Minimum Clique-clique Dominating Laplacian Energy of a Graph
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Abstract—Let \( C(G) \) denotes the set of all cliques of a graph \( G \). Two cliques in \( G \) are adjacent if there is a vertex incident on them. Two cliques \( c_1, c_2 \in C(G) \) are said to clique-clique dominate (cc-dominate) each other if there is a vertex incident with \( c_1 \) and \( c_2 \). A set \( L \subseteq C(G) \) is said to be a cc-dominating set (CCD-set) if every clique in \( G \) is cc-dominated by some clique in \( L \). The cc-domination number \( \gamma_{cc} = \gamma_{cc}(G) \) is the order of a minimum cc-dominating set of \( G \). In this paper we introduce minimum cc-dominating Laplacian energy of the graph denoting it as \( LE_{cc}(G) \). It depends both on underlying graph of \( G \) and its particular minimum cc-dominating set (\( \gamma_{cc} \)-set) of \( G \). Upper and lower bounds for \( LE_{cc}(G) \) are established.

Index Terms—Energy of a graph, Laplacian energy of a graph, clique-clique domination, Clique graph.

I. INTRODUCTION

The terminologies and notations used here are as in [9], [22]. By a graph \( G(V, E) \) we mean a connected finite simple graph of order \( p \) and size \( q \). A set \( D \subseteq V \) is a dominating set of \( G \) if every vertex not in \( D \) is adjacent to some vertex in \( D \). The domination number \( \gamma = \gamma(G) \) is the order of a minimum dominating set of \( G \). The domination number is a well studied parameter in literature and for a survey refer [8], [12], [21], [20], [19], [17], [13], [14]. A maximal complete subgraph is a clique. Let \( C(G) \) denote the set of all cliques of \( G \) with \( \mid C(G) \mid = s \). Two cliques in \( G \) are adjacent if there is a common vertex incident on them. A clique graph \( C_G(G) \) is a graph with vertex set \( C(G) \) and any two vertices in \( C_G(G) \) are adjacent if and only if corresponding cliques are adjacent in \( G \). The number of edges in the clique graph \( C_G(G) \) is denoted as \( q_c \). Smitha G. Bhat et al [4] defined cc-degree and cc-dominating sets as follows. The cc-degree (Clique-clique degree) of a clique \( h \), \( d_{cc}(h) \) is the number of cliques adjacent to \( h \). Two cliques \( c_1, c_2 \in C(G) \) are said to clique dominate each other if there is a vertex incident with \( c_1 \) and \( c_2 \). A set \( L \subseteq C(G) \) is said to be a clique-clique dominating set (CCD-set) if every clique in \( G \) is clique dominated by some clique in \( L \). The clique-clique domination number \( \gamma_{cc} = \gamma_{cc}(G) \) is the order of a minimum clique-clique dominating set of \( G \). The concept of energy of a graph was introduced by I. Gutman [5] in 1978. The eigenvalues of \( G \) are the eigenvalues of its adjacency matrix \( A(G) \). These eigenvalues arranged in an non-increasing order, will be denoted as \( \lambda_1(G), \lambda_2(G), \ldots, \lambda_p(G) \). Then the energy of the graph \( G \) is defined as \( E(G) = \sum_{i=1}^{p} |\lambda_i(G)| \). Various properties of energy of the graph may be found in [6]. In connection with graph energy, energy-like quantities were considered for other matrices such as distance [10], covering [1], incidence [11] and vb-dominating [18]. Gutman and Zhou [7] defined the Laplacian energy of a graph \( G \) in the year 2006. Let \( G \) be a graph with \( p \) vertices and \( q \) edges. The Laplacian matrix of the graph \( G \) is denoted by \( L = [l_{ij}] \) is the square matrix of order \( p \) where

\[
l_{ij} = \begin{cases} 
-1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\
0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\
d_i & \text{if } i = j,
\end{cases}
\]

where \( d_i \) is the degree of the vertex \( v_i \). Let \( \mu_1, \mu_2, \ldots, \mu_p \) be the eigen values of Laplacian matrix \( L = [l_{ij}] \), which are called Laplacian eigen values of \( G \). Then the Laplacian energy of \( G \) is defined as

\[
LE(G) = \sum_{i=1}^{p} |\mu_i(G) - \frac{2q}{p}|.
\]

II. CLIQUE-CLIQUE DOMINATING LAPLACIAN ENERGY OF A GRAPH

Motivated by the definition of clique-clique dominating set and Laplacian energy of a graph, we introduce a new matrix, called minimum cc-dominating Laplacian matrix of a graph and study its energy. Let \( \gamma_{cc} \)-set be a minimum cc-dominating set of a graph \( G \). The minimum cc-dominating Laplacian matrix of \( G \) is the \( s \times s \) matrix

\[
L_{cc} = [l_{ij}],
\]

where

\[
l_{ij} = \begin{cases} 
-1 & \text{if } c_i \text{ and } c_j \text{ are adjacent} \\
d_{cc}(c_i) - 1 & \text{if } i = j \text{ and } c_i \in \gamma_{cc}\text{-set} \\
d_{cc}(c_i) & \text{if } i = j \text{ and } c_i \notin \gamma_{cc}\text{-set} \\
0 & \text{otherwise},
\end{cases}
\]

where \( c_i, c_j \in C(G) \).

The characteristic polynomial of \( L_{cc}(G) \) is denoted by

\[
f\mu(G, \mu) = \det(\mu I - L_{cc}(G)).
\]

The minimum cc-dominating Laplacian eigen values of the graph \( G \) are the eigen values of \( L_{cc}(G) \). Since \( L_{cc}(G) \)
is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \). The minimum cc-dominating Laplacian energy of \( G \) is then defined as \( LE_{cc}(G) = \sum_{i=1}^{s} |\mu_i(G) - \frac{2q_c}{s}| \).

In this section we discuss basic properties of minimum cc-dominating Laplacian energy of a graph \( LE_{cc}(G) \) and derive an upper and lower bound for \( LE_{cc}(G) \).

A. Properties of clique-clique dominating Laplacian energy of a graph

First we compute the minimum cc-dominating Laplacian energy of a graph shown in Figure 1.

**Example II.1.**

Let \( G \) be a graph with 8 cliques \( c_1, c_2, \ldots, c_8 \) (See Fig. 1) with minimum cc-dominating set \( B = \{c_2, c_6, c_8\} \). Then

\[
L_{cc}(G) = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 4 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 3 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The characteristic polynomial of \( L_{cc}(G) \) is

\[
\lambda^8 - 17\lambda^7 + 110\lambda^6 - 328\lambda^5 + 400\lambda^4 - 7\lambda^3 - 222\lambda^2 - 24\lambda
\]

where

\[
\mu_1 = -0.5334, \quad \mu_2 = -0.1112, \quad \mu_3 = 0.0000, \quad \mu_4 = 1.7386, \quad \mu_5 = 3.2084, \quad \mu_6 = 3.4650, \quad \mu_7 = 4.0000 \quad \text{and} \quad \mu_8 = 5.2326.
\]

Therefore the minimum cc-dominating Laplacian eigenvalues of \( G \) are \( -0.5334, -0.1112, 0, 0, 1.7386, 3.2084, 3.4650, 4.0000 \) and \( 5.2326 \). Therefore the minimum cc-dominating Laplacian energy of the graph \( G \) is \( LE_{cc}(G) = 10.8246 \). Note that the minimum cc-dominating Laplacian energy of the graph depends on its cc-dominating set.

**Theorem II.1.** If \( \mu_1(G), \mu_2(G), \ldots, \mu_s(G) \) are the eigenvalues of \( L_{cc}(G) \), then

\[
\sum_{i=1}^{s} \mu_i = 2q_c - \gamma_{cc}(G) \tag{1}
\]

and

\[
\sum_{i=1}^{s} \mu_i^2 = 2q_c + \sum_{i=1}^{s} (d_{cc}(c_i) - k_i)^2 \tag{2}
\]

where \( k_i = \begin{cases} 1 & \text{if } c_i \in \gamma_{cc}\text{-set} \\ 0 & \text{if } c_i \notin \gamma_{cc}\text{-set} \end{cases} \) and \( q_c \) is the number of edges in the clique graph \( C_G \).

**Proof:** (1) The sum of the principal diagonal elements of \( L_{cc}(G) \) is equal to \( \sum_{i=1}^{s} d_{cc}(c_i) - |\gamma_{cc}\text{-set}| = 2q_c - |\gamma_{cc}\text{-set}| \) (By definition). We also know that the sum of the Laplacian eigenvalues of \( L_{cc}(G) \) is equal to trace of \( L_{cc}(G) \). Therefore \( \sum_{i=1}^{s} \mu_i = 2q_c - \gamma_{cc}(G) \).

(2) The sum of the squares of the Laplacian eigenvalues of \( L_{cc}(G) \) is just the trace of \( (L_{cc}(G))^2 \). Therefore

\[
\sum_{i=1}^{s} \mu_i^2 = \sum_{i=1}^{s} \sum_{j=1}^{s} l_{ij}^2 l_{ji}
\]

\[
= 2 \sum_{i<j} (l_{ij})^2 + \sum_{i=1}^{s} (l_{ii})^2
\]

\[
= 2q_c + \sum_{i=1}^{s} (d_{cc}(c_i) - k_i)^2 ,
\]

where \( k_i = \begin{cases} 1 & \text{if } c_i \in \gamma_{cc}\text{-set} \\ 0 & \text{if } c_i \notin \gamma_{cc}\text{-set} \end{cases} \) and \( q_c \) is the number of edges in the clique graph \( C_G \).

**Theorem II.2.** If \( \mu_1(G) \) is the largest eigenvalue of the minimum cc-dominating Laplacian matrix \( L_{cc}(G) \), then

\[
\mu_1(G) \geq \frac{\sum_{i=1}^{s} (d_{cc}(c_i) - k_i)^2 - 2q_c}{s} \tag{4}
\]

where \( k_i = \begin{cases} 1 & \text{if } c_i \in \gamma_{cc}\text{-set} \\ 0 & \text{if } c_i \notin \gamma_{cc}\text{-set} \end{cases} \)

**Proof:** Let \( X \) be any non zero vector, then we have

\[
\mu_1 \left( L_{cc}(G) \right) \geq \max_{X \neq 0} \frac{\left| X^T L_{cc}(G) X \right|}{\left| X^T X \right|} \tag{3}
\]

\[
\mu_1 \left( L_{cc}(G) \right) \geq \frac{
\sum_{i=1}^{s} (d_{cc}(c_i) - k_i)^2 - 2q_c
}{p},
\]

where \( J \) is the all one’s vector.
Let \( M = q_c + \frac{1}{2} \sum_{i=1}^{s} (d_{cc}(c_i) - k_i)^2 \). \hspace{1cm} (5)

Bapat and S. Pati [2] proved that if energy of the graph is a rational number, then it is an even integer. Similar result for minimum cc-dominating Laplacian energy is established in the following theorem.

**Theorem II.3.** Let \( G \) be a graph with cc-domination number \( \gamma_{cc}(G) \). If the sum of the absolute values of minimum cc-dominating Laplacian eigenvalues is a rational number, then

\[
\sum_{i=1}^{s} |\mu_i| \equiv \gamma_{cc}(G) \pmod{2}.
\] \hspace{1cm} (6)

**Proof:** Let \( \mu_1, \mu_2 \ldots, \mu_s \) be the eigenvalues of the minimum cc-dominating Laplacian matrix \( L_{cc}(G) \) of a graph \( G \) of which \( \mu_1, \mu_2 \ldots, \mu_r \) are positive and the rest of the eigenvalues are non-positive, then

\[
\sum_{i=1}^{s} |\mu_i| = (\mu_1 + \mu_2 + \cdots + \mu_r) - (\mu_{r+1} + \mu_{r+2} + \cdots + \mu_s)
\]

\[
= 2(\mu_1 + \mu_2 + \cdots + \mu_r) - (\mu_1 + \mu_2 + \cdots + \mu_s)
\]

\[
= 2(\mu_1 + \mu_2 + \cdots + \mu_r) - \sum_{i=1}^{s} \mu_i
\]

\[
= 2(\mu_1 + \mu_2 + \cdots + \mu_r) - (2q_c - \gamma_{cc}(G))
\]

\[
= 2(\mu_1 + \mu_2 + \cdots + \mu_r - q_c) - \gamma_{cc}(G).
\]

Now, if \( \sum_{i=1}^{s} |\mu_i| \) is rational number, then \( \mu_1 + \mu_2 + \cdots + \mu_r \) is also a rational number. Hence it must be an integer. Therefore

\[
\sum_{i=1}^{s} |\mu_i| \equiv \gamma_{cc}(G) \pmod{2}.
\] \hspace{1cm} (7)

**Theorem II.4.** Let \( G_1 \) and \( G_2 \) be two graphs with \( s \) vertices, \( q_s \) and \( q'_s \) be the number of edges in the clique graph of \( G_1 \) and \( G_2 \) respectively. If \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \) and \( \mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_s \) are the laplacian eigenvalues of \( G_1 \) and \( G_2 \) respectively. Then

\[
\sum_{i=1}^{s} \mu_i \mu'_i \leq 2\sqrt{(M(G_1)M(G_2))}
\]

where \( M \) is the expression defined as in (5).

**Proof:** Using Cauchy-Schwarz inequality

\[
\left( \sum_{i=1}^{s} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{s} a_i^2 \right) \left( \sum_{i=1}^{s} b_i^2 \right)
\]

Taking \( a_i = \mu_i, b_i = \mu'_i \), we get

\[
\left( \sum_{i=1}^{s} \mu_i \mu'_i \right)^2 \leq \left( \sum_{i=1}^{s} \mu_i^2 \right) \left( \sum_{i=1}^{s} (\mu'_i)^2 \right)
\]

\[
= 2M(G_1)(2M(G_2))
\]

then the result follows.

**B. Bounds for minimum cc-dominating Laplacian energy of a graph**

**Theorem II.5.** Let \( G \) be a graph with \( s \) cliques, minimum cc-dominating set \( \gamma_{cc}-set \), and let \( q_c \) be the number of edges in the clique graph \( C_G(G) \). Then

\[
LE_{cc}(G) \leq \sqrt{2Ms} + 2q_c.
\] \hspace{1cm} (8)

where \( M \) is the expression defined as in (5).

**Proof:** Let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \) be the Laplacian eigenvalues of \( L_{cc}(G) \). Using Cauchy-Schwarz inequality,

\[
\left( \sum_{i=1}^{s} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{s} a_i^2 \right) \left( \sum_{i=1}^{s} b_i^2 \right)
\]

choose \( a_i = 1 \) and \( b_i = |\mu_i| \).

\[
\left( \sum_{i=1}^{s} |\mu_i| \right)^2 \leq \left( \sum_{i=1}^{s} 1 \right) \left( \sum_{i=1}^{s} |\mu_i|^2 \right) = s2M,
\]

\[
\sum_{i=1}^{s} |\mu_i| \leq \sqrt{2Ms}.
\]

Using Triangular inequality \( |a - b| \leq |a| + |b| \), we have

\[
|\mu_i - 2q_c| \leq |\mu_i| + 2q_c
\]

\[
\sum_{i=1}^{s} |\mu_i - 2q_c| \leq \sum_{i=1}^{s} |\mu_i| + 2s \frac{2q_c}{s}
\]

\[
LE_{cc}(G) \leq \sum_{i=1}^{s} |\mu_i| + 2q_c
\]

\[
\leq \sqrt{2Ms} + 2q_c.
\] \hspace{1cm} (9)

**Theorem II.6.** Let \( G \) be a graph with \( s \) cliques, minimum cc-dominating set \( \gamma_{cc}-set \), and let \( q_c \) be the number of edges in \( C_G(G) \). Then

\[
LE_{cc}(G) \leq \sqrt{2Ms} + 4q_c(\gamma_{cc}(G) - q_c).
\] \hspace{1cm} (10)

where \( M \) is the expression defined as in (5).

**Proof:** Let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \) be the Laplacian eigenvalues of \( L_{cc}(G) \). Using Cauchy-Schwarz inequality,

\[
\left( \sum_{i=1}^{s} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{s} a_i^2 \right) \left( \sum_{i=1}^{s} b_i^2 \right)
\]
choose $a_i = 1$ and $b_i = |\mu_i - \frac{2q_c}{s}|$, 

$$\left( \sum_{i=1}^{s} |\mu_i - \frac{2q_c}{s}| \right)^2 \leq \left( \sum_{i=1}^{s} \left( \sum_{i=1}^{s} |\mu_i - \frac{2q_c}{s}| \right)^2 \right)$$

$$(LE_{cc}(G))^2 \leq s \left( \sum_{i=1}^{s} |\mu_i|^2 + \sum_{i=1}^{s} \frac{4q_c^2}{s^2} - \frac{4q_c}{s} \sum_{i=1}^{s} \mu_i \right)$$

$$= s \left( 2M + \frac{4q_c^2}{s} - \frac{4q_c}{s} (2q_c - \gamma_{cc}(G)) \right)$$

$$= 2Ms + 4q_c (\gamma_{cc}(G) - q_c) \ .$$

Therefore $LE_{cc}(G) \leq \sqrt{2Ms + 4q_c (\gamma_{cc}(G) - q_c)}$.

**Theorem II.7.** Let $G$ be a graph with $s$ cliques, and let $q_c$ be the number of edges in the clique graph $C_G(G)$. If $R$ is the determinant of $L_{cc}(G)$, then

$$LE_{cc}(G) \leq \sqrt{2M(s-1) + sR^2 + 2q_c} \ .$$

(10)

where $M$ is the expression defined as in (5).

**Proof:** Using Kober’s inequality,

$$s \left[ \sum_{i=1}^{s} |\mu_i|^2 - \left( \prod_{i=1}^{s} |\mu_i|^2 \right)^\frac{1}{s} \right] \leq s \sum_{i=1}^{s} |\mu_i|^2 - \left( \sum_{i=1}^{s} |\mu_i| \right)^2 \ .$$

Putting $\alpha_i = |\mu_i|^2$

$$s \left[ \sum_{i=1}^{s} |\mu_i|^2 - \left( \prod_{i=1}^{s} |\mu_i|^2 \right)^\frac{1}{s} \right] \leq s \sum_{i=1}^{s} |\mu_i|^2 - \left( \sum_{i=1}^{s} |\mu_i| \right)^2 \ .$$

$$2M - sR^2 \leq s(2M) - \left( \sum_{i=1}^{s} |\mu_i| \right)^2 \ .$$

$$\left( \sum_{i=1}^{s} |\mu_i| \right) \leq \sqrt{2M(s-1) + R^2} \ .$$

Now using triangular inequality,

$$|\mu_i - \frac{2q_c}{s}| \leq |\mu_i| + \frac{2q_c}{s} \ .$$

$$\sum_{i=1}^{s} |\mu_i - \frac{2q_c}{s}| \leq \sum_{i=1}^{s} |\mu_i| + \sum_{i=1}^{s} \frac{2q_c}{s} \ \ .$$

$$LE_{cc}(G) \leq \sum_{i=1}^{s} |\mu_i| + 2q_c \ .$$

Therefore

$$LE_{cc}(G) \leq \sqrt{2M(s-1) + sR^2 + 2q_c} \ .$$

**Theorem II.8.** Let $G$ be a graph with $s$ cliques, and let $q_c$ be the number of edges in the clique graph $C_G(G)$. If $R$ is the determinant of $L_{cc}(G)$, then

$$LE_{cc}(G) \geq \sqrt{2M + s(s-1)R^2 - 2q_c} \ .$$

(11)

where $M$ is the expression defined as in (5).

**Proof:**

$$\left( \sum_{i=1}^{s} |\mu_i| \right)^2 = \left( \sum_{i=1}^{s} |\mu_i| \right) \left( \sum_{j=1}^{s} |\mu_j| \right)$$

$$= \sum_{i=1}^{s} |\mu_i|^2 + \sum_{i \neq j} |\mu_i| |\mu_j|$$

$$\sum_{i \neq j} |\mu_i| |\mu_j| = \left( \sum_{i=1}^{s} |\mu_i| \right)^2 - \sum_{i=1}^{s} |\mu_i|^2 \ .$$

$$\frac{1}{s(s-1)} \sum_{i \neq j} |\mu_i| |\mu_j| \geq \left( \prod_{i \neq j} |\mu_i| |\mu_j| \right)^{\frac{1}{s(s-1)}} \ .$$

$$\sum_{i \neq j} |\mu_i| |\mu_j| \geq s(s-1) \left( \prod_{i \neq j} |\mu_i| |\mu_j| \right)^{\frac{1}{s(s-1)}} \ .$$

$$\left( \sum_{i=1}^{s} |\mu_i| \right)^2 - \sum_{i=1}^{s} |\mu_i|^2 \geq s(s-1) \left( \prod_{i=1}^{s} |\mu_i| \right)^{2(s-1)} \ .$$

$$\left( \sum_{i=1}^{s} |\mu_i| \right)^2 \geq 2M \geq s(s-1) R^2 \ .$$

$$\left( \sum_{i=1}^{s} |\mu_i| \right) \geq \sqrt{2M + s(s-1) R^2} \ .$$

Using Triangular inequality $|a - b| \leq |a - b|$, $|\mu_i - \frac{2q_c}{s}| \leq |\mu_i| - \frac{2q_c}{s} \forall i = 1, 2, \ldots s$.

$$\sum_{i=1}^{s} |\mu_i| - \sum_{i=1}^{s} \frac{2q_c}{s} \leq \sum_{i=1}^{s} |\mu_i| - \frac{2q_c}{s} \ .$$

$$\sum_{i=1}^{s} |\mu_i| - 2q_c \leq LE_{vv}(G) \ .$$

$$LE_{vv}(G) \geq \sum_{i=1}^{s} |\mu_i| - 2q_c$$

$$\geq \sqrt{2M + s(s-1) R^2 - 2q_c} \ .$$

**III. Conclusion**

Clique domination and Energy of a graph are two well-studied parameters in literature. We modified and studied Minimum clique clique dominating Laplacian energy of graphs. Few Properties are discussed. Several bounds for Minimum clique clique dominating Laplacian energy are obtained. Characterization of the graphs attaining these bounds are not studied in full and one may take this as an open problem for further research.
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