Some Types of Filters in Pseudo-quasi-MV Algebras

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Abstract—**In the present paper, we introduce the concepts of implicative filters, positive implicative filters, fantastic filters and associative filters in a pseudo-quasi-MV algebra. The properties of these types of filters are investigated and the relationships between them are also discussed.**

Index Terms—**pseudo-quasi-MV algebras, implicative filters, positive implicative filters, fantastic filters, associative filters.**

I. INTRODUCTION

NOn-commutative algebraic models become a popular topic in the non-classical logic research. In the last decades, many non-commutative algebras were introduced as the generalizations of the known commutative algebras. For example, pseudo-hoops [12], pseudo-BL algebras [9], [10] and pseudo-MV algebras [11], they had been introduced as the non-commutative generalization of hoops, BL-algebras and MV-algebras, respectively. In 2006, the notions of quasi-MV algebras were introduced by Ledda et al. in [15]. Quasi-MV algebras arising from quantum computation were closely connected with fuzzy logic. Subsequently, Liu and Chen extended the notions to the non-commutative cases, called pseudo-quasi-MV algebras (pqMV-algebras for short) [17]. More properties of pqMV-algebras can be seen in [4], [5].

The importance of filters in studying the algebraic structures is well-known [7], [18], [19]. Moreover, in the view of logic, a kind of filters correspond to a set of provable formulae. Thus different types of filters were introduced in an algebra and used to characterize the structures of quotient algebras. For example, Haveshki et al. defined the notions of (positive) implicative filters and fantastic filters of a BL-algebra and discussed their relationship [13]. Then Kondo and Dudek investigated the relationship among these filters in a BL-algebra further [14]. The notions of (positive) implicative filters and Boolean filters in a residuated lattice were introduced by Liu and Li in [16]. Moreover, Busneag and Piciu considered their relationship and constructed quotient residuated lattices using these filters [2]. Alavi et al. investigated not only implicative filters and positive implicative filters, but also fantastic filters and associative filters in a pseudo-hoop [1]. In addition, Ciungu introduced involutive filters of pseudo-hoops in [8].

In [6], [17], we have shown the ideal theory in a pqMV-algebra. Since filters are the dual notion of ideals in a pqMV-algebra, some related properties of filters in a pqMV-algebra can be obtained dually. In order to study more properties of filters and characterize the quotient structure of a pqMV-algebra, we want to introduce some kinds of filters in a pqMV-algebra in the present paper. Their properties and the relationships between these filters are also expected to discuss.

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II. PRELIMINARY

This section recalls some basic properties of pqMV-algebras which are used in the following.

Definition 2.1: [17] A *pseudo-quasi-MV algebra* (pqMV-algebra, for short) is an algebra $Q = \langle Q; +, ', 0\rangle$ of type $\langle 2, 1, 0\rangle$ satisfying the following conditions for every $\upsilon, \varpi, \nu \in Q$:

(pqMV1) $(\upsilon + \varpi) + \nu = \upsilon + (\varpi + \nu)$;
(pqMV2) $\upsilon + \varpi + 0 = \upsilon + \varpi$;
(pqMV3) $\upsilon + 0 = 0 + \upsilon$;
(pqMV4) $\upsilon + 0' = 0' = 0' + \upsilon$;
(pqMV5) $(\upsilon + 0)' = \upsilon' + 0$;
(pqMV6) $\varpi+(\upsilon'+\varpi)' = (\varpi+\upsilon')'+\varpi = \upsilon+(\varpi'+\upsilon)' = (\upsilon + \varpi')' + \upsilon$;
(pqMV7) $\upsilon'' = \upsilon$.

On any pqMV-algebra Q, we denote $0' = 1$ and define some operations as follows: for every $\upsilon, \varpi \in Q$, $\upsilon \otimes \varpi = (\upsilon' + \varpi')'$, $\upsilon \sqcup \varpi = \upsilon + (\varpi' + \upsilon)'$, $\upsilon \sqcap \varpi = (\upsilon' \sqcup \varpi')'$, $\upsilon \to^L \varpi = \upsilon' + \varpi$ and $\upsilon \to^R \varpi = \varpi + \upsilon'$. Moreover, one define a relation $\upsilon \le \varpi$ iff $\upsilon \sqcap \varpi = \upsilon + 0$. It is immediate to get that the relation $\le$ is reflexive and transitive. However, it is not always antisymmetric. Below we present some equivalent characterizations and properties of the relation.

Proposition 2.1: [17] Let Q be a pqMV-algebra. Then the following conditions are equivalent for every $\upsilon, \varpi \in Q$:

(1) $\upsilon \le \varpi$;
(2) $\upsilon \sqcup \varpi = \varpi + 0$;
(3) $\upsilon \to^L \varpi = 1 = \upsilon \to^R \varpi$;
(4) $\upsilon \otimes \varpi' = 0 = \varpi' \otimes \upsilon$.

Proposition 2.2: [3], [5] Let Q be a pqMV-algebra. Then for every $\upsilon, \varpi, \nu, \delta \in Q$, we have:

(P1) $0 \le \upsilon \le 1$;
(P2) $\upsilon \le \upsilon + 0 \le \upsilon$;
(P3) if $\upsilon \le \varpi$ and $\nu \le \delta$, then $\upsilon + \nu \le \varpi + \delta$ and $\upsilon \otimes \nu \le \varpi \otimes \delta$;
(P4) $\upsilon \le \upsilon + \varpi$ and $\upsilon \le \varpi + \upsilon$;
(P5) $\upsilon \otimes \varpi \le \upsilon$ and $\varpi \otimes \upsilon \le \upsilon$;
(P6) $\upsilon \otimes \varpi \le \nu$ iff $\upsilon \le \varpi \to^L \nu$ iff $\varpi \le \upsilon \to^R \nu$;
(P7) $\upsilon \sqcap \varpi \le \upsilon \le \upsilon \sqcup \varpi$;
(P8) $\upsilon \le \varpi \to^L \upsilon$ and $\upsilon \le \varpi \to^R \upsilon$;
(P9) if $\upsilon \le \varpi$, then $\varpi \to^L \nu \le \upsilon \to^L \nu$ and $\varpi \to^R \nu \le \upsilon \to^R \nu$, if $\upsilon \le \varpi$, then $\nu \to^L \upsilon \le \nu \to^L \varpi$ and $\nu \to^R \upsilon \le \nu \to^R \varpi$;
(P10) $\upsilon \le (\upsilon \to^L \varpi) \to^R \varpi$ and $\upsilon \le (\upsilon \to^R \varpi) \to^L \varpi$;

(P11) $\upsilon \to^L \varpi \leq (\varpi \to^L \nu) \to^R (\upsilon \to^L \nu)$ and $\upsilon \to^R \varpi \leq (\varpi \to^R \nu) \to^L (\upsilon \to^R \nu)$;

(P12) $(\varpi \to^L \nu) \otimes (\upsilon \to^L \varpi) \leq \upsilon \to^L \nu$ and $(\upsilon \to^R \varpi) \otimes (\varpi \to^R \nu) \leq \upsilon \to^R \nu$;

(P13) $\nu \to^L \upsilon \leq (\varpi \to^L \nu) \to^L (\varpi \to^L \upsilon)$ and $\nu \to^R \upsilon \leq (\varpi \to^R \nu) \to^R (\varpi \to^R \upsilon)$;

(P14) $(\upsilon \sqcup \varpi) \to^L \nu = (\upsilon \to^L \nu) \sqcap (\varpi \to^L \nu)$ and $(\upsilon \sqcup \varpi) \to^R \nu = (\upsilon \to^R \nu) \sqcap (\varpi \to^R \nu)$;

(P15) $\upsilon \to^R (\varpi \to^L \nu) = \varpi \to^L (\upsilon \to^R \nu)$ and $\upsilon \to^L (\varpi \to^R \nu) = \varpi \to^R (\upsilon \to^L \nu)$;

(P16) $((\upsilon \to^L \varpi) \to^R \varpi) \to^L \varpi = \upsilon \to^L \varpi$ and $((\upsilon \to^R \varpi) \to^L \varpi) \to^R \varpi = \upsilon \to^R \varpi$;

(P17) $1 \to^L \upsilon = \upsilon + 0 = 1 \to^R \upsilon$;

(P18) $\upsilon \to^L (1 \to^R \varpi) = \upsilon \to^L \varpi$ and $\upsilon \to^R (1 \to^L \varpi) = \upsilon \to^R \varpi$;

(P19) $(\upsilon \otimes \varpi) \to^L \nu = \upsilon \to^L (\varpi \to^L \nu)$ and $(\upsilon \otimes \varpi) \to^R \nu = \varpi \to^R (\upsilon \to^R \nu)$;

(P20) $(\upsilon \to^L \varpi) \sqcup (\varpi \to^L \upsilon) = 1$ and $(\upsilon \to^R \varpi) \sqcup (\varpi \to^R \upsilon) = 1$.

Suppose that Q is a pqMV-algebra and the set $M \subseteq Q$. Then M is said to be a *filter* in Q if M satisfies: (F1) $1 \in M$; (F2) if $\upsilon, \varpi \in M$, then $\upsilon \otimes \varpi \in M$; (F3) if $\upsilon \in M$ and $\varpi \in Q$ with $\upsilon \leq \varpi$, then $\varpi \in M$. Following from the definition, if M is a filter in Q, then we have $\upsilon + \varpi \in M$ and $\varpi + \upsilon \in M$ by (P4), for every $\upsilon \in M$ and $\varpi \in Q$.

Proposition 2.3: Let Q be a pqMV-algebra and the set $M \subseteq Q$. Then the following statements are equivalent:
(1) M is a filter in Q;
(2) $1 \in M$ and if $\upsilon, \upsilon \to^L \varpi \in M$, then $\varpi \in M$;
(3) $1 \in M$ and if $\upsilon, \upsilon \to^R \varpi \in M$, then $\varpi \in M$.

Proof: (1) $\Leftrightarrow$ (2) Let $\upsilon, \upsilon \to^L \varpi \in M$. Then $\upsilon \sqcap \varpi = (\upsilon \to^L \varpi) \otimes \upsilon \in M$. Because M is a filter in Q and $\upsilon \sqcap \varpi \leq \varpi$, we have $\varpi \in M$ using (F3). Conversely, suppose that $\upsilon, \varpi \in M$. Since $\upsilon \to^L (\varpi \to^L (\upsilon \otimes \varpi)) = (\upsilon \otimes \varpi) \to^L (\upsilon \otimes \varpi) = 1 \in M$ and $\upsilon \in M$, we have $\varpi \to^L (\upsilon \otimes \varpi) \in M$. Note that $\varpi \in M$, we have $\upsilon \otimes \varpi \in M$. If $\upsilon \in M$ and $\varpi \in Q$ with $\upsilon \leq \varpi$, then we have $\upsilon \to^L \varpi = 1 \in M$, it turns out $\varpi \in M$. Thus M is a filter in Q. The proof of (1) $\Leftrightarrow$ (3) is analogous to (1) $\Leftrightarrow$ (2).

Given that Q is a pqMV-algebra. If M is a filter in Q and $\upsilon \to^L \varpi \in M$ iff $\upsilon \to^R \varpi \in M$ for every $\upsilon, \varpi \in Q$, then M is called a *normal filter* in Q; if ϑ is a congruence on Q and $\langle \upsilon \otimes 1, \varpi \otimes 1 \rangle \in \vartheta$ implies $\langle \upsilon, \varpi \rangle \in \vartheta$ for every $\upsilon, \varpi \in Q$, then ϑ is called a *filter congruence* on Q. Following from [6], we can obtain that the mapping from the set of normal filters to the set of filter congruences on a pqMV-algebra is bijective. Assume that M is a normal filter in Q, we denote $[\upsilon] = \{\varpi \in Q | \upsilon \to^L \varpi \in M \text{ and } \varpi \to^L \upsilon \in M\}$ for $\upsilon \in Q$ and $Q/M = \{[\upsilon] | \upsilon \in Q\}$. Define two operations on Q/M as follows: $[\upsilon] + [\varpi] = [\upsilon + \varpi]$ and $[\upsilon]' = [\upsilon']$ for every $[\upsilon], [\varpi] \in Q/M$. Then $(Q/M; +, ', [0])$ is a pseudo-MV algebra [6]. Moreover, if for every $[\upsilon], [\varpi] \in Q/M$, we define $[\upsilon] \sqcup [\varpi] = [\upsilon \sqcup \varpi]$, $[\upsilon] \sqcap [\varpi] = [\upsilon \sqcap \varpi]$ and $[1] = [0]'$, then $(Q/M; \sqcup, \sqcap, [0], [1])$ is a bounded distributive lattice [11].

III. Implicative filters and positive implicative filters

In this section, the concepts of implicative filters and positive implicative filters are introduced in a pqMV-algebra. We discuss their relationship and show some equivalent conditions of an implicative filter in the pqMV-algebra.

Definition 3.1: Let Q be a pqMV-algebra and the set $M \subseteq Q$. Then M is said to be an *implicative filter* in Q if for every $\upsilon, \varpi, \nu \in Q$, M satisfies the following conditions:
(IF1) $1 \in M$;
(IF2) If $\upsilon \in M$ and $\upsilon \to^L ((\varpi \to^L \nu) \to^R \varpi) \in M$, then $\varpi \in M$;
(IF3) If $\upsilon \in M$ and $\upsilon \to^R ((\varpi \to^R \nu) \to^L \varpi) \in M$, then $\varpi \in M$.

Proposition 3.1: Let M be an implicative filter in a pqMV-algebra Q. Then M is a filter in Q.

Proof: Assume that $\upsilon, \upsilon \to^L \varpi \in M$. Because $\varpi \leq 1$, we have $\varpi \to^L 1 = 1$ using Proposition 2.1, it turns out $\upsilon \to^L ((\varpi \to^L 1) \to^R \varpi) = \upsilon \to^L (1 \to^R \varpi) = \upsilon \to^L \varpi \in M$ by (P18). Since $\upsilon \in M$ and M is an implicative filter in Q, we have $\varpi \in M$ by (IF2). Thus M is a filter in Q according to Proposition 2.3.

Definition 3.2: Let Q be a pqMV-algebra and M be a filter in Q. Then M is said to be a *Boolean filter* in Q if for every $\upsilon, \varpi \in Q$, M satisfies the following conditions:
(BF1) if $(\upsilon \to^L \varpi) \to^R \upsilon \in M$, then $\upsilon \in M$;
(BF2) if $(\upsilon \to^R \varpi) \to^L \upsilon \in M$, then $\upsilon \in M$.

Below we give the equivalent characterizations of an implicative filter in a pqMV-algebra.

Theorem 3.1: Let Q be a pqMV-algebra and the set $M \subseteq Q$. Then for every $\upsilon, \varpi, \nu \in Q$, the following statements are equivalent:
(1) M is an implicative filter in Q;
(2) M is a Boolean filter in Q;
(3) M is a filter in Q, $((\upsilon \to^L \varpi) \to^R \upsilon) \to^R \upsilon \in M$ and $((\upsilon \to^R \varpi) \to^L \upsilon) \to^L \upsilon \in M$;
(4) M is a filter in Q, $(\upsilon' \to^R \upsilon) \to^R \upsilon \in M$ and $(\upsilon' \to^L \upsilon) \to^L \upsilon \in M$;
(5) M is a filter in Q, if $(\nu' \otimes \upsilon) \to^R \varpi \in M$ and $\varpi \to^R \nu \in M$, then $\upsilon \to^R \nu \in M$, if $(\upsilon \otimes \nu') \to^L \varpi \in M$ and $\varpi \to^L \nu \in M$, then $\nu \to^L \upsilon \in M$;
(6) if M is a normal filter in Q and $(\varpi' \otimes \upsilon) \to^R \varpi \in M$, then $\upsilon \to^R \varpi \in M$, if M is a normal filter in Q and $(\upsilon \otimes \varpi') \to^L \varpi \in M$, then $\varpi \to^L \upsilon \in M$.

Proof: (1)$\Rightarrow$(2) Suppose that M is an implicative filter in Q. We have that M is a filter in Q by Proposition 3.1. If for every $\upsilon, \varpi \in Q$, $(\upsilon \to^L \varpi) \to^R \upsilon \in M$, because $1 \in M$ and $1 \to^L ((\upsilon \to^L \varpi) \to^R \upsilon) = (\upsilon \to^L \varpi) \to^R \upsilon \in M$, we have $\upsilon \in M$. The condition (BF2) can be proved analogously. Thus M is a Boolean filter in Q.

(2)$\Rightarrow$(1) Assume that M is a Boolean filter in Q. If $\upsilon \in M$ and $\upsilon \to^L ((\varpi \to^L \nu) \to^R \varpi) \in M$, then $(\varpi \to^L \nu) \to^R \varpi \in M$ by Proposition 2.3, it turns out $\varpi \in M$ by (BF1). The condition (IF3) can be proved analogously. Hence M is an implicative filter in Q.

(3)$\Rightarrow$(4) Put $\varpi = 0$ in (3). We have $(\upsilon' \to^R \upsilon) \to^R \upsilon = ((\upsilon' + 0) \to^R \upsilon) \to^R \upsilon = ((\upsilon \to^L 0) \to^R \upsilon) \to^R \upsilon \in M$ and $(\upsilon' \to^L \upsilon) \to^L \upsilon = ((\upsilon' + 0) \to^L \upsilon) \to^L \upsilon = ((\upsilon \to^R 0) \to^L \upsilon) \to^L \upsilon \in M$.

(4)$\Rightarrow$(3) Because $0 \leq \varpi$, we get $\upsilon' \leq \upsilon' + 0 = \upsilon \to^L 0 \leq \upsilon \to^L \varpi$ by (P2) and (P9), it turns out that $(\upsilon \to^L \varpi) \to^R \upsilon \leq \upsilon' \to^R \upsilon$ and $(\upsilon' \to^R \upsilon) \to^R \upsilon \leq ((\upsilon \to^L \varpi) \to^R \upsilon) \to^R \upsilon$ using (P9) again. Because $(\upsilon' \to^R \upsilon) \to^R \upsilon \in M$

and M is a filter in Q, it follows that $((\upsilon \to^L \varpi) \to^R \upsilon) \to^R \upsilon \in M$. The other one can be proved analogously.

(5)⇒(6) Suppose that the filter M in Q is normal and $(\varpi' \otimes \upsilon) \to^R \varpi \in M$. Since $\varpi \to^R \varpi = 1 \in M$, we have $\upsilon \to^R \varpi \in M$ by the assumption. The other can be proved similarly.

(6)⇒(5) Assume that $(\nu' \otimes \upsilon) \to^R \varpi \in M$ and $\varpi \to^R \nu \in M$. Because M is a filter in Q, we have $((\nu' \otimes \upsilon) \to^R \varpi) \otimes (\varpi \to^R \nu) \in M$. By (P12), we have $((\nu' \otimes \upsilon) \to^R \varpi) \otimes (\varpi \to^R \nu) \leq (\nu' \otimes \upsilon) \to^R \nu$, it turns out $(\nu' \otimes \upsilon) \to^R \nu \in M$, so $\upsilon \to^R \nu \in M$. The other can be proved similarly.

(6)⇒(4) By (P19), we have $(\upsilon' \otimes (\upsilon' \to^R \upsilon)) \to^R \upsilon = (\upsilon' \to^R \upsilon) \to^R (\upsilon' \to^R \upsilon) = 1 \in M$, so $(\upsilon' \to^R \upsilon) \to^R \upsilon \in M$. The other can be proved similarly.

(3)⇒(1) Assume that $\upsilon \to^L ((\varpi \to^L \nu) \to^R \varpi) \in M$ and $\upsilon \in M$. Since M is a filter in Q, we have $(\varpi \to^L \nu) \to^R \varpi \in M$. By the assumption, because $((\varpi \to^L \nu) \to^R \varpi) \to^R \varpi \in M$, we get $\varpi \in M$ by Proposition 2.3. The other can be proved similarly. Hence M is an implicative filter in Q.

(1)⇒(6) Let M be an implicative filter in Q. Then $\varpi \leq \upsilon \to^L \varpi$ and $(\upsilon \to^L \varpi) \to^L 0 \leq \varpi \to^L 0$ by (P8) and (P9), it turns out $\varpi' \to^R \varpi \leq (\upsilon \to^L \varpi)' \to^R \varpi$ and then $(\varpi' \otimes \upsilon) \to^R \varpi = \upsilon \to^R (\varpi' \to^R \varpi) \leq \upsilon \to^R ((\upsilon \to^L \varpi)' \to^R \varpi)$ using (P9) and (P19). Because $(\varpi' \otimes \upsilon) \to^R \varpi \in M$ and M is a filter in Q, we get $\upsilon \to^R ((\upsilon \to^L \varpi)' \to^R \varpi) \in M$. Moreover, note that M is normal, we obtain $\upsilon \to^L ((\upsilon \to^L \varpi)' \to^R \varpi) \in M$, thus $(\upsilon \to^L \varpi)' \to^R (\upsilon \to^L \varpi) \in M$ by (P15) and then $1 \to^L (((\upsilon \to^L \varpi) \to^L 0) \to^R (\upsilon \to^L \varpi)) = (\upsilon \to^L \varpi)' \to^R (\upsilon \to^L \varpi) \in M$. By (IF2), we have $\upsilon \to^L \varpi \in M$, and so $\upsilon \to^R \varpi \in M$. The other can be proved similarly.

In order to characterize the quotient algebra, we need more results. The proofs of Lemma 3.1 and Lemma 3.2 are similar to Proposition 2.10 in [20]. Here we omit them. Recall that if $\langle B; +, ^-, ^\sim, 0, 1\rangle$ is a pseudo-MV algebra, we define $\upsilon \sqcup \varpi = \upsilon + (\varpi^- + \upsilon)^\sim$, $\upsilon \sqcap \varpi = (\upsilon^- \sqcup \varpi^-)^\sim$, $\upsilon \to^L \varpi = \upsilon^- + \varpi$ and $\upsilon \to^R \varpi = \varpi + \upsilon^\sim$, then $\upsilon^- = \upsilon \to^L 0$, $\upsilon^\sim = \upsilon \to^R 0$ and $\upsilon \sqcup \varpi = (\upsilon \to^L \varpi) \to^R \varpi = (\upsilon \to^R \varpi) \to^L \varpi = (\varpi \to^L \upsilon) \to^R \upsilon = (\varpi \to^R \upsilon) \to^L \upsilon$.

Lemma 3.1: Let $\langle B; +, ^-, ^\sim, 0, 1\rangle$ be a pseudo-MV algebra. Then for every $\upsilon, \varpi \in B$, the following statements are equivalent:

(1) $(\upsilon \to^L \varpi) \to^R \upsilon = \upsilon$;

(2) $\upsilon \sqcup \upsilon^- = 1$;

(3) $\upsilon^- \to^L \upsilon = \upsilon$;

(4) $\langle B; \sqcup, \sqcap, 0, 1\rangle$ is a Boolean algebra.

Lemma 3.2: Let $\langle B; +, ^-, ^\sim, 0, 1\rangle$ be a pseudo-MV algebra. Then for every $\upsilon, \varpi \in B$, the following statements are equivalent:

(1) $(\upsilon \to^R \varpi) \to^L \upsilon = \upsilon$;

(2) $\upsilon \sqcup \upsilon^\sim = 1$;

(3) $\upsilon^\sim \to^R \upsilon = \upsilon$;

(4) $\langle B; \sqcup, \sqcap, 0, 1\rangle$ is a Boolean algebra.

Corollary 3.1: Let Q be a pqMV-algebra. If a normal filter M in Q satisfies any condition in Theorem 3.1, then $\langle Q/M; \sqcup, \sqcap, [0], [1]\rangle$ is a Boolean algebra.

Proof: According to Lemma 3.1, we only show $([\upsilon] \to^L [\varpi]) \to^R [\upsilon] = [\upsilon]$ for every $[\upsilon], [\varpi] \in Q/M$. Since M is normal and $((\upsilon \to^L \varpi) \to^R \upsilon) \to^R \upsilon \in M$, we have $((\upsilon \to^L \varpi) \to^R \upsilon) \to^L \upsilon \in M$. Meanwhile, because $\upsilon \leq (\upsilon \to^L \varpi) \to^R \upsilon$, we have $\upsilon \to^L ((\upsilon \to^L \varpi) \to^R \upsilon) = 1 \in M$. Thus $[(\upsilon \to^L \varpi) \to^R \upsilon] = [\upsilon]$ and then $([\upsilon] \to^L [\varpi]) \to^R [\upsilon] = [\upsilon]$.

Proposition 3.2: Let Q be a pqMV-algebra and M, M' be normal filters in Q with $M \subseteq M'$. If M is an implicative filter in Q, then M' is also an implicative filter in Q.

Proof: For every $\upsilon, \varpi \in Q$, denote $\nu = (\varpi' \otimes \upsilon) \to^R \varpi \in M'$. Then we have $1 = \nu \to^L \nu = \nu \to^L ((\varpi' \otimes \upsilon) \to^R \varpi) = (\varpi' \otimes \upsilon) \to^R (\nu \to^L \varpi) \in M$ using (P15). Since $(\nu \to^L \varpi) \to^L 0 \leq \varpi \to^L 0$ by (P8) and (P9), we have $(\nu \to^L \varpi)' \otimes \upsilon \leq \varpi' \otimes \upsilon$, then $(\varpi' \otimes \upsilon) \to^R (\nu \to^L \varpi) \leq ((\nu \to^L \varpi)' \otimes \upsilon) \to^R (\nu \to^L \varpi)$, so $((\nu \to^L \varpi)' \otimes \upsilon) \to^R (\nu \to^L \varpi) = 1 \in M$. Because $(\nu \to^L \varpi) \to^L (\nu \to^L \varpi) = 1 \in M$ and M is an implicative filter in Q, we get $\upsilon \to^R (\nu \to^L \varpi) \in M$ following from Theorem 3.1 and then $\nu \to^L (\upsilon \to^R \varpi) \in M$. Because $M \subseteq M'$, we have $\nu \to^L (\upsilon \to^R \varpi) \in M'$. Note that $\nu \in M'$, we get $\upsilon \to^R \varpi \in M'$ using Proposition 2.3 again. The other can be proved similarly. Therefore M' is an implicative filter in Q by Theorem 3.1.

Below we define the positive implicative filter in a pqMV-algebra.

Definition 3.3: Let Q be a pqMV-algebra and the set $M \subseteq Q$. Then M is said to be a *positive implicative filter* in Q if for every $\upsilon, \varpi, \nu \in Q$, M satisfies the following conditions:

(PIF1) $1 \in M$;

(PIF2) if $\upsilon \to^R \varpi \in M$ and $\upsilon \to^L (\varpi \to^L \nu) \in M$, then $\upsilon \to^L \nu \in M$;

(PIF3) if $\upsilon \to^L \varpi \in M$ and $\upsilon \to^R (\varpi \to^R \nu) \in M$, then $\upsilon \to^R \nu \in M$.

Proposition 3.3: Let Q be a pqMV-algebra and M be a positive implicative filter in Q. Then for every $\upsilon, \varpi \in Q$:

(1) if $\upsilon \to^L (\upsilon \to^L \varpi) \in M$, then $\upsilon \to^L \varpi \in M$. Especially, $\upsilon \to^L \upsilon^2 \in M$, where $\upsilon^2 = \upsilon \otimes \upsilon$;

(2) if $\upsilon \to^R (\upsilon \to^R \varpi) \in M$, then $\upsilon \to^R \varpi \in M$. Especially, $\upsilon \to^R \upsilon^2 \in M$, where $\upsilon^2 = \upsilon \otimes \upsilon$.

Proof: (1) Suppose that $\upsilon \to^L (\upsilon \to^L \varpi) \in M$. Because $\upsilon \to^R \upsilon = 1 \in M$ and M is a positive implicative filter in Q, we have $\upsilon \to^L \varpi \in M$ by (PIF2). Especially, putting $\varpi = \upsilon^2$, then $\upsilon \to^L (\upsilon \to^L \upsilon^2) = \upsilon^2 \to^L \upsilon^2 = 1 \in M$, so $\upsilon \to^L \upsilon^2 \in M$.

(2) It is similar to (1).

Proposition 3.4: Let Q be a pqMV-algebra. If a normal filter M in Q satisfies the following conditions for every $\upsilon, \varpi \in Q$:

(1) if $\upsilon \to^L (\upsilon \to^L \varpi) \in M$, then $\upsilon \to^L \varpi \in M$;

(2) if $\upsilon \to^R (\upsilon \to^R \varpi) \in M$, then $\upsilon \to^R \varpi \in M$,

then M is a positive implicative filter in Q.

Proof: Suppose that $\upsilon \to^R \varpi \in M$ and $\upsilon \to^L (\varpi \to^L \nu) \in M$ for every $\upsilon, \varpi, \nu \in Q$. Since M is normal, we have $\upsilon \to^R (\varpi \to^L \nu) \in M$ and $\varpi \to^L (\upsilon \to^R \nu) \in M$ by (P15), so $\varpi \to^R (\upsilon \to^R \nu) \in M$. Note that $\upsilon \to^R \varpi \in M$, we have $(\varpi \to^R (\upsilon \to^R \nu)) \otimes (\upsilon \to^R \varpi) \in M$. Since $(\varpi \to^R (\upsilon \to^R \nu)) \otimes (\upsilon \to^R \varpi) \leq \upsilon \to^R (\upsilon \to^R \nu)$ by (P15), we get $\upsilon \to^R (\upsilon \to^R \nu) \in M$ and then $\upsilon \to^R \nu \in M$. Using M is normal again, we have $\upsilon \to^L \nu \in M$. The condition (PIF3) can be checked similarly. Therefore M is a positive implicative filter in Q.

Proposition 3.5: Let Q be a pqMV-algebra and M, M' be normal filters in Q with $M \subseteq M'$. If M is a positive implicative filter in Q, then M' is also a positive implicative filter in Q.

Proof: For every $\upsilon, \varpi, \nu \in Q$, denote $\upsilon = \upsilon \to^L (\upsilon \to^L \varpi) \in M'$. We calculate $\upsilon \to^L (\upsilon \to^L (\upsilon \to^R \varpi)) = (\upsilon \otimes \upsilon) \to^L (\upsilon \to^R \varpi) = \upsilon \to^R ((\upsilon \otimes \upsilon) \to^L \varpi) = \upsilon \to^R (\upsilon \to^L (\upsilon \to^L \varpi)) = \upsilon \to^R \upsilon = 1 \in M$. Because $\upsilon \to^R \upsilon = 1 \in M$ and M is a positive implicative filter in Q, we have $\upsilon \to^L (u \to^R \varpi) \in M$ by (PIF2), which implies $\upsilon \to^R (\upsilon \to^L \varpi) \in M$. Note that $M \subseteq M'$ and $\upsilon \in M'$, we have $\upsilon \to^L \varpi \in M'$ by Proposition 2.3. The other one can be proved similarly. Therefore M' is a positive implicative filter in Q from Proposition 3.4.

Below we will consider the relationship between positive implicative filters and implicative filters in any pqMV-algebra.

Proposition 3.6: Let Q be a pqMV-algebra and M be an implicative filter in Q. If $\upsilon' \to^R \upsilon \in M$ or $\upsilon' \to^L \upsilon \in M$ for every $\upsilon \in Q$, then M is a positive implicative filter in Q.

Proof: Assume that $\upsilon \to^R \varpi \in M$ and $\upsilon \to^L (\varpi \to^L \nu) \in M$ for every $\upsilon, \varpi, \nu \in Q$. Since M is an implicative filter in Q, we have $(\upsilon' \to^R \upsilon) \to^R \upsilon \in M$ by Theorem 3.1. If $\upsilon' \to^R \upsilon \in M$, then $\upsilon \in M$ by Proposition 2.3, it turns out $\varpi \in M$ and $\varpi \to^L \nu \in M$, so $\nu \in M$. Since M is a filter in Q and $\nu \leq \upsilon \to^L \nu$, we have $\upsilon \to^L \nu \in M$. The condition (PIF3) can be checked similarly. Therefore M is a positive implicative filter in Q.

Proposition 3.7: Let Q be a pqMV-algebra. If M is a normal and positive implicative filter such that $(\upsilon \to^L \varpi) \to^R \varpi \in M$ implies $(\varpi \to^L \upsilon) \to^R \upsilon \in M$ for every $\upsilon, \varpi \in Q$, then M is an implicative filter in Q.

Proof: Assume that $(\upsilon \to^L \varpi) \to^R \upsilon \in M$ for every $\upsilon, \varpi \in Q$. Since $(\upsilon \to^L \varpi) \to^R \upsilon \leq (\upsilon \to^L \varpi) \to^R ((\upsilon \to^L \varpi) \to^R \varpi)$ by (P10) and (P9), we have $(\upsilon \to^L \varpi) \to^R ((\upsilon \to^L \varpi) \to^R \varpi) \in M$. Note that $(\upsilon \to^L \varpi) \to^L (\upsilon \to^L \varpi) = 1 \in M$, so $(\upsilon \to^L \varpi) \to^R \varpi \in M$, which implies $(\varpi \to^L \upsilon) \to^R \upsilon \in M$. Moreover, because $\varpi \leq \upsilon \to^L \varpi$, we have $(\upsilon \to^L \varpi) \to^R \upsilon \leq \varpi \to^R \upsilon$, it follows that $\varpi \to^R \upsilon \in M$. Note that M is normal, we have $\varpi \to^L \upsilon \in M$, so $\upsilon \in M$. Analogously, we can prove that $(\upsilon \to^R \varpi) \to^L \upsilon \in M$ implies $\upsilon \in M$. Therefore M is an implicative filter in Q by Theorem 3.1.

Proposition 3.8: Let Q be a pqMV-algebra with $\upsilon \leq \varpi$ or $\varpi \leq \upsilon$ for every $\upsilon, \varpi \in Q$ and M be a filter in Q. If M is a positive implicative filter in Q, then M is an implicative filter in Q iff $(\upsilon \to^L \varpi) \to^R \varpi \in M$ implies $(\varpi \to^L \upsilon) \to^R \upsilon \in M$ for every $\upsilon, \varpi \in Q$.

Proof: For every $\upsilon, \varpi \in Q$, we have $\upsilon \leq \varpi$ or $\varpi \leq \upsilon$. Now suppose that $(\upsilon \to^L \varpi) \to^R \upsilon \in M$. If $\upsilon \leq \varpi$, then $\upsilon \to^L \varpi = 1$ and $1 \to^R \upsilon = (\upsilon \to^L \varpi) \to^R \upsilon \in M$. Because $1 \to^R \upsilon \leq \upsilon$ and M is a filter in Q, we have $\upsilon \in M$. If $\varpi \leq \upsilon$, since $\upsilon \leq (\upsilon \to^L \varpi) \to^R \varpi$, we have $(\upsilon \to^L \varpi) \to^R \upsilon \leq (\upsilon \to^L \varpi) \to^R ((\upsilon \to^L \varpi) \to^R \varpi)$ by (P9), it turns out $(\upsilon \to^L \varpi) \to^R ((\upsilon \to^L \varpi) \to^R \varpi) \in M$. Since M is a positive implicative filter in Q and $(\upsilon \to^L \varpi) \to^L (\upsilon \to^L \varpi) = 1 \in M$, we have $(\upsilon \to^L \varpi) \to^R \varpi \in M$ by (PIF3). By the assumption, we have $(\varpi \to^L \upsilon) \to^R \upsilon \in M$. Note that $\varpi \to^L \upsilon = 1$ and $(\varpi \to^L \upsilon) \to^R \upsilon = 1 \to^R \upsilon \leq \upsilon$, we have $\upsilon \in M$. The other can be proved similarly. Hence M is an implicative filter in Q by Theorem 3.1. Conversely, if M is an implicative filter in Q and $(\upsilon \to^L \varpi) \to^R \varpi \in M$ for every $\upsilon, \varpi \in Q$, then we have $(\upsilon \to^L \varpi) \to^R \varpi \leq (\upsilon \to^L \varpi) \to^R ((\varpi \to^L \upsilon) \to^R \upsilon)$. Since M is a filter in Q, we get $(\upsilon \to^L \varpi) \to^R ((\varpi \to^L \upsilon) \to^R \upsilon) \in M$. Because $\upsilon \leq (\varpi \to^L \upsilon) \to^R \upsilon$ by (P8), we have $((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi \leq \upsilon \to^L \varpi$ using (P9), it follows that $(\upsilon \to^L \varpi) \to^R ((\varpi \to^L \upsilon) \to^R \upsilon) \leq (((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi) \to^R ((\varpi \to^L \upsilon) \to^R \upsilon)$, so $(((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi) \to^R ((\varpi \to^L \upsilon) \to^R \upsilon) \in M$ and then $(\varpi \to^L \upsilon) \to^R \upsilon \in M$ by Theorem 3.1.

IV. Fantastic filters and associative filters

In this section, the concepts of fantastic filters and associative filters in a pqMV-algebra are introduced. We also discuss their relationship.

Definition 4.1: Let Q be a pqMV-algebra and the set $M \subseteq Q$. Then M is said to be a *fantastic filter* in Q if for every $\upsilon, \varpi, \nu \in Q$, M satisfies the following conditions:

(FF1) $1 \in M$;

(FF2) if $\nu \in M$ and $\nu \to^L (\upsilon \to^L \varpi) \in M$, then $((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi \in M$;

(FF3) if $\nu \in M$ and $\nu \to^R (\upsilon \to^R \varpi) \in M$, then $((\varpi \to^R \upsilon) \to^L \upsilon) \to^R \varpi \in M$.

Proposition 4.1: Let Q be a pqMV-algebra and M be a filter in Q. Then M is a fantastic filter in Q iff $\upsilon \to^L \varpi \in M$ implies $((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi \in M$ and $\upsilon \to^R \varpi \in M$ implies $((\varpi \to^R \upsilon) \to^L \upsilon) \to^R \varpi \in M$ for every $\upsilon, \varpi \in Q$.

Proof: Suppose that M is a fantastic filter in Q and $\upsilon \to^L \varpi \in M$. Then we have $1 \to^L (\upsilon \to^L \varpi) = \upsilon \to^L \varpi \in M$. Since $1 \in M$, we have $((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi \in M$ by (FF2). The other can be proved similarly. Conversely, suppose that $\nu \in M$ and $\nu \to^L (\upsilon \to^L \varpi) \in M$. Since M is a filter in Q, we get $\upsilon \to^L \varpi \in M$ by Proposition 2.3, it follows that $((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi \in M$ by the assumption. The condition (FF3) can be proved similarly. Therefore M is a fantastic filter in Q.

Proposition 4.2: Let Q be a pqMV-algebra and M, M' be two filters in Q with $M \subseteq M'$. If M is a fantastic filter in Q, then M' is also a fantastic filter in Q.

Proof: Suppose that $\upsilon \to^L \varpi \in M'$. Because M is a fantastic filter in Q and $\upsilon \to^L ((\upsilon \to^L \varpi) \to^R \varpi) = (\upsilon \to^L \varpi) \to^R (\upsilon \to^L \varpi) = 1 \in M$, we get $(\upsilon \to^L \varpi) \to^R (((((\upsilon \to^L \varpi) \to^R \varpi) \to^L \upsilon) \to^R \upsilon) \to^L \varpi) = ((((\upsilon \to^L \varpi) \to^R \varpi) \to^L \upsilon) \to^R \upsilon) \to^L ((\upsilon \to^L \varpi) \to^R \varpi) \in M$ by (P15) and Proposition 4.1. Since $M \subseteq M'$, we have $(\upsilon \to^L \varpi) \to^R (((((\upsilon \to^L \varpi) \to^R \varpi) \to^L \upsilon) \to^R \upsilon) \to^L \varpi) \in M'$. Because $\upsilon \to^L \varpi \in M'$ and M' is a filter in Q, we have $((((\upsilon \to^L \varpi) \to^R \varpi) \to^L \upsilon) \to^R \upsilon) \to^L \varpi \in M'$. Put $\nu = ((((\upsilon \to^L \varpi) \to^R \varpi) \to^L \upsilon) \to^R \upsilon) \to^L \varpi$. Then $\nu \in M'$ and we calculate $\nu \to^R (((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi) \geq ((\varpi \to^L \upsilon) \to^R \upsilon) \to^L ((((\upsilon \to^L \varpi) \to^R \varpi) \to^L \upsilon) \to^R \upsilon) \geq (((\upsilon \to^L \varpi) \to^R \varpi) \to^L \upsilon) \to^R (\varpi \to^L \upsilon) \geq \varpi \to^L ((\upsilon \to^L \varpi) \to^R \varpi) = (\upsilon \to^L \varpi) \to^R (\varpi \to^L \varpi) = (\upsilon \to^L \varpi) \to^R 1 = 1$, so $\nu \to^R (((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi) \in M'$. Since $\nu \in M'$, we get $((\varpi \to^L \upsilon) \to^R \upsilon) \to^L \varpi \in M'$. Similarly, if $\upsilon \to^R \varpi \in M'$, we can prove $((\varpi \to^R \upsilon) \to^L \upsilon) \to^R \varpi \in M'$. Therefore M' is a fantastic filter in Q.

Below the relationship among fantastic filters, implicative filters and positive implicative filters in any pqMV-algebra is presented.

Proposition 4.3: Let Q be a pqMV-algebra. If M is an implicative filter in Q, then M is a fantastic filter in Q.

Proof: Suppose that $\upsilon \to^L \varpi \in M$ and M is an implicative filter in Q. Then we have $\varpi \to^L 0 \le \varpi \to^L \upsilon$ and then $(\varpi \to^L \upsilon) \to^R \upsilon \le \varpi' \to^R \upsilon$ by (P9), it turns out that $(\varpi' \otimes ((\varpi \to^L \upsilon) \to^R \upsilon)) \to^R \upsilon = ((\varpi \to^L \upsilon) \to^R \upsilon) \to^R (\varpi' \to^R \upsilon) = 1 \in M$, so we get $((\varpi \to^L \upsilon) \to^R \upsilon) \to^R \varpi \in M$ according to Theorem 3.1. Similarly, if $\upsilon \to^R \varpi \in M$, we can prove $((\varpi \to^R \upsilon) \to^L \upsilon) \to^R \varpi \in M$. Hence M is a fantastic filter in Q by Proposition 4.1.

Proposition 4.4: Let Q be a pqMV-algebra and M be a filter in Q. If M is a fantastic filter and positive implicative filter in Q, then M is an implicative filter in Q.

Proof: If $(\upsilon \to^L \varpi) \to^R \upsilon \in M$ for every $\upsilon, \varpi \in Q$, because M is a fantastic filter in Q, we get $((\upsilon \to^L (\upsilon \to^L \varpi)) \to^R (\upsilon \to^L \varpi)) \to^L \upsilon \in M$ by Proposition 4.1, which implies that $((\upsilon^2 \to^L \varpi) \to^R (\upsilon \to^L \varpi)) \to^L \upsilon \in M$. By (P11), we get $\upsilon \to^L \upsilon^2 \le (\upsilon^2 \to^L \varpi) \to^R (\upsilon \to^L \varpi)$. Because M is a positive implicative filter in Q, we have $\upsilon \to^L \upsilon^2 \in M$ and then $(\upsilon^2 \to^L \varpi) \to^R (\upsilon \to^L \varpi) \in M$, it follows that $\upsilon \in M$. Hence M is an implicative filter in Q by Theorem 3.1.

Definition 4.2: Let Q be a pqMV-algebra and the set $M \subseteq Q$. Then M is said to be an *associative filter* in Q if for every $\upsilon, \varpi, \nu \in Q$, M satisfies the following conditions:

(AF1) $1 \in M$;

(AF2) if $\upsilon \to^R \varpi \in M$ and $\upsilon \to^L (\varpi \to^L \nu) \in M$, then $\nu \in M$;

(AF3) if $\upsilon \to^L \varpi \in M$ and $\upsilon \to^R (\varpi \to^R \nu) \in M$, then $\nu \in M$.

Proposition 4.5: Let Q be a pqMV-algebra. If M is an associative filter in Q, then M is a filter in Q.

Proof: Let $\upsilon, \upsilon \to^L \varpi \in M$. Then we have $\upsilon \to^L (1 \to^L \varpi) = \upsilon \to^L \varpi \in M$. Since $\upsilon \to^R 1 = 1 \in M$, we have $\varpi \in M$ using (AF2). Thus M is a filter in Q by Proposition 2.3.

Lemma 4.1: Let Q be a pqMV-algebra. If M is an associative filter in Q, then $\upsilon' \in M$ for every $\upsilon \in Q$.

Proof: Because $(\upsilon \to^L 0) \to^L (1 \to^L (\upsilon \to^L 0)) = 1 \in M$ and $(\upsilon \to^L 0) \to^R 1 = 1 \in M$, we get $\upsilon \to^L 0 \in M$ using (AF2). Note that M is a filter in Q and $\upsilon \to^L 0 = \upsilon' + 0 \le \upsilon'$, we have $\upsilon' \in M$.

Now we will see the relationship among associative filters, (positive) implicative filters and fantastic filters in any pqMV-algebra.

Proposition 4.6: Let Q be a pqMV-algebra and the set $M \subseteq Q$. If M is an associative filter in Q, then we have

(1) M is an implicative filter in Q;

(2) M is a positive implicative filter in Q;

(3) M is a fantastic filter in Q.

Proof: (1) Let $(\upsilon \to^L \varpi) \to^R \upsilon \in M$. Then we get $(\upsilon \to^L \varpi) \to^R (1 \to^R \upsilon) \in M$ and $(\upsilon \to^L \varpi) \to^L 1 = 1 \in M$. Since M is an associative filter in Q, we have $\upsilon \in M$ by (AF3). Similarly, we can prove that $(\upsilon \to^R \varpi) \to^L \upsilon \in M$ implies $\upsilon \in M$. Therefore M is an implicative filter in Q by Theorem 3.1.

(2) Suppose that $\upsilon \to^L (\varpi \to^L \nu) \in M$ and $\upsilon \to^R \varpi \in M$. We have $\nu \in M$ by (AF2). Since $\nu \le \upsilon \to^L \nu$ and M is a filter in Q, we have $\upsilon \to^L \nu \in M$. Similarly, we can prove that $\upsilon \to^R (\varpi \to^R \nu)$ and $\upsilon \to^L \varpi \in M$ imply $\upsilon \to^R \nu \in M$. Therefore M is a positive implicative filter in Q.

(3) By Proposition 4.3 and (1).

Proposition 4.7: Let Q be a pqMV-algebra. If M is an implicative filter in Q, then for every $\upsilon \in Q$, the following conditions hold:

(1) if $\upsilon' \to^R \upsilon \in M$, then M is an associative filter in Q;

(2) if $\upsilon' \to^L \upsilon \in M$, then M is an associative filter in Q.

Proof: (1) Suppose that $\upsilon \to^L (\varpi \to^L \nu) \in M$ and $\upsilon \to^R \varpi \in M$. Because M is an implicative filter in Q, we have $(\upsilon' \to^R \upsilon) \to^R \upsilon \in M$ from Theorem 3.1. If $\upsilon' \to^R \upsilon \in M$, then $\upsilon \in M$ by Proposition 2.3, it follows that $\varpi \to^L \nu \in M$ and $\varpi \in M$, so $\nu \in M$ using Proposition 2.3 again. Therefore M is an associative filter in Q.

(2) It is similar to (1).

Corollary 4.1: Let Q be a pqMV-algebra and M be a filter in Q. If M is a fantastic and positive implicative filter in Q with $\upsilon' \to^R \upsilon \in M$ or $\upsilon' \to^L \upsilon \in M$ for every $\upsilon \in Q$, then M is an associative filter in Q.

Proof: Follows from Proposition 4.4 and Proposition 4.7.

Proposition 4.8: Let Q be a pqMV-algebra and M, M' be normal filters in Q with $M \subseteq M'$. If M is an associative filter in Q and $\upsilon' \to^L \upsilon \in M'$ for every $\upsilon \in Q$, then M' is also an associative filter in Q.

Proof: By Proposition 4.6, Proposition 3.2 and Proposition 4.7.

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