The Bounded Projector in Weighted Mixed Lebesgue Spaces

Yang Han, Bei Liu*

Abstract—**The approximation theory for non-decaying signals in $L^{p,q}(\mathbb{R}^{d+1})$ is studied recently. In this paper, we prove that $P_{\varphi,h}$ is a bounded projector with the norm estimation from $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ onto $V^{p,q}_{-\alpha,h}(\varphi)$ if φ belongs to an appropriate mixed Wiener amalgam space $W(L^{1,1}_{\alpha})(\mathbb{R}^{d+1})$. This gives the reconstruction of signals which belong to $V^{p,q}_{-\alpha,h}(\varphi)$.**

Index Terms—**Mixed Lebesgue spaces, Bounded projector, Sampling, Approximation, Wiener amalgam spaces.**

I. Introduction

THE sampling and reconstruction theory plays an important role in signal processing since it bridges the modern digital world and the analog world of continuous functions. The sampling states that during conversion of signals, it need to take values at some discrete points. The standard problem in sampling is to recover a signal $f \in V \subset L^2(\mathbb{R})$ from a sequence of sample values $\{f(x_i) : i \in \Lambda\}$, where Λ is a countable indexing set. In other words, sampling converts the continuous signal $f(x)$ into discrete signal $c(k)$.

Reconstruction is the inverse process of sampling. It refers to the process of converting the sampled discrete-time signals into the continuous-time signals. In practical application, we always consider the reconstructed signal which has the translation invariant formula

$$\widetilde{f}(x) = \sum_{k\in\mathbb{Z}^d} c(k)\varphi\left(x/h - k\right).$$

Here φ is the generating kernel that satisfies some certain conditions. This form is very popular in spline theory [1], [2], [3], [4]. Recently, many scholars have done more in-depth research work on sampling and reconstruction [5], [6], [7].

In 1970s, Strang and Fix extended the work of Schoenberg [2] by considering the compactly supported function on $\mathbb{R}^d$ and its multiple integer transformations. In [8], [9], Strang, Fix and Jia gived the concept of controlled approximation. Then they successfully proved that the Strang-Fix condition (SF-condition, for brief) of order k is equivalent to the controlled L^2-approximation property of order k. Strang and Fix gave several equivalent forms of this condition in [9], and their results have been extended in different directions [10], [11], [12], [13]. In [14], [15], Nguyen and Unser extended the classical Strang-Fix theory to two common types: projection and (direct) interpolation in shift-invariant spaces. They proved that if φ has the SF-condition of order k, then the weighted L^p-norm of the error function is bounded when φ belongs to a suitable hybrid-norm space.

The mixed Lebesgue spaces(MLS for brief) are the natural extension of classical Lebesgue spaces. They were first introduced by Benedek and Panzone in [16].When a function which depends on independent quantities with different properties, it may be belong to the MLS. Later in [17], [18] further research was done by Robio de Francia et al. The integrability of each variable can be considered separately when a function belongs the MLS [19]. Under this property assumption, the multi-dimensional non-decaying functions in weighted $L^{p,q}(\mathbb{R}^{d+1})$ can be studied well. Wiener amalgam spaces(WAS for brief) [20] and mixed WAS [21] both were introduced for controlling the local-analytical property of a signal.

In this paper, we mainly give the approximation properties for non-decaying functions in MLS $L^{p,q}(\mathbb{R}^{d+1})$.

We prove that $P_{\varphi,h}$ is a bounded surjective projector from $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ to $V^{p,q}_{-\alpha,h}(\varphi)$ if φ belongs to an appropriate mixed WAS $W(L^{1,1}_{\alpha})(\mathbb{R}^{d+1})$. This gives reconstruction formula for signals which belong to $V^{p,q}_{-\alpha,h}(\varphi)$.

II. Preliminaries

First we introduce the definitions of the important MLS $L^{p,q}(\mathbb{R}^{d+1})$ and its discrete version $\ell^{p,q}(\mathbb{Z}^{d+1})$ ([22]).

Definition 2.1: Let $p, q \in [1, +\infty)$. Then $f \in L^{p,q}(\mathbb{R}^{d+1})$ if

$$\|f\|_{L^{p,q}} = \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d} |f(t_1,t_2)|^q dt_2\right)^{\frac{p}{q}} dt_1\right]^{\frac{1}{p}} < +\infty.$$

The discrete version $\ell^{p,q}(\mathbb{Z}^{d+1})$ is defined as following

$$\ell^{p,q}(\mathbb{Z}^{d+1}) = \left\{c : \|c\|^p_{\ell^{p,q}} = \sum_{n_1\in\mathbb{Z}}\left(\sum_{n_2\in\mathbb{Z}^d} |c(n_1,n_2)|^q\right)^{\frac{p}{q}} < +\infty\right\}.$$

The weighting function is given in the following.

Definition 2.2: Let a weighting function ω on $\mathbb{R}^{d+1}$ be continuous, symmetric and positive. If there is a constant C_ω satisfying that for any $s_1, t_1 \in \mathbb{R}, s_2, t_2 \in \mathbb{R}^d$, we have

$$\omega(s_1 + t_1, s_2 + t_2) \le \mathrm{C}_\omega\ \omega(s_1, s_2)\omega(t_1, t_2). \qquad (1)$$

then it is called $(weakly)$ submultiplicative.
The definitions of weighted MLS $L^{p,q}(\mathbb{R}^{d+1})$ and $\ell^{p,q}(\mathbb{Z}^{d+1})$ are shown in the below.

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Yang Han is with the School of Sciences, Tianjin University of Technology, Tianjin, China e-mail: (mhanyang@sina.com).

Bei Liu is with the School of Sciences, Tianjin University of Technology, Tianjin, China e-mail: (liubei1@mail.nankai.edu.cn).

Corresponding author: e-mail: (liubei1@mail.nankai.edu.cn).

Definition 2.3: For $p, q \in [1, +\infty)$, a function f and a weighting function ω, if $f\omega \in L^{p,q}(\mathbb{R}^{d+1})$, then we call $f \in L^{p,q}_{\omega}(\mathbb{R}^{d+1})$. For a sequence $\{c(n_1, n_2)\}_{n_1\in\mathbb{Z}, n_2\in\mathbb{Z}^d}$, if $\{c(n_1, n_2)\omega(n_1, n_2)\}_{n_1\in\mathbb{Z}, n_2\in\mathbb{Z}^d} \in \ell^{p,q}(\mathbb{Z}^{d+1})$, then we call $\{c(n_1, n_2)\}_{n_1\in\mathbb{Z}, n_2\in\mathbb{Z}^d} \in \ell^{p,q}_{\omega}(\mathbb{Z}^{d+1})$. Their weighted norms are defined as follows

$$\|f\|_{L^{p,q}_{\omega}} := \|f\omega\|_{L^{p,q}};$$

$$\|c\|_{\ell^{p,q}_{\omega}} := \|c\omega\|_{\ell^{p,q}}.$$

Definition 2.4: [21] For $p, q \in [1, +\infty)$, the mixed WAS $W(L^{p,q})(\mathbb{R}^{d+1})$ consists of all functions f which satisfy

$$\|f\|^p_{W(L^{p,q})} := \sum_{k\in\mathbb{Z}} \operatorname{ess\,sup}_{t_1\in[0,1]} \left[\sum_{l\in\mathbb{Z}^d} \operatorname{ess\,sup}_{t_2\in[0,1]^d} |f(t_1 + k, t_2 + l)|^q\right]^{p/q} < \infty. \tag{2}$$

Its weighted norm is defined by

$$\|f\|_{W(L^{p,q}_{\omega})} := \|f\omega\|_{W(L^{p,q})}.$$

We write $\langle\cdot\rangle$ as Sobolev weighting function $\left(1 + \|\cdot\|^2\right)^{1/2}$. When $\omega = \langle\cdot\rangle^{\alpha}$ for some $\alpha \in \mathbb{R}$, we write $L^{p,q}_{\alpha}(\mathbb{R}^{d+1})$ for $L^{p,q}_{\omega}(\mathbb{R}^{d+1})$, $\ell^{p,q}_{\alpha}(\mathbb{Z}^{d+1})$ for $\ell^{p,q}_{\omega}(\mathbb{Z}^{d+1})$, and $W(L^{1,1}_{\alpha})(\mathbb{R}^{d+1})$ for $W(L^{1,1}_{\omega})(\mathbb{R}^{d+1})$. Now we introduce two important properties of this weighting function ω. When $\alpha \geq 0$, according to [14], the weighting function $\omega = \langle\cdot\rangle^{\alpha}$ satisfies

$$\langle s + t\rangle^{\alpha} \leq C_{\alpha}\ \langle s\rangle^{\alpha}\langle t\rangle^{\alpha}, \quad \forall s, t \in \mathbb{R}^{d+1},$$

where C_{α} is a constant. This condition is equivalent to

$$\langle s + t\rangle^{-\alpha} \leq C_{\alpha}\ \langle s\rangle^{\alpha}\langle t\rangle^{-\alpha}, \quad \forall s, t \in \mathbb{R}^{d+1}.$$

The other property of this weighting function ω is that when $\alpha \geq 0$, it has the Gelfand-Raikov-Shilov(GRS) condition [23]

$$\lim_{n\to\infty} \omega(nl)^{\frac{1}{n}} = 1, \quad \forall\, l \in \mathbb{Z}^{d+1}.$$

Definition 2.5: Let $\alpha \geq 0$, and $h > 0$ as a verying scale. The weighted non-decaying shift-invariant subspaces(SIS for brief) $V^{p,q}_{-\alpha,h}(\varphi)$ in mixed WLS are defined by

$$V^{p,q}_{-\alpha,h}(\varphi) := \left\{ f = \sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}^d} c(n_1, n_2)\varphi\left(\frac{\cdot}{h} - n_1, \frac{\cdot}{h} - n_2\right) : c \in \ell^{p,q}_{-\alpha}(\mathbb{Z}^{d+1})\right\}.$$

Note that, when $h = 1$, we write $V^{p,q}_{-\alpha}(\varphi)$ for $V^{p,q}_{-\alpha,1}(\varphi)$; when $\alpha = 0$, we write $V^{p,q}_{h}(\varphi)$ for $V^{p,q}_{0,h}(\varphi)$. So we write $V^{p,q}(\varphi)$ for $V^{p,q}_{0,1}(\varphi)$. It is easy to see that, $V^{p,p}(\varphi) = V^{p}(\varphi)$. According to [24], the SIS is well-defined in $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ and $V^{p,q}_{-\alpha}(\varphi)$ is a closed subspace of $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$.

In the rest of the paper we call σ_h the scaling operator defined by $\sigma_h f := f(\cdot/h)$ with $h > 0$.

III. The main result

We assume that the kernel $\varphi \in W(L^{1,1}_{\alpha})(\mathbb{R}^{d+1})$ and its shifts $\{\varphi(\cdot - n_1, \cdot - n_2)\}_{n_1\in\mathbb{Z}, n_2\in\mathbb{Z}^d}$ is a Riesz basis of $V^2(\varphi)$. According to [26], the dual kernel φ_d exists and is determined by the Fourier domain as following

$$\widehat{\varphi_d}(\omega) = \frac{\widehat{\varphi}(\omega)}{\sum_{k\in\mathbb{Z}^{d+1}} |\widehat{\varphi}(\omega + 2\pi k)|^2}.$$

Let us define the operator

$$P_{\varphi,h} f := \sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}^d} c(n_1, n_2)\varphi\left(\frac{\cdot}{h} - n_1, \frac{\cdot}{h} - n_2\right),$$

where $c(k_1, k_2)$ is given by

$$c(k_1, k_2) = \frac{1}{h^{d+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(y_1, y_2)\varphi_d\left(\frac{y_1}{h} - k_1, \frac{y_2}{h} - k_2\right) dy_2 dy_1.$$

Note that, we write P_{φ} for $P_{\varphi,1}$.

The main result proves that the projector $P_{\varphi,h}$ is bounded and surjective from $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ to $V^{p,q}_{-\alpha,h}(\varphi)$, then we can approximate f by $P_{\varphi,h} f$.

Theorem 3.1: Let $p, q \in [1, +\infty)$ and $\alpha \geq 0$. If $\varphi \in W(L^{1,1}_{\alpha})(\mathbb{R}^{d+1})$ and $\{\varphi(\cdot - n_1, \cdot - n_2)\}_{n_1\in\mathbb{Z}, n_2\in\mathbb{Z}^d}$ is a Riesz basis for $V^2(\varphi)$, then, for each $h > 0$, $V^{p,q}_{-\alpha,h}(\varphi)$ is a closed subspace of $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ and $P_{\varphi,h}$ is a projector from $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ onto $V^{p,q}_{-\alpha,h}(\varphi)$. Furthermore, there is a constant $C_{\varphi,\alpha}$ such that for any $f \in L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ and $h \in (0, 1)$,

$$\|P_{\varphi,h} f\|_{L^{p,q}_{-\alpha}} \leq C_{\varphi,\alpha}\ \|f\|_{L^{p,q}_{-\alpha}}. \tag{3}$$

Proof: See section VI. ■

IV. Example

In this section, we give an example to show the reconstruction of function.

Let $x = (x_1, \cdots, x_{d+1}) \in \mathbb{R}^{d+1}$,

$$\begin{aligned} \varphi(x) &= \varphi(x_1, \cdots, x_{d+1}) \\ &= \chi_{[0,1]}(x_1)\chi_{[0,1]}(x_2) \cdots \chi_{[0,1]}(x_{d+1}) \\ &= \chi_{[0,1]^{d+1}}(x), \end{aligned}$$

then $\varphi \in W(L^{1,1}_{\alpha})(\mathbb{R}^{d+1})$ and its shift $\{\varphi(\cdot - n_1, \cdot - n_2)\}_{n_1\in\mathbb{Z}, n_2\in\mathbb{Z}^d}$ is a Riesz basis of $V^2(\varphi)$. Since

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^{d+1}} |\widehat{\varphi}(\omega + 2\pi k)|^2 \\ &= \sum_{k\in\mathbb{Z}^{d+1}} \left|\int_{\mathbb{R}^{d+1}} \varphi(x)e^{-ix\cdot(\omega+2k\pi)} dx\right|^2 \\ &= \sum_{i=1}^{d+1} \sum_{k_i\in\mathbb{Z}} \left|\prod_{i=1}^{d+1} \left(\int_{\mathbb{R}} \varphi(x_i)e^{-ix_i(\omega_i+2k_i\pi)} dx_i\right)\right|^2 \\ &= \sum_{i=1}^{d+1} \sum_{k_i\in\mathbb{Z}} \prod_{i=1}^{d+1} \left|\int_0^1 e^{-ix_i(\omega_i+2k_i\pi)} dx_i\right|^2 \\ &= \sum_{i=1}^{d+1} \sum_{k_i\in\mathbb{Z}} \prod_{i=1}^{d+1} \left|\frac{e^{-i(\omega_i+2k_i\pi)} - 1}{\omega_i + 2k_i\pi}\right|^2 \\ &= \prod_{i=1}^{d+1} \sum_{k_i\in\mathbb{Z}} \left|\frac{e^{-i(\omega_i+2k_i\pi)} - 1}{\omega_i + 2k_i\pi}\right|^2 \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{d+1} \sum_{k_i\in\mathbb{Z}} \left| \frac{e^{\frac{-i}{2}(\omega_i+2k_i\pi)}\left(e^{\frac{-i}{2}(\omega_i+2k_i\pi)} - e^{\frac{i}{2}(\omega_i+2k_i\pi)}\right)}{\omega_i+2k_i\pi} \right|^2 \\
&= \prod_{i=1}^{d+1} \sum_{k_i\in\mathbb{Z}} \left| \frac{sin(\frac{\omega_i}{2}+k_i\pi)}{\frac{\omega_i}{2}+k_i\pi} \right|^2 \\
&= 1
\end{aligned}
$$

we have the dual kernel

$$\widehat{\varphi_d}(\omega) = \frac{\widehat{\varphi}(\omega)}{\sum_{k\in\mathbb{Z}^{d+1}} |\widehat{\varphi}(\omega+2\pi k)|^2} = \widehat{\varphi}(\omega).$$

Let $f = e^{\frac{-\|x\|^2}{2}}$, then $f \in L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ when $\alpha \geq 0$.

$$
\begin{aligned}
c(k) &= \frac{1}{h^{d+1}} \int_{\mathbb{R}^{d+1}} f(x)\varphi_d\left(\frac{x}{h}-k\right) dx \\
&= \frac{1}{h^{d+1}} \int_{\mathbb{R}^{d+1}} e^{\frac{-\|x\|^2}{2}} \chi_{[0,1]^{d+1}}\left(\frac{x}{h}-k\right) dx.
\end{aligned}
$$

Let $d = 1$. Figure 1 and Figure 2 are the graphs of the functions $f = e^{\frac{-\|x\|^2}{2}}$, $P_{\varphi,0.1}f$, $P_{\varphi,0.05}f$ and $P_{\varphi,0.01}f$.

V. Conclusion

The reconstruction of a signal from the sampling is very important in signal processing. As a result, the sampling is studied in various spaces and the approximation theory of sampling is also concerned. In this paper, we consider the approximation property for non-decaying signals in MLS. We prove that $P_{\varphi,h}$ is a bounded projector from the MLS onto mixed shift-invariant space when the generator belongs to an appropriate mixed WAS. This gives the reconstruction of signals which belong to mixed shift-invariant space.

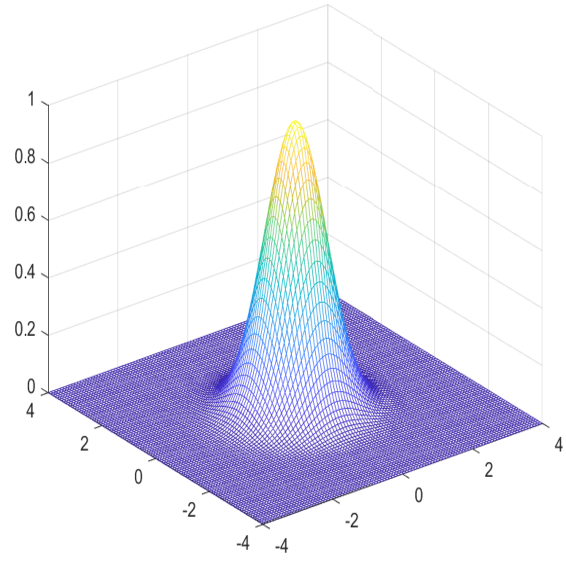
VI. Proofs of Theorem 3.1

Proof: First, we prove that $V^{p,q}_{-\alpha,h}(\varphi)$ is a subspace of $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$, for each $h > 0$. Let $f \in V^{p,q}_{-\alpha,h}(\varphi)$, then

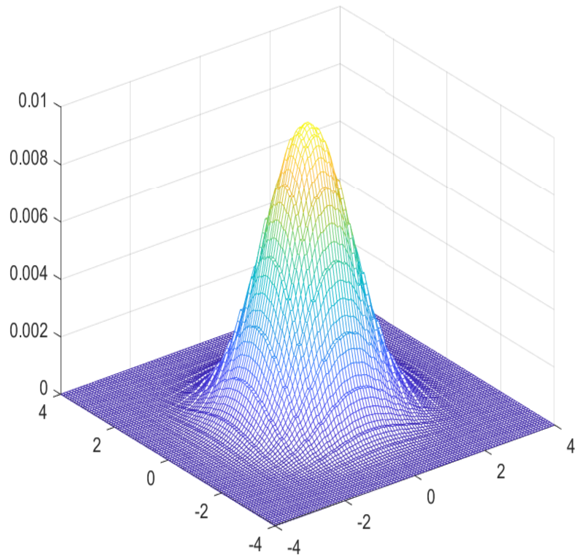
$$\sigma_{1/h}f = \sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}^d} c(n_1,n_2)\varphi(\cdot - n_1, \cdot - n_2) \in V^{p,q}_{-\alpha}(\varphi).$$

According to Theorem 3.7 in [24], $V^{p,q}_{-\alpha}(\varphi)$ is a closed subspace of $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$. Thus, $\sigma_{1/h}f \in L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$. For the convenience of writing, let $\langle x\rangle^\alpha = \langle x_1, x_2\rangle^\alpha$, where $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^d$, then

$$
\begin{aligned}
&\|f\|^p_{L^{p,q}_{-\alpha}} \\
= \;& \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |\langle x_1, x_2\rangle^{-\alpha} f(x_1,x_2)|^q dx_2 \right)^{\frac{p}{q}} dx_1 \\
= \;& \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \left|\langle hx_1, hx_2\rangle^{-\alpha} f(hx_1,hx_2)\right|^q d(hx_2) \right)^{\frac{p}{q}} \\
& \qquad \times d(hx_1) \\
= \;& \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \langle hx_1, hx_2\rangle^{-\alpha q} \left|\left(\sigma_{1/h}f\right)(x_1,x_2)\right|^q d(hx_2) \right)^{\frac{p}{q}} \\
& \qquad \times d(hx_1) \\
= \;& h^{\frac{dp}{q}+1} \times \\
& \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \langle hx_1, hx_2\rangle^{-\alpha q} \left|\left(\sigma_{1/h}f\right)(x_1,x_2)\right|^q dx_2 \right)^{\frac{p}{q}} dx_1 \\
\leq \;& h^{\frac{dp}{q}+1} \cdot \max\left(1, h^{-\alpha q}\right) \\
& \times \int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \langle x_1, x_2\rangle^{-\alpha q} \left|\left(\sigma_{1/h}f\right)(x_1,x_2)\right|^q dx_2 \right)^{\frac{p}{q}} dx_1 \\
= \;& h^{\frac{dp}{q}+1} \cdot \max\left(1, h^{-\alpha q}\right) \cdot \left\|\sigma_{1/h}f\right\|^p_{L^{p,q}_{-\alpha}}. \qquad (4)
\end{aligned}
$$



(a)



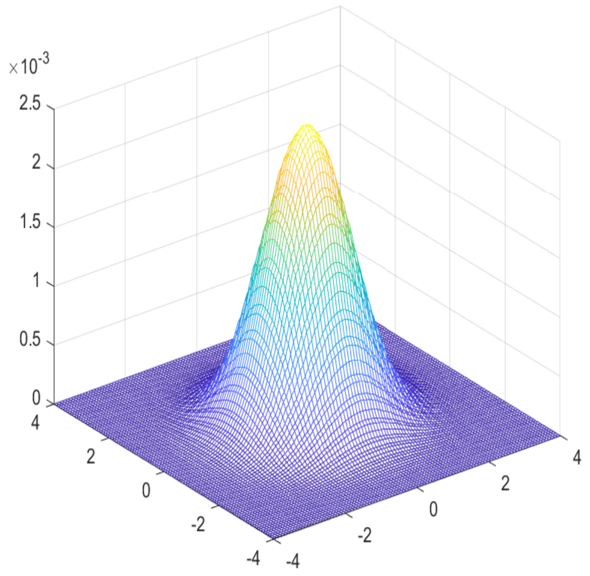
(b)

Fig. 1. (a) is the graph of the function $f = e^{\frac{-\|x\|^2}{2}}$. (b) is the graph of the function $P_{\varphi,0.1}f$.

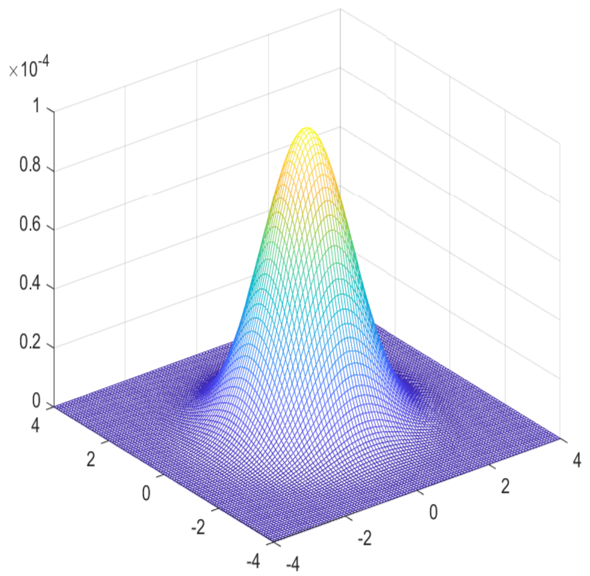
Thus, $f \in L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$. This implies that $V^{p,q}_{-\alpha,h}(\varphi)$ is a subspace of $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$.

Second, we prove that $V^{p,q}_{-\alpha,h}(\varphi)$ is closed under the norm of $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$, when $h > 0$. Assume that $\{f_n\}$ is a sequence in $V^{p,q}_{-\alpha,h}(\varphi)$ which satisfies $f_n \to f$ in $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ as $n \to \infty$. Then

$$
\begin{aligned}
&\left\|\sigma_{1/h}f_n - \sigma_{1/h}f\right\|_{L^{p,q}_{-\alpha}} \\
= \;& \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |\langle x_1, x_2\rangle^{-\alpha} \right.\right. \\
& \times \left.\left. (f_n(hx_1,hx_2) - f(hx_1,hx_2))|^q dx_2 \right)^{\frac{p}{q}} dx_1 \right]^{\frac{1}{p}} \\
= \;& \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |\langle x_1/h, x_2/h\rangle^{-\alpha} \right.\right.
\end{aligned}
$$



(c)



(d)

Fig. 2. (c) is the graph of the function $P_{\varphi,0.05}f$. (d) is the graph of the function $P_{\varphi,0.01}f$.

$$\begin{aligned}
& \times\ \left(f_n(x_1,x_2)-f(x_1,x_2))\right|^q d\left(x_2/h\right)\big)^{\frac{p}{q}}\, d\left(x_1/h\right)\Big]^{\frac{1}{p}} \\
= \ & h^{-d/q-1/p}\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\langle x_1/h,x_2/h\rangle^{-\alpha}\right.\right.\right. \\
& \times\ \left(f_n(x_1,x_2)-f(x_1,x_2))\right|^q dx_2\big)^{\frac{p}{q}}\, dx_1\Big]^{\frac{1}{p}} \\
\leq \ & h^{-d/q-1/p}\cdot\max(1,h^\alpha)\cdot\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\langle x_1,x_2\rangle^{-\alpha}\right.\right.\right. \\
& \times\ \left(f_n(x_1,x_2)-f(x_1,x_2))\right|^q dx_2\big)^{\frac{p}{q}}\, dx_1\Big]^{\frac{1}{p}} \\
= \ & h^{-d/q-1/p}\cdot\max(1,h^\alpha)\cdot\|f_n-f\|_{L^{p,q}_{-\alpha}}, \qquad (5)
\end{aligned}$$

which shows that $\sigma_{1/h}f_n \to \sigma_{1/h}f$ in $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ as $n\to\infty$. Since $\{\sigma_{1/h}f_n\}$ is a sequence in $V^{p,q}_{-\alpha}(\varphi)$, it is known from the Theorem 3.7 in [24] that $\sigma_{1/h}f\in V^{p,q}_{-\alpha}(\varphi)$ (i.e.$f\in V^{p,q}_{-\alpha,h}(\varphi)$). This implies that $V^{p,q}_{-\alpha,h}(\varphi)$ is closed in $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ norm.

Third, we prove that $P_{\varphi,h}$ is a projector mapping $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ to $V^{p,q}_{-\alpha,h}(\varphi)$, for each $h>0$. Inequality (4) implies that $\sigma_{1/h}$ maps $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ to itself. According to the definition of $V^{p,q}_{-\alpha,h}(\varphi)$, σ_h maps $V^{p,q}_{-\alpha}(\varphi)$ to $V^{p,q}_{-\alpha,h}(\varphi)$. We can known from Theorem 3.5 in [24] that P_φ maps $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ to $V^{p,q}_{-\alpha}(\varphi)$. Since $P_{\varphi,h}=\sigma_h P_\varphi\sigma_{1/h}$, we have $P_{\varphi,h}$ maps $L^{p,q}_{-\alpha}(\mathbb{R}^{d+1})$ to $V^{p,q}_{-\alpha,h}(\varphi)$, then we get

$$\begin{aligned}
P^2_{\varphi,h} &= \sigma_h P_\varphi\sigma_{1/h}\sigma_h P_\varphi\sigma_{1/h} \\
&= \sigma_h P^2_\varphi\sigma_{1/h}=\sigma_h P_\varphi\sigma_{1/h}=P_{\varphi,h}.
\end{aligned}$$

Finally, we prove the bound (3). Let

$$\omega_h(x_1,x_2):=\langle hx_1,hx_2\rangle^\alpha,$$

using the submultiplicative of ω_h, one has for each $x_1,y_1\in\mathbb{R}$, $x_2,y_2\in\mathbb{R}^d$ and $h>0$,

$$\omega_h(x_1+y_1,x_2+y_2)\leq C_\alpha\ \omega_h(x_1,x_2)\omega_h(y_1,y_2). \qquad (6)$$

Then

$$\begin{aligned}
& \|P_{\varphi,h}f\|_{L^{p,q}_{-\alpha}} \\
= \ & \|\sigma_h P_\varphi\sigma_{1/h}f\|_{L^{p,q}_{-\alpha}} \\
= \ & \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\langle x_1,x_2\rangle^{-\alpha}\right.\right.\right. \\
& \times\ \left(\sigma_h P_\varphi\sigma_{1/h}f\right)(x_1,x_2)\big|^q dx_2\big)^{\frac{p}{q}}\, dx_1\Big]^{\frac{1}{p}} \\
= \ & \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\langle hx_1,hx_2\rangle^{-\alpha}\right.\right.\right. \\
& \times\ \left(\sigma_h P_\varphi\sigma_{1/h}f\right)(hx_1,hx_2)\big|^q d(hx_2)\big)^{\frac{p}{q}}\, d(hx_1)\Big]^{\frac{1}{p}} \\
= \ & h^{d/q+1/p}\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\frac{1}{\omega_h(x_1,x_2)}\right.\right.\right. \\
& \times\ \left(P_\varphi\sigma_{1/h}f\right)(x_1,x_2)\big|^q dx_2\big)^{\frac{p}{q}}\, dx_1\Big]^{\frac{1}{p}} \\
= \ & h^{d/q+1/p}\left\|P_\varphi(\sigma_{1/h}f)\right\|_{L^{p,q}_{1/\omega_h}} \\
\leq \ & h^{d/q+1/p}\ C^2_\alpha\ \|\varphi\|_{W(L^{1,1}_{\omega_h})} \\
& \times\|\varphi_d\|_{W(L^{1,1}_{\omega_h})}\left\|\sigma_{1/h}f\right\|_{L^{p,q}_{1/\omega_h}} \qquad (7) \\
= \ & h^{d/q+1/p}\ C^2_\alpha\ \|\varphi\|_{W(L^{1,1}_{\omega_h})} \\
& \times\|\varphi_d\|_{W(L^{1,1}_{\omega_h})}\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\langle hx_1,hx_2\rangle^{-\alpha}\right.\right.\right. \\
& \times\ \left(\sigma_{1/h}f\right)(x_1,x_2)\big|^q dx_2\big)^{\frac{p}{q}}\, dx_1\Big]^{\frac{1}{p}} \\
= \ & C^2_\alpha\cdot\|\varphi\|_{W(L^{1,1}_{\omega_h})}\|\varphi_d\|_{W(L^{1,1}_{\omega_h})} \\
& \times\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\langle hx_1,hx_2\rangle^{-\alpha}\right.\right.\right. \\
& \times\ \left(\sigma_{1/h}f\right)(x_1,x_2)\big|^q d(hx_2)\big)^{\frac{p}{q}}\, d(hx_1)\Big]^{\frac{1}{p}} \\
= \ & C^2_\alpha\cdot\|\varphi\|_{W(L^{1,1}_{\omega_h})}\|\varphi_d\|_{W(L^{1,1}_{\omega_h})} \\
& \times\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}\left|\langle x_1,x_2\rangle^{-\alpha}f(x_1,x_2)\right|^q dx_2\right)^{\frac{p}{q}}dx_1\right]^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned} &= C_\alpha^2 \cdot \|\varphi\|_{W(L^{1,1}_{\omega_h})} \\ &\quad \times \|\varphi_d\|_{W(L^{1,1}_{\omega_h})} \|f\|_{L^{p,q}_{-\alpha}} , \end{aligned} \tag{8}$$

where C_α is a constant in (6), φ_d is the dual generator of φ and (7) follows by the proof of Theorem 3.5 in [24].

According to Lemma 3.6 in [24], φ and φ_d are both in $W(L^{1,1}_\alpha)(\mathbb{R}^{d+1})$. Since $h \in (0,1)$,

$$\begin{aligned} &\|\varphi\|_{W(L^{1,1}_{\omega_h})} \\ = &\sum_{n_1\in\mathbb{Z}} \operatorname{ess\,sup}_{x_1\in[0,1]} \sum_{n_2\in\mathbb{Z}^d} \operatorname{ess\,sup}_{x_2\in[0,1]^d} \\ &|\varphi(x_1+n_1, x_2+n_2)\omega_h(x_1+n_1, x_2+n_2)| , \end{aligned}$$

and

$$\begin{aligned} &\|\varphi\|_{W(L^{1,1}_{\alpha})} \\ = &\sum_{n_1\in\mathbb{Z}} \operatorname{ess\,sup}_{x_1\in[0,1]} \sum_{n_2\in\mathbb{Z}^d} \operatorname{ess\,sup}_{x_2\in[0,1]^d} \\ &|\varphi(x_1+n_1, x_2+n_2)\langle x_1+n_1, x_2+n_2\rangle^\alpha| . \end{aligned}$$

Then

$$\|\varphi\|_{W(L^{1,1}_{\omega_h})} \le \|\varphi\|_{W(L^{1,1}_{\alpha})} < \infty, \tag{9}$$

and

$$\|\varphi_d\|_{W(L^{1,1}_{\omega_h})} \le \|\varphi_d\|_{W(L^{1,1}_{\alpha})} < \infty. \tag{10}$$

Combining (8)-(10), one has

$$\begin{aligned} &\|P_{\varphi,h} f\|_{L^{p,q}_{-\alpha}} \\ &\le \underbrace{C_\alpha^2 \cdot \|\varphi\|_{W(L^{1,1}_{\alpha})} \|\varphi_d\|_{W(L^{1,1}_{\alpha})}}_{C_{\varphi,\alpha}} \\ &\quad \times \|f\|_{L^{p,q}_{-\alpha}} . \end{aligned}$$

Therefore, one completes this proof. ■

REFERENCES

[1] De Boor Carl, "A practical guide to splines", Springer-Verlag, New York, NY, 1978.

[2] I.J. Schoenberg, "Contributions to the problem of approximation of equidistant data by analytic functions", Quarterly of Applied Mathematics. vol.4 pp 45-99, 1946.

[3] I.J. Schoenberg, "Cardinal spline interpolation", Society of Industrial and Applied Mathematics, Philadelphia, PA, 1973.

[4] L.L. Schumaker, "Spline functions: Basic theory", Wiley, New York, NY, 1981.

[5] Kamrul Hasan TalukderI and Koichi Harada, "Haar Wavelet Based Approach for Image Compression and Quality Assessment of Compressed Image", IAENG International Journal of Applied Mathematics. vol.36, no.1, pp49-56, 2007.

[6] Kittipat Savetratanakaree, Kingkarn Sookhanaphibarn, Sarun Intakosum, and Ruck Thawonmas. "Borderline Over-sampling in Feature Space for Learning Algorithms in Imbalanced Data Environments", IAENG International Journal of Computer Science. vol.43, no.3, pp363-373, 2016.

[7] Rui-Meng Jing, and Bing-Zhao Li, "Higher Order Derivatives Sampling of Random Signals Related to the Fractional Fourier Transform", IAENG International Journal of Applied Mathematics. vol.48, no.3, pp330-336, 2018.

[8] R.Q. Jia, "A counterexample to a result concerning controlled approximation", Proceedings of the American Mathematical Society. vol.97, no.4, pp647-654, 1986.

[9] G. Strang, G. Fix, "A Fourier analysis of the finite element variational method", In: Geymonat, G.(ed.) Constructive aspects of functional analysis, , pp796-830, 1971.

[10] C. de Boor, R.Q. Jia, "Controlled approximation and a characterization of the local approximation order", Proceedings of the American Mathematical Society. vol.95, no.4, pp 547-553, 1985.

[11] C. de Boor, R.A. DeVore, A. Ron, "Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$", Transactions of the American Mathematical Society. vol.341, no.2, pp 787-806, 1994.

[12] T. Blu, M. Unser, "Approximation error for quasi-interpolators and (multi-) wavelet expansions". Applied and Computational Harmonic Analysis. vol.6, no.2, pp219-251, 1999.

[13] Qingyue Zhang, "The Computation of Fourier Transform via Gauss-Regularized Fourier Series", IAENG International Journal of Applied Mathematics. vol.48, no.1, pp40-44,2018.

[14] H.Q. Nguyen, M. Unser, "Approximation of non-decaying signals from shift-invariant subspaces", Journal of Fourier Analysis and Applications. vol.25, no.3, pp633-660,2019.

[15] H.Q. Nguyen, M. Unser, "A sampling theory for non-decaying signals", Applied and Computational Harmonic Analysis. vol.43, no.1, pp76-93, 2017.

[16] A. Benedek, R. Panzone, "The space L_p with mixed norm", Duke Mathematical Journal. vol.28, no.3, pp301-324, 1961.

[17] J.L. Francia, F.J. Ruiz, J.L. Torrea, "Calderón–Zygmund theory for operator-valued kernels", Advances in Mathematics. vol.62, no.1, pp 7-48, 1986.

[18] R. Torres, E. Ward, Leibniz's Rule, "Sampling and wavelets on mixed Lebesgue spaces", Journal of Fourier Analysis and Applications. vol.21, no.5, pp1053-1076, 2015.

[19] A. Benedek, A.P. Calderón, R. Panzone, "Convolution operators on Banach space valued functions", Proceedings of the National Academy of Sciences of the United States of America, vol.48, no.3, pp356-365,1962 .

[20] C. Heil, "An introduction to weighted Wiener amalgams", in: M. Krishna, R. Radha, S. Thangavelu (Eds.), Wavelets and Their Applications, Allied Publishers, New Delhi, pp183-216, 2003.

[21] R. Li, B. Liu, R. Liu, Q.Y. Zhang, "Nonuniform sampling in principal shift-invariant subspaces of mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{d+1})$", Journal of Mathematical Analysis and Applications. vol.453, no.2, pp 928-941, 2017.

[22] R. Li, B. Liu, R. Liu, Q.Y. Zhang, "The $L^{p,q}$-stability of the shifts of finitely many functions in mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{d+1})$", Acta. Math. Sinica. vol.34, no.6, pp1001-1014, 2018.

[23] I. Gel'fand, D. Raikov, G. Shilov, "Commutative normed rings", Amer. Math. Soc. Transl. vol.5, no.2, pp115-220, 1957.

[24] Y. Han, B. Liu, Q.Y. Zhang, "A sampling theory for non-decaying signals in mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{d+1})$", Applicable Analysis, Published online:2020, https://doi.org/10.1080/00036811.2020.1736286.

[25] J.J.F. Fournier, "Mixed norms and rearrangements: Sobolev's inequality and Littlewood's inequality". Annali di Matematica Pura ed Applicata, vol.148 , no.1, pp51-76, 1987.

[26] M. Unser, "Sampling-50 years after Shannon", Proceedings of the IEEE, vol.88, no.4, pp569-587, 2000.

Y. Han Yang Han was born in Hebei, China. She received the B.S. degree in Mathematics from Langfang Normal University, Langfang, China in 2017. She is currently a graduate student in College of Science, at Tianjin University of Technology.

B. Liu Bei Liu was born in Hebei, China. She received the B.S. degree in Mathematics from Hebei Normal University, ShiJiazhuang, China in 2004. She received the M.S. degree and D.S. degree in Mathematics from Nankai University, Tianjin, China, in 2007 and 2010. She is currently a teacher in College of Science, at Tianjin University of Technology.