lemma 13;
   // construct Hamiltonian (s,t)-path when $m-k \geqslant 2$ and $n-l \geqslant 2$
3. **if** $(L(m,n;k,l),s,t)$ satisfies one of conditions (F1), (F4), and (F5), **then output** the longest (s,t)-path based on Lemma 18.
   // construct the longest (s,t)-path when $L(m,n;k,l)$ contains no Hamiltonian (s,t)-path

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