# Fuzzy Expected Value-Deviation Portfolio Selection with Riskless Asset Based on Credibility Measures

Xue Deng, Yongkang Yuan

***Abstract*—This paper explores a problem whether a portfolio should contain a riskless asset in fuzzy environment based on credibility theory. From a novel perspective, we use the fuzzy deviation together with value function (value-deviation) as a novel portfolio risk measurement. In the light of credibility theory in this paper, three credibilistic expected value-deviation models with fuzzy returns are proposed for the first time. In addition, genetic algorithm is adopted to solve our presented fuzzy nonlinear portfolio models. Furthermore, a comparative numerical example with or without a riskless asset is given to verify the validity of our presented portfolio models. The final results present that no matter how much return investors want to realize, the maximum value deviation of portfolio with a riskless asset is more than the maximum value deviation of portfolio without a riskless asset forever. That enriches portfolio contents in theory and provides investors with more choices in practice.**

***Index Terms*—Credibility theory; Deviation; Portfolio selection; Riskless asset; Value function**

## I. Introduction

The key problem of portfolio selection is to consider the trade-off between return and risk according to investors' preference. Portfolio management is a key and important component in finance. In 1952, the well-known mean-variance model (M-V model) was put forward by Markowitz [1]. In his M-V model, the mean is used as the measure of portfolio return. The variance is adopted to quantify portfolio risk. As a milestone in modern finance theory, Markowitz initially combined portfolio selection with mathematics in his M-V model.

However, the M-V model is rather difficult for calculating the covariance with the increasing of dimensionality. Meanwhile, compared with the expected value, the different influences from gain and loss are ignored when the variance is adopted to measure portfolio risk. On the one hand, several means were proposed to improve the effectiveness of the M-V models by using different approximation schemes by Sharpe and Stone in [2, 3, 4]. On the other hand, several researchers proposed some other measures of risk such as semi-variance by Huang in [5] and deviation by Konno et al. in [6]. Yet the semi-variance only considers those risks below the expected value and overlooks the 'compensation' from those risks above the expected value. Konno et al. [6] put forward a new portfolio model in which the mean absolute deviation was adopted to quantify risk instead of variance. Compared to variance, the mean-absolute deviation can remove most of the difficulties. Because it removes the complex calculation of covariance and replaces with a linear program. Meanwhile, it maintains the advantages of the M-V model. For the sake of making up for the drawbacks of previous distance measures and pursuing more accuracy and effectiveness, Chen and Deng [7] put forward a new inclusion relation of IFSs and a new definition called strict distance measure. On the basis of traditional M-V models, some practical approaches were presented by Deng and Zhao [8], who derived more exact results about risk value ranges. Daniel Kahneman and Tversky [9] proposed prospect theory, which considered different attitudes of investor towards gain and loss. They used value function to measure different attitudes towards gain and loss based on prospect theory. Denault et al. [10] presented simulation-and-regression methods and compared value function recursion with portfolio weight recursion. Gong et al. [11] combined cumulative prospect theory to deeply explore portfolio problem and solved it by coupling scenario generation techniques with a genetic algorithm.

In the M-V model, Markowitz assumed that the future return of each security could be predicted by historical data, which would surely generate great inaccuracy. In order to reduce the inaccuracy, we adopted fuzzy numbers to characterize the security returns. Since the possibility measurement of fuzzy numbers is not self-dual in [12], we proposed three new credibilistic fuzzy portfolio models.

## II. Preliminaries

### A. Credibility Theory

The triangular fuzzy variable $\xi$ is a kind of common fuzzy numbers. It generally described by a triplet $(a,b,c)$, where $c > b > a$. Additionally, the membership function is adopted to describe the distribution of fuzzy numbers. Specially, the specific membership function formula of triangular number $\xi$ is as follows.

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$$\mu(x)=\begin{cases}\dfrac{a-x}{a-b}, & \text{if } a\le x<b,\\ \dfrac{c-x}{c-b}, & \text{if } b\le x<c,\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

We let $r$ be a real number. The possibility of a fuzzy event $\{\xi\ge r\}$ is given a calculation formula $\mathrm{Pos}\{\xi\ge r\}=\sup_{x\ge r}\mu(x)$. However, the possibility measure mentioned above is not self-dual, which means a theoretically inevitable event may not happen. From the example above, we can obtain $\mathrm{Pos}\{\xi\ge r\}=1$ when $r\le b$, but the event $\{\xi\ge r\}$ is not a certain event. To seek the solution to this problem, a self-dual credibility measure was by defined Liu [13] in 2002.

**Definition 1:** We assume $\xi$ is a fuzzy variable, then Liu [13] assigned the credibility value calculation formula of $\{\xi\ge r\}$ as follows.

$$\mathrm{Cr}\{\xi\ge r\}=\frac{1}{2}\left(\sup_{x\ge r}\mu(x)+1-\sup_{x<r}\mu(x)\right). \tag{2}$$

**Example 1:** We assume fuzzy triangular variable $\xi$ as $(a,b,c)$. From the definition 1, the credibility of $\xi\ge r$ is

$$\mathrm{Cr}\{\xi\ge r\}=\begin{cases}1, & r\in(-\infty,a),\\ \dfrac{2b-a-r}{2(b-a)}, & r\in[a,b),\\ \dfrac{c-r}{2(c-b)}, & r\in[b,c),\\ 0, & r\in[c,\infty).\end{cases} \tag{3}$$

When using the credibility measure, the credibility of $\xi\ge r$ ranges from 0 to 1. More importantly, the credibility value drops as the probability of an event decreases, and the credibility value of an event is 0 if and only if it is an impossible event, and vice versa. Therefore, we stipulate credibility as the measurement to quantify happening odds of a fuzzy event.

**Definition 2:** We assume $\xi$ is a fuzzy variable. Then we can compute the expected value by

$$E[\xi]=\int_0^{+\infty}\mathrm{Cr}\{\xi\ge r\}\mathrm{d}r-\int_{-\infty}^0\mathrm{Cr}\{\xi\le r\}\mathrm{d}r. \tag{4}$$

Supposed that one of the two parts is convergent at least.

**Example 2:** According to Definition 2, we obtain $E[\xi]=(a+2b+c)/4$ easily if $\xi$ is a triangular variable.

**Definition 3:** We assume $\xi$ is a fuzzy variable, and Liu [15] defined the expected value of a function about fuzzy variable as

$$E[f(\xi)]=\int_0^{+\infty}\mathrm{Cr}\{f(\xi)\ge r\}\mathrm{d}r-\int_{-\infty}^0\mathrm{Cr}\{f(\xi)\le r\}\mathrm{d}r. \tag{5}$$

**Definition 4:** With the definition of the expected value of fuzzy variables, the distance between fuzzy variables $\xi$ and $\eta$ is given by Liu in [14] as

$$d(\xi,\eta)=E[|\xi-\eta|]. \tag{6}$$

It has the following properties:

(a) $d(\xi,\eta)\ge 0$;

(b) $d(\xi,\eta)=0$; if and only if $\xi=\eta$ ↵

(c) $d(\xi,\eta)=d(\eta,\xi)$;

(d) $d(\xi,\eta)\le d(\xi,\tau)+d(\tau,\eta)$, where $\tau$ is also a fuzzy variable.

*B. Absolute Deviation:*

**Definition 4**: If we regard returns as random variable $X$, then the absolute deviation of $X$ was defined by Konno in [6] as

$$E|X-m|=\int f(x)|X-m|\mathrm{d}x. \tag{7}$$

Where the function $f$ is on behalf of the probability function, and $m$ is on behalf of the mean of $X$.

**Definition 5:** If $\xi$ is a fuzzy variable whose $E[\xi]$ exists. Then we define the fuzzy absolute deviation of $\xi$ as

$$AD[\xi]=E[|\xi-E[\xi]|]. \tag{8}$$

**Theorem 1:** We let $\xi$ and $\eta$ be two uncertain variables, which are independent. If $E[\xi]$ and $E[\eta]$ exit, then for real numbers $a$ and $b$, we can derive the theorem from Liu [16]

$$E[a\xi+b\eta]=aE[\xi]+bE[\eta]. \tag{9}$$

**Theorem 2:** Let $\xi=(a_\xi,b_\xi,c_\xi,d_\xi),\eta=(a_\eta,b_\eta,c_\eta,d_\eta)$. Then according to Liu in [11], we have

$$\xi+\eta=(a_\xi+a_\eta,b_\xi+b_\eta,c_\xi+c_\eta,d_\xi+d_\eta).$$

*C. Value Function*

As is mentioned in Section 1, people regard gain and loss differently. More importantly, people tend to have loss aversion, which means people pay more attention to loss rather than gain. From the Prospect Theory by Kahneman et al. in [8], the first step of choice process is editing phase. We have to define $r_i'$, which can be regarded as the "pure risk" for investors. It can be defined as $r_i'=r_i-\overline{r_i}$, where $\overline{r_i}$ is the reference of return rate of security $S_i$. In this paper, we suppose that the reference of return is the expected return namely, $\overline{r_i}=E[r_i]$. Then the "pure risk" can be described as $r_i'=r_i-E[r_i]$. The second step is the evaluation. It can be measured as the value function, which shows the difference of subjective evaluation between gain and loss from the investors. Now there are three common value functions given in [17,18,19]:

$$f\left(r_i'\right)=\begin{cases}ar_i'+c\left(r_i'\right)^\theta, & r_i'\ge 0,\\ br_i'-d\left|r_i'\right|^\theta, & r_i'\le 0.\end{cases} \tag{10}$$

where are constants, and $a\ge 0,b\ge 0,b\ge a$.

$$f\left(r_i'\right)=\begin{cases}ar_i'+\omega\left(1-\mathrm{e}^{-cr_i'}\right), & r_i'\ge 0,\\ br_i'-\theta\left(1-\mathrm{e}^{dr_i'}\right), & r_i'\le 0.\end{cases} \tag{11}$$

where $a,b,c,d,\theta$ are constants, and $a\ge 0,b\ge 0,b\ge a$.

$$f\left(r_i'\right)=\begin{cases}m\ln\left(pr_i'+p\right), & r_i'\ge 0,\\ -n\ln\left|qr_i'-q\right|, & r_i'\le 0.\end{cases} \tag{12}$$

where $m,n,p,q$ are constants, and $m\ge 0,n\ge 0,n\ge m$.

And in the real world, a bigger deviation means a smaller risk (which is different from absolute deviation), therefore we regard minus value deviation $-f(r_i')$ as risk. (Here we assume the value deviations of investors are direct additive, then the total deviation is $R^v=\sum_{i=1}^n f\left(r_i'\right)x_i$ ).

## III. Fuzzy Expected Value-Deviation Model

*A. The Fuzzy Expected Value-Absolute Deviation Model*

In a portfolio selection, investors always pursue the largest return and the least risk, which can be described as a bi-objective model. Hence, a single objective model is presented with a minimum return constraint.

We use $\xi_i$ to denote the fuzzy return rate of security $S_i$. $x_i$ represents the percentage of security $S_i$ in portfolio without considering the transaction fee. Then we have

$$\begin{cases} \min \quad AD\left[\sum_{i=1}^{n}\xi_i x_i\right] \\ \text{s.t.} \quad E\left[\sum_{i=1}^{n}\xi_i x_i\right] > \alpha, \\ \qquad \sum_{i=1}^{n} x_i = 1, \\ \qquad 0 \le x_i \le u_i \le 1, i = 1, 2, \cdots, n. \end{cases} \tag{13}$$

$\alpha$ denotes the lowest return rate the investor required, $u$ is the maximum permitted amount of buying security i. And here we stipulate $x_i \ge 0$, which means there is no oversell. Besides, there is no transaction fee.

In fact, there are also some other assets like bank deposit, which have low return but nearly no risk. Considering that, let $R_f$ be the return rate of a riskless asset, whose percentage is $x_f$. Related to actual life, it is obvious that the riskless asset will reduce risk. Here we can obtain a better model:

$$\begin{cases} \min \quad AD\left[\sum_{i=1}^{n}\xi_i x_i\right] \\ \text{s.t.} \quad E\left[\sum_{i=1}^{n}\xi_i x_i\right] + R_f x_f > \alpha, \\ \qquad \sum_{i=1}^{n} x_i + x_f = 1, \\ \qquad 0 \le x_i \le u_i \le 1, i = 1, 2, \cdots, n, \\ \qquad 0 \le x_f \le 1. \end{cases} \tag{14}$$

All marks are the same as in model (13).

*B. The Fuzzy Absolute-Value-Deviation-Based Model*

With the basis of model (14), now we take the influences from gain and loss into account, which are described by the absolute value deviation applying value function.

We regard $\xi$ as a fuzzy variable whose $E[\xi]$ exists. Then the absolute value deviation of $\xi$ is defined by $AVD[\xi] = E[|f(\xi')|]$, and we obtain model (15) (where $\xi' = \xi - E[\xi]$).

Marks are the same as model (14). Model (15) is an improvement of (14). Here we used the value function to describe the investors' different attitudes towards gain and loss.

$$\begin{cases} \min \quad AVD\left[\sum_{i=1}^{n}\xi_i x_i\right] \\ \text{s.t.} \quad E\left[\sum_{i=1}^{n}\xi_i x_i\right] + R_f x_f > \alpha, \\ \qquad \sum_{i=1}^{n} x_i + x_f = 1, \\ \qquad 0 \le x_i \le u_i \le 1, i = 1, 2, \cdots, n, \\ \qquad 0 \le x_f \le 1. \end{cases} \tag{15}$$

*C. The Fuzzy Max-Value-Deviation-Based Model*

Model (15) really reflects different weights from gain and loss, but it ignores the fact that people have a predilection towards gain and an aversion towards loss. In other words, loss can be compensated by gain. Based on this, we proposed a better model.

We set $\xi$ to be a fuzzy variable. And its Value Deviation is defined by $VD[\xi] = -E[f(\xi')]$, where $\xi' = \xi - E[\xi]$.

Here is a minus because $E[f(\xi')]$ is better as it grows while variance is worse when it grows. The variables are the same as model (14). We can obtain models (16) and (17)

$$\begin{cases} \min \quad VD\left[\sum_{i=1}^{n}\xi_i x_i\right] \\ \text{s.t.} \quad E\left[\sum_{i=1}^{n}\xi_i x_i\right] + R_f x_f > \alpha, \\ \qquad \sum_{i=1}^{n} x_i + x_f = 1, \\ \qquad 0 \le x_i \le u \le 1, i = 1, 2, ..., n, \\ \qquad 0 \le x_f \le 1. \end{cases} \tag{16}$$

and

$$\begin{cases} \min \quad VD\left[\sum_{i=1}^{n}\xi_i x_i\right] \\ \text{s.t.} \quad E\left[\sum_{i=1}^{n}\xi_i x_i\right] > \alpha, \\ \qquad \sum_{i=1}^{n} x_i = 1, \\ \qquad 0 \le x_i \le u_i \le 1, i = 1, 2, \cdots, n. \end{cases} \tag{17}$$

Obviously, model (16) and (17) used a different measure of risk from model (15). In model (15), we expect the total absolute value deviation to be minimum. In comparison, we consider the minus value deviation and the plus one to ensure the former be the largest and the latter be the lowest in model (16). It can be seen that model (16) is more appropriate.

## IV. Solution Method and Algorithms

*A. The Basic Ideas of Tolerantly Complete Layering Method*

To simplify the calculation, we chose the third function from Section 2.4 (assuming $m = p = q = 1, n = 2$) and applied it to model (16) and (17) and stipulated that $\xi_i$ is a group of independent triangular fuzzy variables determined by the triplets $(a_i, b_i, c_i)$. The object function turns to be $-E[f(\sum_{i=1}^{n}(\xi_i - E[\xi_i]))]$. According to Definition 3, we can simplify the object function as:

$$\begin{aligned} VD[\xi] &= -E[f(\textstyle\sum_{i=1}^{n}(\xi_i - E[\xi_i]))] \\ &= -E[f(\xi - E[\xi])] \\ &= -\int_0^{+\infty} \text{Cr}\{f(\xi - E[\xi]) \ge r\}dr \\ &\quad + \int_{-\infty}^{0} \text{Cr}\{f(\xi - E[\xi]) < r\}dr. \end{aligned} \tag{18}$$

Apparently, $\int_{-\infty}^{0} \text{Cr}\{f(\xi - E[\xi]) < r\}\text{d}r = 0$, so we have:

$$VD[\xi] = -\int_0^{+\infty} \text{Cr}\{f(\xi - E[\xi]) \ge r\}\text{d}r, \tag{19}$$

where $f(\xi - E[\xi]) > 0$.

And at the same time, we suppose

$$f(r')=\begin{cases} m\ln(pr_i'+p), & r_i'\ge 0,\\ -n\ln|qr_i'-q|, & r_i'\le 0,\end{cases} \tag{20}$$

where $m=p=q=1, n=2$.

Then when $r_i'\ge 0$ we have $f(r')>0$, so $f(\xi-E[\xi])=\ln(\xi-E[\xi]+1)$, we obtain

$$\begin{aligned}&\int_0^{+\infty}\text{Cr}\{f(\xi-E[\xi])\ge r\}\text{d}r\\ &=\int_0^{+\infty}\text{Cr}\{\ln(\xi-E[\xi]+1)\ge r\}\text{d}r\\ &=\int_0^{+\infty}\text{Cr}\{\xi-E[\xi]+1\ge e^r\}\text{d}r\\ &=\int_0^{+\infty}\text{Cr}\{\xi\ge e^r+E[\xi]-1\}\text{d}r.\end{aligned} \tag{21}$$

When $b<E[\xi]$, we have

$$\begin{aligned}VD[\xi]&=-\int_0^{\ln(1-E[\xi]+c)}\frac{1}{2}(\sup_{\xi\ge e^r+E[\xi]-1}\mu(\xi)\ge r+1-\sup_{\xi<e^r+E[\xi]-1}\mu(\xi))\text{d}r-\int_{\ln(1-E[\xi]+c)}^{+\infty}\frac{1}{2}(\sup_{\xi\ge e^r+E[\xi]-1}\mu(\xi)\ge r+1-\sup_{\xi<e^r+E[\xi]-1}\mu(\xi))\text{d}r\\ &=-\int_0^{\ln(1-E[\xi]+c)}\frac{1}{2}(\frac{e^r+E[\xi]-1-c}{b-c}+1-1)\text{d}r-\int_{\ln(1-E[\xi]+c)}^{+\infty}\frac{1}{2}(0+1-1)\text{d}r\\ &=-\int_0^{\ln(1-E[\xi]+c)}\frac{e^r-1+E[\xi]-c}{2x(b-c)}\text{d}r-\int_{\ln(1-E[\xi]+c)}^{+\infty}0\text{d}r\\ &=-\frac{E[\xi]-1-c}{2(b-c)}\ln(1-E[\xi]+c)-\frac{1-E[\xi]+c-1}{2(b-c)}\\ &=\frac{1+c-E[\xi]}{2(b-c)}\ln(1-E[\xi]+c)-\frac{c-E[\xi]}{2(b-c)}.\end{aligned} \tag{22}$$

The situation $b>E[\xi]$ can be deduced in the same way. Then the objective function can be simplified as:

$$VD[\xi]=\begin{cases}-\ln(1-e+b)+\dfrac{(e-a)-1}{2(b-a)}\ln(1-e+b)+\dfrac{b-e}{2(b-a)}\\ -\dfrac{e-c-1}{2(b-c)}\ln(\dfrac{1-e+c}{1-e+b})+\dfrac{1}{2}+\dfrac{1+e-a}{b-a}\ln(1+e-a)+\dfrac{a-e}{b-a}, \text{when } b\ge e,\\ 2\ln(1+e-b)+\dfrac{e-b}{b-c}+\dfrac{1+(e-c)}{b-c}\ln(1+e-b)-1\\ +\dfrac{1+e-a}{b-a}\ln(\dfrac{1+e-a}{1+e-b})+\dfrac{1+c-e}{2(b-c)}\ln(1-e+c)-\dfrac{c-e}{2(b-c)}, \text{ when } b<e.\end{cases} \tag{23}$$

Where $\xi=\sum_{i=1}^n\xi_i$, $a=\sum_{i=1}^n a_ix_i$, $b=\sum_{i=1}^n b_ix_i$, $c=\sum_{i=1}^n c_ix_i$, $e=E[\xi]=(a+2b+c)/4$. Meanwhile, the constraints condition $E[\sum_{i=1}^n\xi_ix_i]+R_fx_f>\alpha$ can be simplified as $\sum_{i=1}^n x_iE[\xi_i]+R_fx_f>\alpha$. And according to the Theorem 1, the former formula be rewritten as

$$R_fx_f+\sum_{i=1}^n\frac{a_i+2b_i+c_i}{4}x_i\ge\alpha. \tag{24}$$

Then we have simplified models (16) and (17) to models (25) and (26).

In this paper we solve the models (25) and (26) with Genetic Algorithm (GA).

$$\begin{cases}\min & VD[\xi]\\ \text{s.t.} & R_fx_f+\sum_{i=1}^n\dfrac{a_i+2b_i+c_i}{4}x_i>\alpha,\\ & \sum_{i=1}^{10}x_i+x_f=1,\\ & 0\le x_i\le u_i\le 1, i=1,2,\cdots,n,\\ & 0\le x_f\le 1.\end{cases} \tag{25}$$

and

$$\begin{cases}\min & VD[\xi]\\ \text{s.t.} & \sum_{i=1}^n\dfrac{a_i+2b_i+c_i}{4}x_i>\alpha,\\ & \sum_{i=1}^{10}x_i=1,\\ & 0\le x_i\le u_i\le 1, i=1,2,\cdots,n.\end{cases} \tag{26}$$

GA was firstly proposed by Holland in [20] to deal with the complex calculating problems. Then it was developed by Koza [21], Gen and Cheng [22], Li [23] and Li [24]. A hybrid intelligent algorithm was proposed by Huang for fuzzy variables.

GA initially generates many chromosomes (solutions) until there are *popsize* feasible chromosomes (solutions meet the constraints). And with a comparison among different objective values, the chromosomes are ranked, selected and updated with crossover and mutation. After *generation* cycles, the chromosome that ranks the first is regarded as the more satisfactory solution. Here the GA toolbox in Matlab is used to seek the solution to the problem.

**Step 1:** Generate *popsize* feasible chromosomes, set $popsize=20$;

**Step 2:** Compute the objective value in initial group;

**Step 3:** Pick out the chromosomes and update the population with crossover and mutation operation. The crossover probability $P_c$ is valued as 0.8. The mutation probability $P_m$ is valued 0.2;

**Step 4:** It is not until *generation* times that we repeat Step 2 to Step 3. Additionally we set $generation = 2000$;

**Step 5:** Take the chromosome ranking first at last as the solution.

## V. NUMERICAL EXAMPLES

To see whether the max value deviation can really be a measure of risk, we assume 10 fuzzy returns of securities as shown in Table I.

TABLE I
FUZZY RETURNS OF 10 SECURITIES

| Security $i$ | Fuzzy return | Security $i$ | Fuzzy return |
|---|---|---|---|
| 1 | $(-0.2, 2.1, 2.5)$ | 6 | $(-0.2, 2.5, 3.0)$ |
| 2 | $(-0.1, 1.9, 3.0)$ | 7 | $(-0.2, 3.0, 3.5)$ |
| 3 | $(-0.4, 3.0, 4.0)$ | 8 | $(-0.4, 2.5, 4.0)$ |
| 4 | $(-0.1, 2.0, 2.5)$ | 9 | $(-0.3, 2.8, 3.2)$ |
| 5 | $(-0.6, 3.0, 4.0)$ | 10 | $(-0.3, 2.0, 2.5)$ |

Also we assume $\alpha = 1.7, u = 0.3$ and $R_f = 1$, then we have the following model from model (16):

$$\begin{cases} \min \quad VD[\xi] = -E[f(\sum_{i=1}^{10}(\xi_i - E[\xi_i]))] \\ \text{s.t.} \quad x_f + \sum_{i=1}^{10} \frac{a_i + 2b_i + c_i}{4} x_i \ge 1.7, \\ \quad \sum_{i=1}^{10} x_i + x_f = 1, \\ \quad 0 \le x_i \le 0.3, i = 1, 2, \cdots, n, \\ \quad 0 \le x_f \le 1. \end{cases} \tag{27}$$

Where we take an example from model (25) to give a specific numerical form (where $x_f = 1 - \sum_{i=1}^{10} x_i$ )

$$\begin{cases} \min \quad VD[\xi] = -E[f(\sum_{i=1}^{10}(\xi_i - E[\xi_i]))] \\ \text{s.t.} \quad \sum_{i=1}^{10} \frac{a_i + 2b_i + c_i}{4} x_i \ge 1.7, \\ \quad \sum_{i=1}^{10} x_i = 1, \\ \quad 0 \le x_i \le 0.3, i = 1, 2, \cdots, 10. \end{cases} \tag{28}$$

In order to maximize the max value deviation and ensure the total expected value is higher than 1.7, the investors should allocate money as Table 2 according to the results of GA. With this allocation, the max value deviation is 0.1932. Obviously, the result meets the definition of the max value deviation. When the max value deviation is above zero, it means there is a return that exceeds its value at risk.

Keep $\alpha$ and $u$ unchanged and then we get the following model from model (17)

$$\begin{cases} \min \ f = VD[\xi] = -E[f(\sum_{i=1}^{10}(\xi_i - E[\xi_i]))] \\ \text{s.t.} \ 0.625x_1 + 0.675x_2 + 1.4x_3 + 0.6x_4 + 1.35x_5 + 0.95x_6 + 1.325x_7 \\ \quad + 1.15x_8 + 1.125x_9 + 0.55x_{10} \ge 1.7, \\ \quad \sum_{i=1}^{10} x_i = 1, \\ \quad 0 \le x_i \le 0.3, i = 1, \cdots, 10. \end{cases} \tag{29}$$

Also, the result is shown in Table 3 after running GA. In this model, the max value deviation is 0.2260. Compared with model (25), we can see that when adding the riskless asset, the max value deviation increased by 0.0327. There is no doubt that adding riskless asset to the portfolio can increase the max value deviation.

Now we select a series of $\alpha$ for different situations and run them in both model (25) and model (26). The results are in Table IV and Table V. Then we draw a figure with its x-axis the lowest return rate the investor required and its y-axis the max value deviation as Fig. 1.

From the results in Tables II-IV, we can see that with the growth of $\alpha$, the max value deviation continues decreasing whatever with a riskless asset or not, which means that the risk grows with a higher required return.

TABLE II
THE OPTIMAL PORTFOLIO WHEN $\alpha = 1.7$, $u = 0.3$ AND $R_f = 1$ WITH RISKLESS ASSET FROM MODEL (27)

| Security $i$ | Investment | Security $i$ | Investment |
|---|---|---|---|
| $x_1$ | 0.173 | $x_6$ | 0.192 |
| $x_2$ | 0.139 | $x_7$ | 0.178 |
| $x_3$ | 0.055 | $x_8$ | 0 |
| $x_4$ | 0 | $x_9$ | 0.001 |
| $x_5$ | 0.002 | $x_{10}$ | 0 |
| | | $x_f$ | 0.260 |

TABLE III
THE OPTIMAL PORTFOLIO WHEN $\alpha = 1.7$, $u = 0.3$ AND $R_f = 1$ WITHOUT RISKLESS ASSET FROM MODEL (28)

| Security $i$ | Investment | Security $i$ | Investment |
|---|---|---|---|
| $x_1$ | 0.262 | $x_6$ | 0.049 |
| $x_2$ | 0.049 | $x_7$ | 0.089 |
| $x_3$ | 0.049 | $x_8$ | 0.049 |
| $x_4$ | 0.300 | $x_9$ | 0.049 |
| $x_5$ | 0.057 | $x_{10}$ | 0.049 |

TABLE IV
THE OPTIMAL PORTFOLIO FOR 6 DIFFERENT $\alpha$ VALUES WITH A RISKLESS ASSET ( $u = 0.3$ ) IN MODEL (25)

| $\alpha$ | 1.7 | 1.8 | 1.9 | 2.0 | 2.1 | 2.2 |
|---|---|---|---|---|---|---|
| $-VD^*$ | 0.1932 | 0.2011 | 0.2208 | 0.2335 | 0.2397 | 0.2522 |
| $x_1$ | 0.173 | 0 | 0.124 | 0.001 | 0 | 0.032 |
| $x_2$ | 0.139 | 0 | 0 | 0.109 | 0.004 | 0.03 |
| $x_3$ | 0.055 | 0.099 | 0.04 | 0.297 | 0.272 | 0.3 |
| $x_4$ | 0 | 0.076 | 0.106 | 0.037 | 0.053 | 0.027 |
| $x_5$ | 0.002 | 0.004 | 0.058 | 0.084 | 0 | 0.015 |
| $x_6$ | 0.192 | 0.205 | 0.025 | 0.075 | 0.269 | 0.005 |
| $x_7$ | 0.178 | 0.207 | 0.289 | 0.164 | 0.068 | 0.298 |
| $x_8$ | 0 | 0 | 0.045 | 0.014 | 0 | 0.085 |
| $x_9$ | 0.001 | 0.079 | 0.091 | 0.062 | 0.3 | 0.184 |
| $x_{10}$ | 0 | 0.095 | 0.116 | 0 | 0 | 0 |
| $x_f$ | 0.260 | 0.235 | 0.106 | 0.157 | 0.034 | 0.024 |

TABLE V
THE OPTIMAL PORTFOLIO FOR 6 DIFFERENT $\alpha$ VALUES WITHOUT A RISKLESS ASSET ( $u = 0.3$ ) IN MODEL (26)

| $\alpha$ | 1.7 | 1.8 | 1.9 | 2.0 | 2.1 | 2.2 |
|---|---|---|---|---|---|---|
| $-VD^*$ | 0.2260 | 0.2314 | 0.2355 | 0.2404 | 0.2512 | 0.2596 |
| $x_1$ | 0.262 | 0.278 | 0.3 | 0.296 | 0.3 | 0.221 |
| $x_2$ | 0.049 | 0.127 | 0.077 | 0.039 | 0 | 0 |
| $x_3$ | 0.049 | 0.069 | 0.079 | 0.123 | 0.193 | 0.3 |
| $x_4$ | 0.3 | 0.078 | 0.076 | 0.019 | 0 | 0 |
| $x_5$ | 0.057 | 0.082 | 0.079 | 0.117 | 0.172 | 0.3 |
| $x_6$ | 0.049 | 0.077 | 0.078 | 0.071 | 0.006 | 0 |
| $x_7$ | 0.089 | 0.119 | 0.079 | 0.114 | 0.162 | 0.179 |
| $x_8$ | 0.049 | 0 | 0.078 | 0.043 | 0.089 | 0 |
| $x_9$ | 0.049 | 0.091 | 0.078 | 0.153 | 0.079 | 0 |
| $x_{10}$ | 0.049 | 0.078 | 0.076 | 0.025 | 0 | 0 |

Note that, in Tables IV and V, $VD^*$ refers to the optimal $VD$ value. According to the data in Tables IV and V, we can draw the line of the lowest required return and max value deviation.

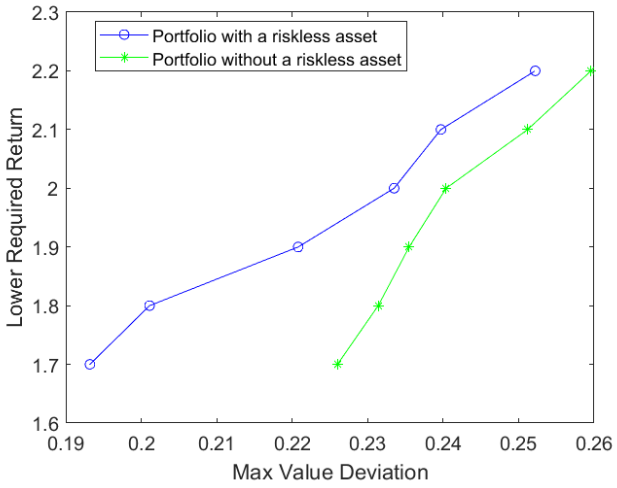


**Fig.1:** The lowest required return and the max value deviation in models (25) and (26)

There are two significant features in Fig. 1. The green line is to the left of the blue line. That means the portfolio with a riskless asset is better than the one without forever. Besides, both the two broken lines showed an upward trend. This is in line with the equilibrium principle of return and risk. That means our proposed value deviation function is a reasonable measurement to risk.

## VI. Conclusion

The portfolio selection is a balancing problem between return and risk. This paper proposes a new measure of risk with a combination of absolute deviation, credibility theory and value function. Based on this measure, several models are given. After running GA to solve the models, a series of results show that the models proposed are robust enough for a portfolio selection. From Fig. 1 we can find out that the corresponding max value deviation will increases with the increase of the lowest required return. On the other hand, when the lowest required return is fixed, the max value deviation of portfolio with a riskless asset is less than the max value deviation of portfolio without a riskless asset at the same lower required return. That means portfolio with a riskless asset can improve investors' investment status. These conclusions have important empirical significance for investors to make portfolio decisions.

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