Minimum Clique-clique Dominating Laplacian Energy of a Graph

Sayinath Udupa N. V.

***Abstract*—Let $C(G)$ denotes the set of all cliques of a graph G. Two cliques in G are adjacent if there is a vertex incident on them. Two cliques $c_1, c_2 \in C(G)$ are said to clique-clique dominate (cc-dominate) each other if there is a vertex incident with c_1 and c_2. A set $L \subseteq C(G)$ is said to be a cc-dominating set (CCD-set) if every clique in G is cc-dominated by some clique in L. The cc-domination number $\gamma_{cc} = \gamma_{cc}(G)$ is the order of a minimum cc-dominating set of G. In this paper we introduce minimum cc-dominating Laplacian energy of the graph denoting it as $LE_{cc}(G)$. It depends both on underlying graph of G and its particular minimum cc-dominating set (γ_{cc}-set) of G. Upper and lower bounds for $LE_{cc}(G)$ are established.**

***Index Terms*—Energy of a graph, Laplacian energy of a graph, clique-clique domination, Clique graph.**

I. Introduction

The terminologies and notations used here are as in [9], [22]. By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q. A set $D \subseteq V$ is a dominating set of G if every vertex not in D is adjacent to some vertex in D. The domination number $\gamma = \gamma(G)$ is the order of a minimum dominating set of G. The domination number is a well studied parameter in literature and for a survey refer [8], [12], [21], [20], [19], [17], [13], [14]. A maximal complete subgraph is a clique. Let C(G) denote the set of all cliques of G with $|C(G)| = s$. Two cliques in G are adjacent if there is a common vertex incident on them. A clique graph $C_G(G)$ is a graph with vertex set $C(G)$ and any two vertices in $C_G(G)$ are adjacent if and only if corresponding cliques are adjacent in G. The number of edges in the clique graph $C_G(G)$ is denoted as q_c. Smitha G. Bhat et al [4] defined cc-degree and cc-dominating sets as follows. The cc-degree (Clique-clique degree) of a clique h, $d_{cc}(h)$ is the number of cliques adjacent to h. Two cliques $c_1,\ c_2 \in C(G)$ are said to clique dominate each other if there is a vertex incident with c_1 and c_2. A set $L \subseteq C(G)$ is said to be a clique-clique dominating set (CCD-set) if every clique in G is clique dominated by some clique in L. The clique-clique domination number $\gamma_{cc} = \gamma_{cc}(G)$ is the order of a minimum clique-clique dominating set of G. The concept of energy of a graph was introduced by I. Gutman [5] in 1978. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. These eigenvalues arranged in an non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G) \ldots, \lambda_p(G)$. Then the energy of the graph G is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i(G)|$. Various properties of energy of the graph may be found in [6]. In connection with graph energy, energy-like quantities were considered for other matrices such as distance [10], covering [1], incidence [11] and vb-dominating [18]. Gutman and Zhou [7] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with p vertices and q edges. The Laplacian matrix of the graph G is denoted by $L = [l_{ij}]$ is the square matrix of order p where

$$l_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i & \text{if } i = j. \end{cases}$$

where d_i is the degree of the vertex v_i. Let $\mu_1, \mu_2, \ldots, \mu_p$ be the eigen values of Laplacian matrix $L = [l_{ij}]$, which are called Laplacian eigen values of G. Then the Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^{p} \left|\mu_i(G) - \frac{2q}{p}\right|$.

II. Clique-clique dominating Laplacian energy of a graph

Motivated by the definition of clique-clique dominating set and Laplacian energy of a graph, we introduce a new matrix, called minimum cc-dominating Laplacian matrix of a graph and study its energy. Let γ_{cc}-set be a minimum cc-dominating set of a graph G. The minimum cc-dominating Laplacian matrix of G is the $s \times s$ matrix $L_{cc} = [l_{ij}]$, where

$$l_{ij} = \begin{cases} -1 & \text{if } c_i \text{ and } c_j \text{ are adjacent} \\ d_{cc}(c_i) - 1 & \text{if } i = j \text{ and } c_i \in \gamma_{cc}\text{-set} \\ d_{cc}(c_i) & \text{if } i = j \text{ and } c_i \notin \gamma_{cc}\text{-set} \\ 0 & \text{otherwise.} \end{cases}$$

where $c_i, c_j \in C(G)$.
The characteristic polynomial of $L_{cc}(G)$ is denoted by $f_s(G, \mu) = \det(\mu I - L_{cc}(G))$.
The minimum cc-dominating Laplacian eigenvalues of the graph G are the eigenvalues of $L_{cc}(G)$. Since $L_{cc}(G)$

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is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. The minimum cc-dominating Laplacian energy of G is then defined as $LE_{cc}(G) = \sum_{i=1}^{s} \left|\mu_i(G) - \frac{2q_c}{s}\right|$. In this section we discuss basic properties of minimum cc-dominating Laplacian energy of a graph $LE_{cc}(G)$ and derive an upper and lower bound for $LE_{cc}(G)$.

A. Properties of clique-clique dominating Laplacian energy of a graph

First we compute the minimum cc-dominating Laplacian energy of a graph shown in Figure 1.

Example II.1.

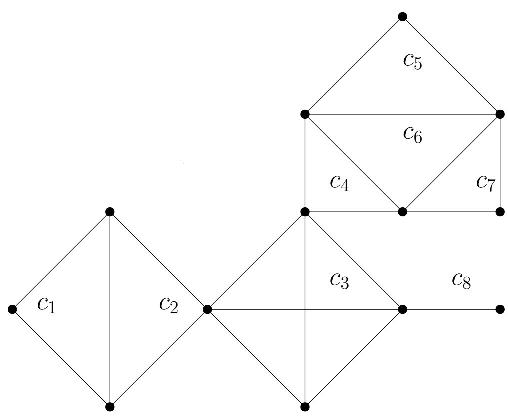

Fig. 1. Graph G

Let G be a graph with 8 cliques $c_1, c_2, \ldots, c_8$ (See Fig. 1) with minimum cc-dominating set $B = \{c_2, c_6, c_8\}$. Then

$$L_{cc}(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 4 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of $L_{cc}(G)$ is $\lambda^8 - 17\lambda^7 + 110\lambda^6 - 328\lambda^5 + 400\lambda^4 - 7\lambda^3 - 222\lambda^2 - 24\lambda$, the minimum cc-dominating Laplacian eigenvalues are $-0.5334, -0.1112, 0.0000, 1.7386,$ $3.2084, 3.4650, 4.0000$ and 5.2326. Therefore the minimum cc-dominating Laplacian energy of the graph G is $LE_{cc}(G) = 10.8246$. Note that the minimum cc-dominating Laplacian energy of the graph depends on its cc-dominating set.

Theorem II.1. *If $\mu_1(G), \mu_2(G) \ldots, \mu_s(G)$ are the eigenvalues of $L_{cc}(G)$, then*

$$\sum_{i=1}^{s} \mu_i = 2q_c - \gamma_{cc}(G) \tag{1}$$

$$\sum_{i=1}^{s} \mu_i^2 = 2q_c + \sum_{i=1}^{s} \left(d_{cc}(c_i) - k_i\right)^2 \tag{2}$$

where $k_i = \begin{cases} 1 & \text{if } c_i \in \gamma_{cc}\text{-set} \\ 0 & \text{if } c_i \notin \gamma_{cc}\text{-set} \end{cases}$ and q_c is the number of edges in the clique graph $C_G(G)$.

Proof: (1) The sum of the principal diagonal elements of $L_{cc}(G)$ is equal to $\sum_{i=1}^{s} d_{cc}(c_i)$ - $|\gamma_{cc}$-set$|$ $=$ $2q_c - |\gamma_{cc}$-set$|$ (By definition). We also know that the sum of the Laplacian eigenvalues of $L_{cc}(G)$ is equal to trace of $L_{cc}(G)$. Therefore $\sum_{i=1}^{s} \mu_i = 2q_c - \gamma_{cc}(G)$.

(2) The sum of the squares of the Laplacian eigenvalues of $L_{cc}(G)$ is just the trace of $(L_{cc}(G))^2$.
Therefore

$$\begin{aligned} \sum_{i=1}^{s} \mu_i^2 &= \sum_{i=1}^{s} \sum_{j=1}^{s} l_{ij} l_{ji} \\ &= 2\sum_{i<j} (l_{ij})^2 + \sum_{i=1}^{s} (l_{ii})^2 \\ &= 2q_c + \sum_{i=1}^{s} \left(d_{cc}(c_i) - k_i\right)^2, \end{aligned}$$

where $k_i = \begin{cases} 1 & \text{if } c_i \in \gamma_{cc}\text{-set} \\ 0 & \text{if } c_i \notin \gamma_{cc}\text{-set} \end{cases}$ ■

$$\text{Let } M = q_c + \frac{1}{2} \sum_{i=1}^{s} \left(d_{cc}(c_i) - k_i\right)^2. \tag{3}$$

Theorem II.2. *If $\mu_1(G)$ is the largest eigenvalue of the minimum cc-dominating Laplacian matrix $L_{cc}(G)$, then*

$$\mu_1(G) \geq \frac{\sum_{i=1}^{s} \left(d_{cc}(c_i) - k_i\right)^2 - 2q_c}{s}. \tag{4}$$

where $k_i = \begin{cases} 1 & \text{if } c_i \in \gamma_{cc}\text{-set} \\ 0 & \text{if } c_i \notin \gamma_{cc}\text{-set} \end{cases}$

Proof: Let X be any non zero vector, then we have $\mu_1\left(L_{cc}(G)\right) = \begin{matrix} \max \\ X \neq 0 \end{matrix} \left[\frac{X' L_{cc}(G) X}{X' X}\right]$ (see [3])

$\mu_1(L_{cc}(G)) \geq \left[\frac{J' L_{cc}(G) J}{J' J}\right]$

$= \frac{\sum_{i=1}^{s} \left(d_{cc}(c_i) - k_i\right)^2 - 2q_c}{p}$, where J is the all one's vector. ■

$$\text{Let } M = q_c + \frac{1}{2}\sum_{i=1}^{s}\left(d_{cc}(c_i) - k_i\right)^2. \tag{5}$$

Bapat and S. Pati [2] proved that if energy of the graph is a rational number, then it is an even integer. Similar result for minimum cc-dominating Laplacian energy is established in the following theorem.

Theorem II.3. *Let G be a graph with cc-domination number $\gamma_{cc}(G)$. If the sum of the absolute values of minimum cc-dominating Laplacian eigenvalues is a rational number, then*

$$\sum_{i=1}^{s}|\mu_i| \equiv \gamma_{cc}(G)\,(\mathrm{mod}\,2). \tag{6}$$

Proof: Let $\mu_1, \mu_2 \ldots, \mu_s$ be the eigenvalues of the minimum cc-dominating Laplacian matrix $L_{cc}(G)$ of a graph G of which $\mu_1, \mu_2 \ldots, \mu_r$ are positive and the rest of the eigenvalues are non-positive, then

$$\begin{aligned}\sum_{i=1}^{s}|\mu_i| &= (\mu_1+\mu_2+\cdots+\mu_r) - (\mu_{r+1}+\mu_{r+2}\cdots+\mu_s)\\ &= 2(\mu_1+\mu_2+\cdots+\mu_r) - (\mu_1+\mu_2+\cdots+\mu_s)\\ &= 2(\mu_1+\mu_2+\cdots+\mu_r) - \sum_{i=1}^{s}\mu_i\\ &= 2(\mu_1+\mu_2+\cdots+\mu_r) - (2q_c - \gamma_{cc}(G))\\ &= 2(\mu_1+\mu_2+\cdots+\mu_r - q_c) - \gamma_{cc}(G).\end{aligned}$$

Now, if $\sum_{i=1}^{s}|\mu_i|$ is rational number, then $\mu_1+\mu_2+\cdots+\mu_r$ is also a rational number. Hence it must be an integer. Therefore

$$\sum_{i=1}^{s}|\mu_i| \equiv \gamma_{cc}(G)\,(\mathrm{mod}\,2). \tag{7}$$

■

Theorem II.4. *Let G_1 and G_2 be two graphs with s vertices , q_s and $q_{s'}$ be the number of edges in the clique graph of G_1 and G_2 respectively. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$ and $\mu_1^{'} \geq \mu_2^{'} \geq \cdots \geq \mu_s^{'}$ are the laplacian eigenvalues of G_1 and G_2 respectively. Then*

$$\sum_{i=1}^{s}\mu_i\mu_i^{'} \leq 2\sqrt{(M(G_1)M(G_2))}$$

where M is the expression defined as in (5).

Proof: Using Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{s}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{s}a_i^2\right)\left(\sum_{i=1}^{s}b_i^2\right)$$

Taking $a_i = \mu_i, b_i = \mu_i^{'}$, we get

$$\begin{aligned}\left(\sum_{i=1}^{s}\mu_i\mu_i^{'}\right)^2 &\leq \left(\sum_{i=1}^{s}\mu_i^2\right)\left(\sum_{i=1}^{s}(\mu_i^{'})^2\right)\\ &= 2M(G_1)(2M(G_2))\end{aligned}$$

then the result follows. ■

B. Bounds for minimum cc-dominating Laplacian energy of a graph

Theorem II.5. *Let G be a graph with s cliques, minimum cc-dominating set γ_{cc}-set, and let q_c be the number of edges in the clique graph $C_G(G)$. Then*

$$LE_{cc}(G) \leq \sqrt{2Ms} + 2q_c. \tag{8}$$

where M is the expression defined as in (5).

Proof: Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s$ be the Laplacian eigenvalues of $L_{cc}(G)$.
Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{s}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{s}a_i^2\right)\left(\sum_{i=1}^{s}b_i^2\right)$$

choose $a_i = 1$ and $b_i = |\mu_i|$,

$$\left(\sum_{i=1}^{s}|\mu_i|\right)^2 \leq \left(\sum_{i=1}^{s}1\right)\left(\sum_{i=1}^{s}|\mu_i|^2\right) = s2M,$$

$$\sum_{i=1}^{s}|\mu_i| \leq \sqrt{2Ms}.$$

Using Triangular inequality $|a-b| \leq |a|+|b|$, we have

$$\begin{aligned}\left|\mu_i - \frac{2q_c}{s}\right| &\leq |\mu_i| + \frac{2q_c}{s}\\ \sum_{i=1}^{s}\left|\mu_i - \frac{2q_c}{s}\right| &\leq \sum_{i=1}^{s}|\mu_i| + \sum_{i=1}^{s}\frac{2q_c}{s}\\ LE_{cc}(G) &\leq \sum_{i=1}^{s}|\mu_i| + 2q_c\\ &\leq \sqrt{2Ms} + 2q_c.\end{aligned}$$

■

Theorem II.6. *Let G be a graph with s cliques, minimum cc-dominating set γ_{cc}-set, and let q_c be the number of edges in $C_G(G)$. Then*

$$LE_{cc}(G) \leq \sqrt{2Ms + 4q_c(\gamma_{cc}(G) - q_c)}. \tag{9}$$

where M is the expression defined as in (5).

Proof: Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s$ be the Laplacian eigenvalues of $L_{cc}(G)$.
Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{s}a_ib_i\right)^2 \leq \left(\sum_{i=1}^{s}a_i^2\right)\left(\sum_{i=1}^{s}b_i^2\right)$$

choose $a_i = 1$ and $b_i = \left|\mu_i - \frac{2q_c}{s}\right|$,

$$\left(\sum_{i=1}^{s}\left|\mu_i - \frac{2q_c}{s}\right|\right)^2 \leq \left(\sum_{i=1}^{s}1\right)\left(\sum_{i=1}^{s}\left|\mu_i - \frac{2q_c}{s}\right|^2\right)$$

$$(LE_{cc}(G))^2 \leq s\left(\sum_{i=1}^{s}\mu_i^2 + \sum_{i=1}^{s}\frac{4q_c^2}{s^2} - \frac{4q_c}{s}\sum_{i=1}^{s}\mu_i\right)$$

$$= s\left(2M + \frac{4q_c^2}{s} - \frac{4q_c}{s}\left(2q_c - \gamma_{cc}(G)\right)\right)$$

$$= 2Ms + 4q_c\left(\gamma_{cc}(G) - q_c\right).$$ Therefore

$$LE_{cc}(G) \leq \sqrt{2Ms + 4q_c(\gamma_{cc}(G) - q_c)}.$$

■

Theorem II.7. *Let G be a graph with s cliques, and let q_c be the number of edges in the clique graph $C_G(G)$. If R is the determinant of $L_{cc}(G)$, then*

$$LE_{cc}(G) \leq \sqrt{2M(s-1) + sR^{\frac{2}{s}}} + 2q_c. \tag{10}$$

where M is the expression defined as in (5).

Proof: Using Kober's inequality,

$$s\left[\frac{1}{s}\sum_{i=1}^{s}a_i - \left(\prod_{i=1}^{s}a_i\right)^{\frac{1}{s}}\right] \leq s\sum_{i=1}^{s}a_i - \left(\sum_{i=1}^{s}\sqrt{a_i}\right)^2$$

Putting $a_i = |\mu_i|^2$

$$s\left[\frac{1}{s}\sum_{i=1}^{s}|\mu_i|^2 - \left(\prod_{i=1}^{s}|\mu_i|^2\right)^{\frac{1}{s}}\right] \leq s\sum_{i=1}^{s}|\mu_i|^2 - \left(\sum_{i=1}^{s}|\mu_i|\right)^2$$

$$2M - sR^{\frac{2}{s}} \leq s(2M) - \left(\sum_{i=1}^{s}|\mu_i|\right)^2.$$

$$\left(\sum_{i=1}^{s}|\mu_i|\right) \leq \sqrt{2M(s-1) + R^{\frac{2}{s}}}.$$

Now using triangular inequality,

$$\left|\mu_i - \frac{2q_c}{s}\right| \leq |\mu_i| + \left|\frac{2q_c}{s}\right|$$

$$\sum_{i=1}^{s}\left|\mu_i - \frac{2q_c}{s}\right| \leq \sum_{i=1}^{s}|\mu_i| + \sum_{i=1}^{s}\frac{2q_c}{s}$$

$$LE_{cc}(G) \leq \sum_{i=1}^{s}|\mu_i| + 2q_c.$$ Therefore

$$LE_{cc}(G) \leq \sqrt{2M(s-1) + sR^{\frac{2}{s}}} + 2q_c.$$

■

Theorem II.8. *Let G be a graph with s cliques, and let q_c be the number of edges in the clique graph $C_G(G)$. If R is the determinant of $L_{cc}(G)$, then*

$$LE_{cc}(G) \geq \sqrt{2M + s(s-1)R^{\frac{2}{s}}} - 2q_c. \tag{11}$$

where M is the expression defined as in (5).

Proof:

$$\left(\sum_{i=1}^{s}|\mu_i|\right)^2 = \left(\sum_{i=1}^{s}|\mu_i|\right)\left(\sum_{j=1}^{s}|\mu_j|\right)$$

$$= \sum_{i=1}^{s}|\mu_i|^2 + \sum_{i\neq j}|\mu_i||\mu_j|$$

$$\sum_{i\neq j}|\mu_i||\mu_j| = \left(\sum_{i=1}^{s}|\mu_i|\right)^2 - \sum_{i=1}^{s}|\mu_i|^2$$

$$\frac{1}{s(s-1)}\sum_{i\neq j}|\mu_i||\mu_j| \geq \left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{s(s-1)}}$$

$$\sum_{i\neq j}|\mu_i||\mu_j| \geq s(s-1)\left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{s(s-1)}}.$$

$$\left(\sum_{i=1}^{s}|\mu_i|\right)^2 - \sum_{i=1}^{s}|\mu_i|^2 \geq s(s-1)\left(\prod_{i=1}^{s}|\mu_i|^{2(s-1)}\right)^{\frac{1}{s(s-1)}}$$

$$\left(\sum_{i=1}^{s}|\mu_i|\right)^2 - 2M \geq s(s-1)\left(\prod_{i=1}^{s}|\mu_i|\right)^{\frac{2}{s}}$$

$$\left(\sum_{i=1}^{s}|\mu_i|\right)^2 \geq 2M + s(s-1)R^{\frac{2}{s}}$$

$$\left(\sum_{i=1}^{s}|\mu_i|\right) \geq \sqrt{2M + s(s-1)R^{\frac{2}{s}}}$$

Using Triangular inequality $|a| - |b| \leq |a-b|$,

$$|\mu_i| - \left|\frac{2q_c}{s}\right| \leq \left|\mu_i - \frac{2q_c}{s}\right| \forall i = 1, 2, \ldots s.$$

$$\sum_{i=1}^{s}|\mu_i| - \sum_{i=1}^{s}\frac{2q_c}{s} \leq \sum_{i=1}^{s}\left|\mu_i - \frac{2q_c}{s}\right|$$

$$\sum_{i=1}^{s}|\mu_i| - 2q_c \leq LE_{vv}(G)$$

$$LE_{vv}(G) \geq \sum_{i=1}^{s}|\mu_i| - 2q_c$$

$$\geq \sqrt{2M + s(s-1)R^{\frac{2}{s}}} - 2q_c.$$

■

III. Conclusion

Clique domination and Energy of a graph are two well-studied parameters in literature. We modified and studied Minimum clique clique dominating Laplacian energy of graphs. Few Properties are discussed. Several bounds for Minimum clique clique dominating Laplacian energy are obtained. Characterization of the graphs attaining these bounds are not studied in full and one may take this as an open problem for further research.

REFERENCES

[1] C. Adiga, A. Bayad, I. Gutman and S. A. Srinivas, "The minimum covering energy of a graph" *Kragujevac J. Sci*, vol. 34, pp. 39–56, 2012.
[2] R. B. Bapat and S. Pati, "Energy of a graph is never be an odd integer" *Bull. Kerala Math. Assoc.* , vol. 1, pp. 129–132, 2011.
[3] R. B. Bapat, "Graphs and Matrices", Hindustan Book Agency, New Delhi, 2011.
[4] Smitha G. Bhat, "A study of independent set and clique related parameters of a graph", PhD Thesis, Manipal Academy of Higher education, Manipal, 2019.
[5] I. Gutman, "The Energy of a Graph", *Ber. Math. Stat. Sekt. Forschungszentrum Graz.* vol. 103, pp. 1–22, 1978.
[6] I. Gutman, "The energy of a graph : old and new results", *Algebraic Combinatorics and Applications*. Springer-Verlag, Berlin, pp. 196–211, 2001.
[7] I. Gutman and B. Zhou "Laplacian energy of a graph", *Lin. Algebra Appli*, vol. 414, pp. 29–37, 2004.
[8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, "Fundamentals of Domination in Graphs", Marcel Dekker, New York, 1998.
[9] F. Harary, "Graph Theory", Addison Wesley, Reading, Massachusetts, 1969.
[10] G. Indulal, I. Gutman and A. Vijaykumar, "On distance energy of graphs", *MATCH Commun. Math. Comput. Chem*, vol. 60, pp. 461–472, 2008.
[11] M. R. Jooyandeh, D. Kiani and M. Mirzakhah, "Incidence energy of a graph", *MATCH Commun. Math. Comput. Chem*, vol. 62 pp. 561–572, 2009.
[12] S. S. Kamath and R. S. Bhat, "On Strong (Weak) Independent sets and Vertex Coverings of a Graph", *Discrete Mathematics*, vol. 307, pp. 1136–1145, 2007.
[13] C. li, "On the shephard type problems for general Lp-Projection Bodies", *IAENG International journal of Applied Mathematics*, vol. 49, no.1, pp. 122–126, 2019.
[14] Q. Liu, "Resistance distance and Kirchoff Index of Two edge-subdivision Corona Graphs", *IAENG International journal of Applied Mathematics*, vol. 49, no.1, pp. 127–133, 2019.
[15] J. H. Koolen and V. Moulton, "Maximal energy graphs", *Adv. Appl. Math*, vol. 26, pp. 47–52, 2001.
[16] B. J. McClelland, "Properties of the latent roots of a matrix : The estimation of π-electron energies", *J. Chem. Phys*, vol. 54, pp. 640–643, 1971.
[17] Sabitha D'souza, Gowtham H. J and Pradeep G. Bhat, "Energy of Generalized Complements of a Graph", *Engineering Letters*, vol. 28, no. 1, pp. 131–136, 2020.
[18] Sayinath Udupa, R. S. Bhat and Vinay Madhusudanan, "The minimum vertex-block dominating energy of the graph", *Proceedings of the Jangeon Mathematical Society*, vol. 22, no. 1, pp. 593–608, 2019 .
[19] Sayinath Udupa, R. S. Bhat and Prathviraj .N, "Minimum Clique-clique dominating energy of a graph", *Advances in Mathematics : Scientific Journal*, vol. 9, no. 6, pp. 3237–3246, 2020 .
[20] Sayinath Udupa and R. S. Bhat, "Strong (Weak) vb-Dominating and vb-Independent sets of a Graph", *Discrete Mathematics, algorithms and applications*, vol. 20, no. 1, pp. 300–315, 2020.
[21] Sayinath Udupa and R. S. Bhat, "Strong (weak) vv-dominating set of a graph", *Malaysian Journal of Science*, vol. 38, no. 3 , pp. 24–33, 2019.
[22] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, New Jersey, 1996.

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