Adaptive Stabilization for Fractional-order System with Nonsymmetrical Dead-zone Input via Backstepping-based Sliding Mode Control

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***Abstract*—A new backstepping-based sliding mode control strategy to realize the stabilization of a class of fractional-order system is introduced in this paper. We assume that the system is fluctuated by unmodeled dynamics and external disturbances, meanwhile, the bounds of these uncertainties and the system parameters are unknown in advance. The effect of nonsymmetrical dead-zone input is taken into account in this paper. To deal with these unknown parameters, a sliding mode surface is proposed and some adaptive update laws are assigned. Then the indirect Lyapunov theory is given to analyze the stability of every subsystem. Simulation results are presented to prove the correctness and feasibility of the given control strategy.**

***Index Terms*—Backstepping-based sliding mode control, Fractional-order system, Nonsymmetrical dead-zone, Unmodeled dynamics.**

I. Introduction

FRACTIONAL calculus is a generalization of traditional integer order calculus, the history of fractional calculus is more than 300 years, which can be traced back to the contribution by Euler, Leibniz and other famous mathematicians [1,2]. It was found that, with the help of fractional calculus, many systems in interdisciplinary fields can be described more accurately, such as viscoelastic system [3], dielectric polarization [4], electrode-electrolyte polarization [5], finance systems and electromagnetic waves [6]. That is to say, fractional calculus provides a superb instrument for the description of memory and hereditary properties of various materials and processes.

At present, The study of fractional-order system has become an active research field. In particular, the control and stabilization of fractional-order systems has attracted extensive attention in various scientific fields. The results show that the fractional-order controller applied to the fractional-order system can obtain better control effect than the integer controller, such as, fractional-order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ control [7], fractional-order terminal sliding mode control [8], fractional-order fuzzy control [9], fractional-order finite-time sliding mode control [10], and so on.

Backstepping is a recursive controller design method, if virtual controller and part of Lyapunov function are designed step by step, then a common Lyapunov function of the whole system can be derived from the above operations. This method can guarantee the global stability, tracking and transient performance of nonlinear systems [11]. Take into consideration of the excellent performance of backstepping method, more and more researchers began to pay attention to this potential problem. It is reported that there are many preeminent literatures for the backstepping-based control or synchronization of fractional-order chaotic system have been existed. For instance, Luo [12] applied adding one power integrator to the robust control and synchronization of fractional-order system. Shukla [13,14] used backstepping method to realize the stabilization and synchronization of fractional-order chaotic system. Wei [15,16] adopted backstepping technique to investigate the stability of fractional-order nonlinear system.

However, almost all the above mentioned methods for control or stabilizing of fractional-order system assume that the system parameters are know in advance. As a matter of fact, many systems parameters cannot be exactly known in advance. The control object will be not achieved under the effect of unknown uncertainties. Therefore, it is urgent to consider the influence of unknown parameters in control or stabilizing fractional-order systems. Another important problem encountered in practice is the input nonlinearity. This kind of control input nonlinearity can be regarded as a cause of performance degradation or even worse, instability of a system. So, it is clear that the effect of input nonlinearity must be taken into account when analyzing and implementing a control scheme.

Motivated by the above discussions, in this paper, the stabilization problem of fractional-order system with nonsymmetric dead-zone nonlinear input by using adaptive backstepping-based sliding mode control technique is investigated. For compensation the nonlinear input, a fractional-

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order auxiliary system is constructed to generate necessary signal. Some appropriate estimation rules are given to deal with the system parameters and the unknown upper bound of uncertainties. The frequency distributed model of fractional integrator and indirect Lyapunov stability theory are used to verify the stability and design virtual controller for every subsystem. Through designing virtual controllers step by step, a comprehensive actual controller is finally determined.

The remaining part of this paper is organized as follows: Section 2 introduces the relevant definitions, lemmas, and frequency distributed model. Main results are presented in Section 3. Some simulation results are provided in Section 4 to show the effectiveness of the proposed method. Finally, conclusions are given in Section 5.

II. Preliminaries

The Riemann-Liouville, Caputo definition are main definitions of fractional calculus

Definition 1 The α th-order Riemann-Liouville fractional integration of function $f(t)$ is given by

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 For $n-1 < \alpha \leq n$, $n \in R$, the α th-order Riemann-Liouville fractional derivative of function $f(t)$ is defined as

$$ \begin{aligned} {}_{t_0}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \\ &= \frac{d^n}{dt^n} I^{n-\alpha} f(t) \end{aligned} \tag{2} $$

Definition 3 The α th-order Caputo fractional derivative of function $f(t)$ is defined as

$$ {}_{t_0}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \tag{3} $$

where m is the smallest integer number, larger than α.

Lemma 1 (see [17]). Let $x = 0$ be an equilibrium point for either Caputo or RL fractional nonautonomous system:

$$ D^q x(t) = f(x,t) \tag{4} $$

where $q \in (0,1)$ and $f(x,t)$ satisfies the Lipschitz condition with Lipschitz constant $l > 0$. Assume that there exists a Lyapunov function $V(t, x(t))$ satisfying

$$ \begin{aligned} \alpha_1 ||x||^a &\leq V(t,x(t)) \leq \alpha_2 ||x|| \\ \dot{V}(t,x(t)) &\leq -\alpha_3 ||x|| \end{aligned} \tag{5} $$

where α_1, α_2, α_3 and a are positive constants and $||\cdot||$ denotes an arbitrary norm. Then the equilibrium point of system (4) is asymptotically stable.

Lemma 2 (see [18]). Consider a nonlinear fractional-order system

$$ D^{\alpha} x(t) = f(x(t)) \tag{6} $$

where $\alpha \in (0,1)$. Then the system can be equivalently converted to the following continuous frequency distributed model

$$ \begin{aligned} \frac{\partial z(\omega,t)}{\partial t} &= -\omega z(\omega,t) + f(x(t)) \\ x(t) &= \int_0^{\infty} \mu_{\alpha}(\omega) z(\omega,t) d\omega \end{aligned} \tag{7} $$

where $\mu_{\alpha}(\omega) = \frac{sin(\alpha\pi)}{\pi\omega^{\alpha}}$ and $z(\omega,t)$ is the true state of the system.

III. Main results

Backstepping technique is suitable for research strict feedback system, which can be described as follows

$$ \begin{aligned} D^{\alpha} x_1 &= g_1(x_1,t)x_2 + \delta_1^T F_1(x_1,t) + f_1(x_1,t) \\ D^{\alpha} x_2 &= g_2(x_1,x_2,t)x_3 + \delta_2^T F_2(x_1,x_2,t) + f_2(x_1,x_2,t) \\ &\vdots \\ D^{\alpha} x_{n-1} &= g_{n-1}(x_1,x_2,...,x_{n-1},t)x_n + \delta_{n-1}^T F_{n-1}(x_1,x_2, \\ &\quad ...,x_{n-1},t) + f_{n-1}(x_1,x_2,...,x_{n-1},t) \\ D^{\alpha} x_n &= g_n(x_1,x_2,...,x_n,t)u + \delta_n^T F_n(x_1,x_2,...,x_n,t) \\ &\quad + f_n(x_1,x_2,...,x_n,t) \end{aligned} \tag{8} $$

where δ_i is the system parameters vector of the i-th state equation, $g_i(\cdot)$, $F_i(\cdot)$, $f_i(\cdot)$ for $i = 1,2,...,n$ are known, smooth nonlinear functions. This paper investigates a class of typical fractional-order strict feedback system, it has the following form

$$ \begin{aligned} D^{\alpha} x_1 &= x_2 + \delta_1^T F_1(x_1) + f_1(x_1) + \Delta f_1(X) + d_1(t) \\ D^{\alpha} x_2 &= x_3 + \delta_2^T F_2(x_1,x_2) + f_2(x_1,x_2) + \Delta f_2(X) + d_2(t) \\ &\vdots \\ D^{\alpha} x_n &= k\Psi(u(t)) + \delta_n^T F_n(X) + f_n(X) + \Delta f_n(X) + d_n(t) \end{aligned} \tag{9} $$

where $\alpha \in (0,1)$, $X = [x_1, x_2, ..., x_n]^T$ is state variables vector, k is non-zero constant, δ_i is unknown system parameters vector, $F_i(\cdot)$ and $f_i(\cdot)$ are system nonlinear parts, $\Delta f_i(X)$ and $d_i(t)$ for $i = 1,2,...,n$ respectively are unmodeled dynamics and external disturbance. $\Psi(u(t))$ is nonsymmetrical dead-zone input.

Assumption 1. The nonsymmetrical dead-zone function is described as follows:

$$ \Psi(u(t)) = \begin{cases} \rho_+(u(t) - u_+), & u(t) \geq u_+ \\ 0, & -u_- < u(t) < u_+ \\ \rho_-(u(t) + u_-), & u(t) \leq -u_- \end{cases} \tag{10} $$

where ρ_+, ρ_-, u_+, u_- are strictly positive parameters, and the slope parameters ρ_+ and ρ_- are bounded, i.e., there exist known constants ρ_1 and ρ_2 such that $max\{\rho_+, \rho_-\} = \rho_1$ and $min\{\rho_+, \rho_-\} = \rho_2$. Further, the nonsymmetrical dead-zone function can be rewritten as

$$ \Psi(u(t)) = \rho(t)u(t) + \Delta u(t) \tag{11} $$

where

$$ \rho(t) = \begin{cases} \rho_+, & u(t) > 0 \\ \rho_-, & u(t) \leq 0 \end{cases} \tag{12} $$

$$ \Delta u(t) = \begin{cases} -\rho_+ u_+, & u(t) \geq u_+ \\ -\rho(t)u(t), & -u_- < u(t) < u_+ \\ \rho_- u_-, & u(t) \leq -u_- \end{cases} \tag{13} $$

A typical nonsymmetrical dead-zone function is depicted in Figure 1.

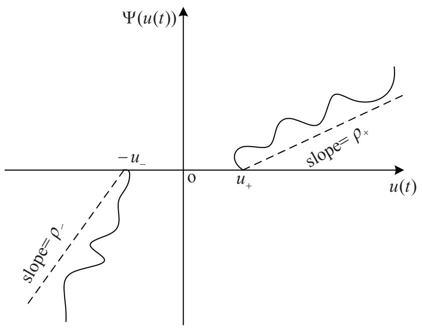

Figure 1. A typical nonsymmetrical dead-zone function

Remark 1. Many fractional-order systems can be described as equation (9), such as, fractional-order gyro system, fractional-order Genesio-Tesi system, fractional-order Arneodo system, and so on.

For using backstepping-based sliding mode control strategy, transformation variables are firstly assigned as

$$\begin{aligned}\xi_1 &= x_1 - \sigma_1\\ \xi_i &= x_i - \vartheta_{i-1} - \sigma_i, \quad i = 2,3,...,n.\end{aligned} \tag{14}$$

where $\vartheta_j(j = 1,2,...,n-1)$ is virtual controller to be determined later. $\sigma_j(j = 1,2,...,n)$ is the virtual signal generated by the following auxiliary fractional-order system to compensate the nonlinear input

$$\begin{aligned}D^\alpha\sigma_i &= \sigma_{i+1} - c_i\sigma_i, \quad i = 1,2,...,n-1.\\ D^\alpha\sigma_n &= k\Delta u(t) - c_n\sigma_n\end{aligned} \tag{15}$$

where $c_i > 0$, $c_n > 0$.

For handling the unknown bounded uncertainties, the following assumption is given.

Assumption 2. The unmodeled dynamics and external disturbance are unknown bounded, which satisfy the following conditions

$$\begin{aligned}|\Delta f_i(X)| &\leq \beta_{i1}|\xi_i|\\ |d_i(t)| &\leq \beta_{i2}\end{aligned} \tag{16}$$

where β_{i1} and β_{i2} are unknown positive constants. In this paper, the sliding mode surface can be constructed as

$$s_p = D^{\alpha-1}\xi_p + \int_0^t \big(\xi_p + sgn(\xi_p)\big)d\tau \tag{17}$$

in which $p = 1,2,...,n$. Taking the derivative of s with respect time, we have

$$\dot{s}_p = D^\alpha\xi_p + \big(\xi_p + sgn(\xi_p)\big) \tag{18}$$

when system trajectories arrived at the sliding mode surface, we have $\dot{s} = 0$, that is

$$D^\alpha\xi_p = -\xi_p - sgn(\xi_p) \tag{19}$$

according the sliding mode dynamics, the virtual controller can be determined as

$$\begin{aligned}\vartheta_1 &= -m_1s_1 - \xi_2 - \hat{\delta}_1^TF_1 - f_1 - (\hat{\beta}_{11}|\xi_1| + \hat{\beta}_{12})sgn(s_1)\\ &\quad -c_1\sigma_1 - (\xi_1 + sgn(\xi_1))\\ \vartheta_j &= -m_js_j - \xi_{j+1} - \hat{\delta}_j^TF_j - f_j - (\hat{\beta}_{j1}|\xi_j| + \hat{\beta}_{j2})sgn(s_j)\\ &\quad +D^\alpha\vartheta_{j-1} - c_j\sigma_j - (\xi_j + sgn(\xi_j))\end{aligned} \tag{20}$$

where $j = 2,3,...,n-1$. $m_i > 0$, F_i and f_i are the abbreviations of $F_i(\cdot)$ and $f_i(\cdot)$. $\hat{\delta}_i$, $\hat{\beta}_{i1}$ and $\hat{\beta}_{i2}$ are estimations of δ_i, β_{i1} and β_{i2} for $i = 1,2,...,n$, respectively. Denote $\widetilde{\delta}_i = \hat{\delta}_i - \delta_i$, $\widetilde{\beta}_{i1} = \hat{\beta}_{i1} - \beta_{i1}$, $\widetilde{\beta}_{i2} = \hat{\beta}_{i2} - \beta_{i2}$ as parameters estimation errors, which adaptive update laws are designed as

$$\begin{aligned}D^\alpha\widetilde{\delta}_i &= D^\alpha\hat{\delta}_i = F_is_i\\ D^\alpha\widetilde{\beta}_{i1} &= D^\alpha\hat{\beta}_{i1} = \eta_{i1}|\xi_i||s_i|, \quad \eta_{i1} > 0\\ D^\alpha\widetilde{\beta}_{i2} &= D^\alpha\hat{\beta}_{i2} = \eta_{i2}|s_i|, \quad \eta_{i2} > 0\end{aligned} \tag{21}$$

Theorem 2. Consider the system (9) with nonsymmetrical dead-zone nonlinear input, if the system is controlled by the following controller

$$\begin{aligned}u(t) = \frac{1}{k\rho_2}\Big(&-m_n|s_n| - |\hat{\delta}_n|^T|F_n| - |f_n| - |\hat{\beta}_{n1}||\xi_n| - |\hat{\beta}_{n2}|\\ &-|D^\alpha\vartheta_{n-1}| - c_n|\sigma_n| - |\xi_n + sgn(\xi_n)|\Big)sgn(s_n)\end{aligned} \tag{22}$$

then the system trajectories can converge to the sliding surface $s_i(i = 1,2,...,n) = 0$ asymptotically.

Proof. Step 1: The first new subsystem can be obtain according to equations (9), (14) and (15)

$$\begin{aligned}D^\alpha\xi_1 &= D^\alpha x_1 - D^\alpha\sigma_1\\ &= x_2 + \delta_1^TF_1 + f_1 + \Delta f_1(X) + d_1(t) - \sigma_2 + c_1\sigma_1\\ &= \xi_2 + \vartheta_1 + \delta_1^TF_1 + f_1 + \Delta f_1(X) + d_1(t) + c_1\sigma_1\end{aligned} \tag{23}$$

according to Lemma 2, the parameters adaptation laws (21) can transform into the frequency distributed model, that is

$$\begin{aligned}\frac{\partial z_{\widetilde{\delta}_1}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\delta}_1}(\omega,t) + F_1s_1\\ \widetilde{\delta}_1 &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\delta}_1}(\omega,t)d\omega\\ \frac{\partial z_{\widetilde{\beta}_{11}}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\beta}_{11}}(\omega,t) + \eta_{11}|\xi_1||s_1|\\ \widetilde{\beta}_{11} &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{11}}(\omega,t)d\omega\\ \frac{\partial z_{\widetilde{\beta}_{12}}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\beta}_{12}}(\omega,t) + \eta_{12}|s_1|\\ \widetilde{\beta}_{12} &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{12}}(\omega,t)d\omega\end{aligned} \tag{24}$$

selecting the Lyapunov function as

$$\begin{aligned}V_1 &= \frac{1}{2}s_1^2 + \frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\delta}_1}^T(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega\\ &\quad+\frac{1}{2\eta_{11}}\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{11}}^2(\omega,t)d\omega\\ &\quad+\frac{1}{2\eta_{12}}\int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{12}}^2(\omega,t)d\omega\end{aligned} \tag{25}$$

taking the derivative of V_1 with respect to time, it yields

$$\begin{aligned}\dot{V}_1 &= s_1\dot{s}_1 - \int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\delta}_1}^T(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega + \widetilde{\delta}_1^TF_1s_1\\ &\quad -\frac{1}{\eta_{11}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{11}}^2(\omega,t)d\omega + \widetilde{\beta}_{11}|\xi_1||s_1|\\ &\quad -\frac{1}{\eta_{12}}\int_0^\infty \omega\mu_\alpha(\omega)z_{\widetilde{\beta}_{12}}^2(\omega,t)d\omega + \widetilde{\beta}_{12}|s_1|\end{aligned} \tag{26}$$

substituting $\dot{s}_1$ from equation (18) into equation (26), one has

$$\begin{aligned}\dot{V}_1 &= s_1\big[D^\alpha\xi_1 + (\xi_1 + sgn(\xi_1))\big]\\ &\quad -\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_1}(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega + \widetilde{\delta}_1^T F_1 s_1\\ &\quad -\frac{1}{\eta_{11}}\int_0^\infty\omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{11}}(\omega,t)d\omega + \widetilde{\beta}_{11}|\xi_1||s_1|\\ &\quad -\frac{1}{\eta_{12}}\int_0^\infty\omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{12}}(\omega,t)d\omega + \widetilde{\beta}_{12}|s_1| \qquad (27)\end{aligned}$$

substituting the new subsystem (23) into the above equation

$$\begin{aligned}\dot{V}_1 &= s_1\big[\xi_2 + \vartheta_1 + \delta_1^T F_1 + f_1 + \Delta f_1(X)\\ &\quad + d_1(t) + c_1\sigma_1 + (\xi_1 + sgn(\xi_1))\big]\\ &\quad -\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_1}(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega + \widetilde{\delta}_1^T F_1 s_1\\ &\quad -\frac{1}{\eta_{11}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{11}}(\omega,t)d\omega + \widetilde{\beta}_{11}|\xi_1||s_1|\\ &\quad -\frac{1}{\eta_{12}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{12}}(\omega,t)d\omega + \widetilde{\beta}_{12}|s_1| \qquad (28)\end{aligned}$$

replacing ϑ_1 from equation (20) into equation (28) and using Assumption 2, we have

$$\begin{aligned}\dot{V}_1 &\le s_1\big[-m_1 s_1 - \widetilde{\delta}_1^T F_1 - (\hat{\beta}_{11}|\xi_1| + \hat{\beta}_{12})sgn(s_1)\big] + \beta_{11}\\ &\quad \times|\xi_1||s_1| + \beta_{12}|s_1| - \int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_1}(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega\\ &\quad + \widetilde{\delta}_1^T F_1 s_1 - \frac{1}{\eta_{11}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{11}}(\omega,t)d\omega + \widetilde{\beta}_{11}|\xi_1||s_1|\\ &\quad -\frac{1}{\eta_{12}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{12}}(\omega,t)d\omega + \widetilde{\beta}_{12}|s_1|\\ &= -\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_1}(\omega,t)z_{\widetilde{\delta}_1}(\omega,t)d\omega\\ &\quad -\frac{1}{\eta_{11}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{11}}(\omega,t)d\omega\\ &\quad -\frac{1}{\eta_{12}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{12}}(\omega,t)d\omega - m_1 s_1^2 < 0 \qquad (29)\end{aligned}$$

because of $\dot{V}_1 < 0$, then s_1, $\widetilde{\delta}_1$, $\widetilde{\beta}_{11}$, $\widetilde{\beta}_{12}$ are all asymptotically converge to zero.

Step 2: The second subsystem about ξ_2 can be established as

$$\begin{aligned}D^\alpha\xi_2 &= D^\alpha x_2 - D^\alpha\vartheta_1 - D^\alpha\sigma_2\\ &= x_3 + \delta_2^T F_2 + f_2 + \Delta f_2(X) + d_2(t) - D^\alpha\vartheta_1\\ &\quad -\sigma_3 + c_2\sigma_2\\ &= \xi_3 + \vartheta_2 + \delta_2^T F_2 + f_2 + \Delta f_2(X) + d_2(t)\\ &\quad -D^\alpha\vartheta_1 + c_2\sigma_2 \qquad (30)\end{aligned}$$

similar to the step 1, the frequency distributed model of adaptive estimation laws can be constructed as

$$\begin{aligned}\frac{\partial z_{\widetilde{\delta}_2}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\delta}_2}(\omega,t) + F_2 s_2\\ \widetilde{\delta}_2 &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\delta}_2}(\omega,t)d\omega\\ \frac{\partial z_{\widetilde{\beta}_{21}}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\beta}_{21}}(\omega,t) + \eta_{21}|\xi_2||s_2|\\ \widetilde{\beta}_{21} &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{21}}(\omega,t)d\omega\\ \frac{\partial z_{\widetilde{\beta}_{22}}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\beta}_{22}}(\omega,t) + \eta_{22}|s_2|\\ \widetilde{\beta}_{22} &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{22}}(\omega,t)d\omega \qquad (31)\end{aligned}$$

selecting the Lyapunov function as

$$\begin{aligned}V_2 &= V_1 + \frac{1}{2}s_2^2 + \frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z^T_{\widetilde{\delta}_2}(\omega,t)z_{\widetilde{\delta}_2}(\omega,t)d\omega\\ &\quad + \frac{1}{2\eta_{21}}\int_0^\infty \mu_\alpha(\omega)z^2_{\widetilde{\beta}_{21}}(\omega,t)d\omega\\ &\quad + \frac{1}{2\eta_{22}}\int_0^\infty \mu_\alpha(\omega)z^2_{\widetilde{\beta}_{22}}(\omega,t)d\omega \qquad (32)\end{aligned}$$

thus its derivative can be described as

$$\begin{aligned}\dot{V}_2 &\le -\sum_{j=1}^2\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^2\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^2\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega\\ &\quad -m_1 s_1^2 + s_2\dot{s}_2 + \widetilde{\delta}_2^T F_2 s_2 + \widetilde{\beta}_{21}|\xi_2||s_2| + \widetilde{\beta}_{22}|s_2| \qquad (33)\end{aligned}$$

substituting $\dot{s}_2$ from equation (18) into (33) and according to equation (30), one obtains

$$\begin{aligned}\dot{V}_2 &\le -\sum_{j=1}^2\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^2\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^2\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega\\ &\quad -m_1 s_1^2 + s_2\big[D^\alpha\xi_2 + (\xi_2 + sgn(\xi_2))\big]\\ &\quad + \widetilde{\delta}_2^T F_2 s_2 + \widetilde{\beta}_{21}|\xi_2||s_2| + \widetilde{\beta}_{22}|s_2|\\ &= -\sum_{j=1}^2\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^2\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^2\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega - m_1 s_1^2\\ &\quad + s_2\big[\xi_3 + \vartheta_2 + \delta_2^T F_2 + f_2 + \Delta f_2(X) + d_2(t)\\ &\quad - D^\alpha\vartheta_1 + c_2\sigma_2 + (\xi_2 + sgn(\xi_2))\big]\\ &\quad + \widetilde{\delta}_2^T F_2 s_2 + \widetilde{\beta}_{21}|\xi_2||s_2| + \widetilde{\beta}_{22}|s_2| \qquad (34)\end{aligned}$$

replace ϑ_2 from equation (20) into the above equation, we have

$$\begin{aligned}\dot{V}_2 &\le -\sum_{j=1}^2\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\ &\quad -\sum_{j=1}^2\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{2}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega - m_1 s_1^2\\
&+s_2\big[-m_2 s_2 - \widetilde{\delta}_2^T F_2 - (\hat{\beta}_{21}|\xi_2| + \hat{\beta}_{22})sgn(s_2)\big]\\
&+\beta_{21}|\xi_2||s_2| + \beta_{22}|s_2| + \widetilde{\delta}_2^T F_2 s_2 + \widetilde{\beta}_{21}|\xi_2||s_2|\\
&+\widetilde{\beta}_{22}|s_2|\\
\le\ &-\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\
&-\sum_{j=1}^{2}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\
&-\sum_{j=1}^{2}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega\\
&-m_1 s_1^2 - m_2 s_2^2 \qquad (35)
\end{aligned}
$$

since $\dot{V}_2 < 0$, then s_2, $\widetilde{\delta}_2$, $\widetilde{\beta}_{21}$, $\widetilde{\beta}_{22}$ are all asymptotically converge to zero.

Step i: We continue to investigate the i-th new subsystem with transformation variables, that is

$$
\begin{aligned}
D^\alpha \xi_i &= D^\alpha x_i - D^\alpha\vartheta_{i-1} - D^\alpha\sigma_i\\
&= x_{i+1} + \delta_i^T F_i + f_i + \Delta f_i(X) + d_i(t)\\
&\quad -D^\alpha\vartheta_{i-1} - \sigma_{i+1} + c_i\sigma_i\\
&= \xi_{i+1} + \vartheta_i + \delta_i^T F_i + f_i + \Delta f_i(X) + d_i(t)\\
&\quad -D^\alpha\vartheta_{i-1} + c_i\sigma_i \qquad (36)
\end{aligned}
$$

similar to the above steps, the frequency distributed model of adaptive estimation laws can be constructed as

$$
\begin{aligned}
\frac{\partial z_{\widetilde{\delta}_i}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\delta}_i}(\omega,t) + F_i s_i\\
\widetilde{\delta}_i &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\delta}_i}(\omega,t)d\omega\\
\frac{\partial z_{\widetilde{\beta}_{i1}}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\beta}_{i1}}(\omega,t) + \eta_{i1}|\xi_i||s_i|\\
\widetilde{\beta}_{i1} &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{i1}}(\omega,t)d\omega\\
\frac{\partial z_{\widetilde{\beta}_{i2}}(\omega,t)}{\partial t} &= -\omega z_{\widetilde{\beta}_{i2}}(\omega,t) + \eta_{i2}|s_i|\\
\widetilde{\beta}_{i2} &= \int_0^\infty \mu_\alpha(\omega)z_{\widetilde{\beta}_{i2}}(\omega,t)d\omega \qquad (37)
\end{aligned}
$$

selecting the Lyapunov function as

$$
\begin{aligned}
V_i &= V_{i-1} + \frac{1}{2}s_i^2 + \frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z^T_{\widetilde{\delta}_i}(\omega,t)z_{\widetilde{\delta}_i}(\omega,t)d\omega\\
&\quad +\frac{1}{2\eta_{i1}}\int_0^\infty \mu_\alpha(\omega)z^2_{\widetilde{\beta}_{i1}}(\omega,t)d\omega\\
&\quad +\frac{1}{2\eta_{i2}}\int_0^\infty \mu_\alpha(\omega)z^2_{\widetilde{\beta}_{i2}}(\omega,t)d\omega \qquad (38)
\end{aligned}
$$

taking the derivative of V_i, and using the deduce results of the above steps, one has

$$
\begin{aligned}
\dot{V}_i \le\ &-\sum_{j=1}^{i}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega\\
&-\sum_{j=1}^{i-1}m_j s_j^2 + s_i\dot{s}_i + \widetilde{\delta}_i^T F_i s_i\\
&+\widetilde{\beta}_{i1}|\xi_i||s_i| + \widetilde{\beta}_{i2}|s_i| \qquad (39)
\end{aligned}
$$

substituting $\dot{s}_i$ from equation (18) into (39), and considering equations (16), (20) and (36), it yields

$$
\begin{aligned}
\dot{V}_i \le\ &-\sum_{j=1}^{i}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega\\
&-\sum_{j=1}^{i-1}m_j s_j^2 + s_i\big[D^\alpha\xi_i + (\xi_i + sgn(\xi_i))\big]\\
&+\widetilde{\delta}_i^T F_i s_i + \widetilde{\beta}_{i1}|\xi_i||s_i| + \widetilde{\beta}_{i2}|s_i|\\
\le\ &-\sum_{j=1}^{i}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega - \sum_{j=1}^{i-1}m_j s_j^2\\
&+s_i\big[-m_i s_i - \widetilde{\delta}_i^T F_i - (\hat{\beta}_{i1}|\xi_i| + \hat{\beta}_{i2})sgn(s_i)\big]\\
&+\beta_{i1}|\xi_i||s_i| + \beta_{i2}|s_i| + \widetilde{\delta}_i^T F_i s_i + \widetilde{\beta}_{i1}|\xi_i||s_i| + \widetilde{\beta}_{i2}|s_i|\\
\le\ &-\sum_{j=1}^{i}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega\\
&-\sum_{j=1}^{i}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega - \sum_{j=1}^{i}m_j s_j^2 \qquad (40)
\end{aligned}
$$

because of $\dot{V}_i < 0$, then s_i, $\widetilde{\delta}_i$, $\widetilde{\beta}_{i1}$, $\widetilde{\beta}_{i2}$ can converge to zero asymptotically.

Step n: The last subsystem with transformation variable ξ_n is determined as

$$
\begin{aligned}
D^\alpha \xi_n &= D^\alpha x_n - D^\alpha\vartheta_{n-1} - D^\alpha\sigma_n\\
&= k\Psi(u(t)) + \delta_n^T F_n + f_n + \Delta f_n(X) + d_n(t)\\
&\quad -D^\alpha\vartheta_{n-1} - k\Delta u(t) + c_n\sigma_n\\
&= k\rho(t)u(t) + \delta_n^T F_n + f_n + \Delta f_n(X)\\
&\quad +d_n(t) - D^\alpha\vartheta_{n-1} + c_n\sigma_n \qquad (41)
\end{aligned}
$$

the overall Lyapunov function is selected as

$$
\begin{aligned}
V_n &= V_{n-1} + \frac{1}{2}s_n^2 + \frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z^T_{\widetilde{\delta}_n}(\omega,t)z_{\widetilde{\delta}_n}(\omega,t)d\omega\\
&\quad +\frac{1}{2\eta_{n1}}\int_0^\infty \mu_\alpha(\omega)z^2_{\widetilde{\beta}_{n1}}(\omega,t)d\omega
\end{aligned}
$$

$$+\frac{1}{2\eta_{n2}}\int_0^\infty \mu_\alpha(\omega)z^2_{\widetilde{\beta}_{n2}}(\omega,t)d\omega \tag{42}$$

taking the derivative of V_n with respect time, and according to Assumption 2, one has

$$\begin{aligned}\dot{V}_n \leq & -\sum_{j=1}^{n}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ & -\sum_{j=1}^{n}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega \\ & -\sum_{j=1}^{n}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega-\sum_{j=1}^{n-1}m_js_j^2 \\ & +s_n\big[k\rho(t)u(t)+\delta_n^TF_n+f_n+\Delta f_n(X)+d_n(t) \\ & -D^\alpha\vartheta_{n-1}+c_n\sigma_n+(\xi_n+sgn(\xi_n))\big] \\ & +\widetilde{\delta}_n^TF_ns_n+\widetilde{\beta}_{n1}|\xi_n||s_n|+\widetilde{\beta}_{n2}|s_n| \\ \leq & -\sum_{j=1}^{n}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ & -\sum_{j=1}^{n}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega \\ & -\sum_{j=1}^{n}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega-\sum_{j=1}^{n-1}m_js_j^2 \\ & +k\rho(t)s_nu(t)+|f_n||s_n|+|D^\alpha\vartheta_{n-1}||s_n| \\ & +c_n|\sigma_n||s_n|+|\xi_n+sgn(\xi_n)||s_n| \\ & +|\hat{\delta}_n|^T|F_n||s_n|+|\hat{\beta}_{n1}||\xi_n||s_n|+|\hat{\beta}_{n2}||s_n| \end{aligned}\tag{43}$$

since $\rho(t)/\rho_2=1+a(t)$, where $a(t)$ is a some piecewise positive function, we have

$$\begin{aligned}k\rho(t)s_nu(t) = & (1+a(t))\big[-m_ns_n^2-|\hat{\delta}_n|^T|F_n||s_n| \\ & -|f_n||s_n|-|\hat{\beta}_{n1}||\xi_n||s_n|-|\hat{\beta}_{n2}||s_n| \\ & -|D^\alpha\vartheta_{n-1}||s_n|-c_n|\sigma_n||s_n| \\ & -|\xi_n+sgn(\xi_n)||s_n|\big] \\ \leq & -m_ns_n^2-|\hat{\delta}_n|^T|F_n||s_n|-|f_n||s_n| \\ & -|\hat{\beta}_{n1}||\xi_n||s_n|-|\hat{\beta}_{n2}||s_n| \\ & -|D^\alpha\vartheta_{n-1}||s_n|-c_n|\sigma_n||s_n| \\ & -|\xi_n+sgn(\xi_n)||s_n| \end{aligned}\tag{44}$$

substituting (44) into (43), it yields

$$\begin{aligned}\dot{V}_n \leq & -\sum_{j=1}^{n}\int_0^\infty \omega\mu_\alpha(\omega)z^T_{\widetilde{\delta}_j}(\omega,t)z_{\widetilde{\delta}_j}(\omega,t)d\omega \\ & -\sum_{j=1}^{n}\frac{1}{\eta_{j1}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j1}}(\omega,t)d\omega \\ & -\sum_{j=1}^{n}\frac{1}{\eta_{j2}}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\widetilde{\beta}_{j2}}(\omega,t)d\omega \\ & -\sum_{j=1}^{n}m_js_j^2 \end{aligned}\tag{45}$$

because of $\dot{V}_n<0$, that is the system trajectories (9) with nonsymmetical dead-zone input can reach to sliding mode surface gradually, thus the proof is completed.

Theorem 2. Consider the sliding mode dynamics (19), the system is stable and its state trajectories converge to zero asymptotically.

Proof. Step 1: For the first sliding mode dynamic, the corresponding frequency distributed model is

$$\begin{aligned}\frac{\partial z_{\xi_1}(\omega,t)}{\partial t} &= -\omega z_{\xi_1}(\omega,t)-\big(\xi_1+sgn(\xi_1)\big) \\ \xi_1 &= \int_0^\infty \mu_\alpha(\omega)z_{\xi_1}(\omega,t)d\omega\end{aligned}\tag{46}$$

selecting the following Lyapunov function

$$W_1=\frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z^2_{\xi_1}(\omega,t)d\omega \tag{47}$$

taking the derivation of W_1 with respect time, one has

$$\begin{aligned}\dot{W}_1 &= -\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\xi_1}(\omega,t)d\omega+\xi_1\big(-\xi_1-sgn(\xi_1)\big) \\ &= -\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\xi_1}(\omega,t)d\omega-\xi_1^2-|\xi_1|\end{aligned}\tag{48}$$

Obviously, $\dot{W}_1<0$, according to Lemma 1, the first sliding mode dynamics is asymptotical stable, that is $\xi_1\to 0$ as $t\to\infty$.

Step 2: According to the second sliding mode dynamics in equation (19), its frequency distributed model can be written as

$$\begin{aligned}\frac{\partial z_{\xi_2}(\omega,t)}{\partial t} &= -\omega z_{\xi_2}(\omega,t)-\big(\xi_2+sgn(\xi_2)\big) \\ \xi_2 &= \int_0^\infty \mu_\alpha(\omega)z_{\xi_2}(\omega,t)d\omega\end{aligned}\tag{49}$$

selecting the following Lyapunov candidate function for

$$W_2=W_1+\frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z^2_{\xi_2}(\omega,t)d\omega \tag{50}$$

taking the derivation of W_2, and according to the above deduced results, we have

$$\dot{W}_2\leq-\sum_{j=1}^{2}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\xi_j}(\omega,t)d\omega-\sum_{j=1}^{2}\xi_j^2-\sum_{j=1}^{2}|\xi_j| \tag{51}$$

Similarly, $\dot{W}_2<0$, according to Lemma 1, the second sliding mode dynamics is asymptotical stable, that is $\xi_2\to 0$ as $t\to\infty$.

Step i: We continue to investigate the stability of the i-th sliding mode dynamics, which frequency distributed model is

$$\begin{aligned}\frac{\partial z_{\xi_i}(\omega,t)}{\partial t} &= -\omega z_{\xi_i}(\omega,t)-\big(\xi_i+sgn(\xi_i)\big) \\ \xi_i &= \int_0^\infty \mu_\alpha(\omega)z_{\xi_i}(\omega,t)d\omega\end{aligned}\tag{52}$$

choosing the following form Lyapunov function

$$W_i=W_{i-1}+\frac{1}{2}\int_0^\infty \mu_\alpha(\omega)z^2_{\xi_i}(\omega,t)d\omega \tag{53}$$

taking the time derivation of W_i, one has

$$\dot{W}_i\leq-\sum_{j=1}^{i}\int_0^\infty \omega\mu_\alpha(\omega)z^2_{\xi_j}(\omega,t)d\omega-\sum_{j=1}^{i}\xi_j^2-\sum_{j=1}^{i}|\xi_j| \tag{54}$$

Indeed, $\dot{W}_i<0$, then ξ_i is asymptotically converge to zero.

Step n: In the last step, the stability of the whole sliding mode dynamics is demonstrated, the corresponding frequency distributed model is

$$\begin{aligned} \frac{\partial z_{\xi_n}(\omega,t)}{\partial t} &= -\omega z_{\xi_n}(\omega,t) - \big(\xi_n + sgn(\xi_n)\big) \\ \xi_n &= \int_0^{\infty} \mu_\alpha(\omega) z_{\xi_n}(\omega,t) d\omega \end{aligned} \quad (55)$$

we select the following Lyapunov function

$$W_n = W_{n-1} + \frac{1}{2}\int_0^{\infty} \mu_\alpha(\omega) z_{\xi_n}^2(\omega,t) d\omega \quad (56)$$

its fractional order derivation is satisfying

$$\dot{W}_n \leq -\sum_{j=1}^{n}\int_0^{\infty} \omega\mu_\alpha(\omega) z_{\xi_j}^2(\omega,t) d\omega - \sum_{j=1}^{n}\xi_j^2 - \sum_{j=1}^{n}|\xi_j| \quad (57)$$

therefore $\dot{W}_n < 0$, the whole sliding mode dynamics (19) is asymptotically stale. This completes the proof.

IV. Simulation Results

In this section, simulation results are given to demonstrate the effectiveness and feasibility of the proposed control strategy. Consider the fractional-order Genesio-Tesi system with nonlinear input, which is described as

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= x_3 \\ D^\alpha x_3 &= \Psi(u(t)) - a_1x_1 - a_2x_2 - a_3x_3 \\ &\quad + a_4x_1^2 + \Delta f(X) + d(t) \end{aligned} \quad (58)$$

where $a_1 = 1$, $a_2 = 1.1$, $a_3 = -0.232$, $a_4 = 1$, $\delta_3 = [a_1, a_2, a_3, a_4]^T$, $F_3 = [-x_1, -x_2, -x_3, x_1^2]^T$, $\Delta f(X) = -0.01cos(x_3)$ and $d(t) = 0.02sin(3t)$ are unmodeled dynamics and external disturbance, respectively. Firstly, considering $\Psi(u(t))$ as nonsymmetrical dead-zone nonlinear input, that is

$$\Psi(u(t)) = \begin{cases} 2(u(t) - 1.5), & u(t) \geq 1.5 \\ 0, & -1 \leq u(t) \leq 1.5 \\ 4(u(t) + 1), & u(t) \leq -1 \end{cases} \quad (59)$$

the parameters $m_1 = m_2 = m_3 = 5$, $c_1 = c_2 = c_3 = 2$, $\eta_{31} = 5$, $\eta_{32} = 2$, the initial conditions are chosen as $x_1(0) = -0.3$, $x_2(0) = 0.1$, $x_3(0) = -0.2$, $\hat{\delta}_3(0) = [0.1, 0.1, 0.1, 0.1]^T$, $\hat{\beta}_{31}(0) = 0$, $\hat{\beta}_{32}(0) = 0$, $\alpha = 0.8$. Once the actual controller $u(t)$ is activated, the state trajectories of subsystem with transformation variables are presented in Figure 2, it is clearly that all state trajectories tends to zero gradually, which demonstrates that using the proposed control scheme, the adaptive stabilization of the controlled system with nonsymmetrical dead-zone nonlinear input is achieved.

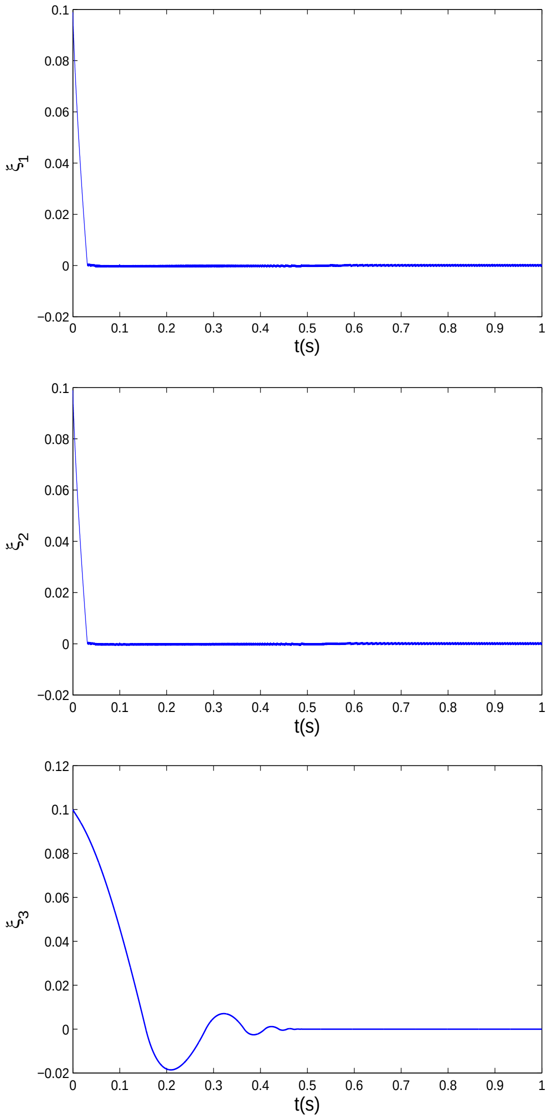

Figure 2. Time history of transformation system with nonsymmetrical dead-zone input

All above simulation results sufficiently demonstrated that the proposed control scheme is effective in stabilizing this kind of uncertain fractional-order nonlinear system with nonsymmetrical dead-zone input.

V. Conclusions

This paper investigated a backstepping-based sliding mode control scheme for adaptive stabilization of a class of fractional-order system. The system is perturbed by unknown bounded uncertainties, and system parameters are unknown in advance. The effect of nonsymmetrical dead-zone nonlinear input is taken into account in the design of actual controller. In order to compensate the influence of nonlinear input, an auxiliary fractional-order system is introduced to generate the necessary virtual signal. To deal with the unknown parameters and unknown uncertainties, a proper sliding mode surface is established to determine the adaptive update laws. For verify the stability of the controlled system, the frequency distributed model is used so that indirect Lyapunov function can be applied. Simulation results demonstrated the feasibility and effectiveness of the proposed control scheme.

References

[1] I. Podlubny, "Fractional Differential Equations," San Diego: Academic Press, 1999.

[2] S. Victor, P. Meclchior, J. Levine, and A. Oustaloup, "Flatness for linear fractional systems with application to a thermal system," Automatica, vol. 50, no.12, pp3173-3181, 2014.

[3] R. L. Bagley and R. A. Calico, "Fractional order state equations for the control of viscoelastically damped structure," Journal of Guidance, Control, Dynamics, vol. 14, pp304-311, 1991.

[4] H. H. Sun, A. A. Abdelwahad, and B. Oharal, "Linear approximation of transfer function with a pole of fractional power," IEEE Transaction on Automation Control, vol. 29, pp441-444, 1984.

[5] M. Ichise, Y. Nagayanagi, and T. Kojima, "An analog simulation of non-integer order transfer functions for analysis of electrode process," Journal of Electroanalytical Chemistry and Interfacial Electrochemistry, vol. 33, pp253-265, 1971.

[6] O. Heaviside, "Electromagnetic Theory," London: Chelsea Publishing Company, 1971.

[7] A. Dumlu and K. Erenturl, "Trajectory tracking control for a 3-DOF parallel manipulator using fractional-order $PI^{\lambda}D^{\mu}$ control," IEEE Transactions on Industrial Electronics, vol. 61, pp 3417-3426, 2014.

[8] J. Ni, L. Liu, C. Liu, and X. Hu, "Fractional order fixed-time nonsingular terminal sliding mode synchronization and control of fractional order chaotic systems," Nonlinear Dynamics, vol. 89, pp2065-2083, 2017.

[9] H. Liu, Y. Pan, and Y. Chen, "Adaptive fuzzy backstepping control of fractional-order nonlinear systems," IEEE Transactions on Systems, Man and Cybernetics, vol. 47, pp2209-2217, 2017.

[10] M. P. Aghababa, S. Khanmohammadi, and G. Alizadeh, "Finite-time synchronization of two different chaotic systems with unknown parameters via sliding mode technique," Applied Mathematical Modelling, vol. 35, pp3080-3091, 2011.

[11] N. Bigdeli and H. A. Ziazi, "Finite-time fractional-order adaptive intelligent backsstepping sliding mode control of uncertain fractional-order chaotic systems," Journal of the Franklin Institute, vol. 354, pp160-183, 2017.

[12] R. Z. Luo, M. C. Huang, and H. P. Su, "Robust control and synchronization of 3-D uncertain fractional-order chaotic systems with external disturbances cia adding one power integrator control," Complexity, 8417536, 2019.

[13] M. K. Shukla and B. B. Sharma, "Backstepping based stabilizaiton and synchronizaiton of a class of fractional order chaotic systems," Chaos Solitons and Fractals, vol. 102, pp274-284, 2017.

[14] M. K. Shukla and B. B. Sharma, "Stabilizaiton of a class of fractional order chaotic systems via backstepping approach," Chaos Solitons and Fractals, vol. 98, pp56-62, 2017.

[15] Y. H. Wei, D. Sheng, Y. Q. Chen, and Y. Wang, "Fractional order chattering-free robust adaptive backstepping control technique," Nonlinear Dynamics, vol. 95, pp2383-2394, 2019.

[16] Y. H. Wei, Y. Q. Chen, S. Liang, and Y. Wang, "A novel algorithm on adaptive backstepping control of fractional order systems," Neurocomputing, vol. 165, pp395-402, 2015.

[17] Y. Li, Y.Q. Chen, and I. Podlubny, "Mittag-Leffler stability of frational order nonlinear dynamic systems," Automatica, vol. 45, pp1965-1969, 2009.

[18] J. C. Trigeassou, N. Maamri, J. Sabatier, and A. Oustaloup, "A Lyapunov approach to the stability of fractional differential equations," Signal Processing, vol. 91, pp437-445, 2011.

[19] Y. Wang, L. Liu, C.X. Liu, Z. W. Zhu, and Z. Q. Sun, "Fractional-order adaptive backstepping control of a noncommensurate fractional-order ferroresonance system," Mathematical Problems in Engineering, 8091757, 2018.

[20] S. M. Ha, H. Liu, S. G. Li, and A. J. Liu, "Backstepping-based adaptive fuzzy synchronization control for a class of fractional-order chaotic systems with input saturation," International Journal of Fuzzy Systems, vol. 21, pp1571-1584, 2019.

[21] Y. Q. Chen, Y. H. Wei, X. Zhou, and Y. Wang, "Stability for nonlinear fractional order system: an indirect approach," Nonlinear Dynamics, vol. 89, pp1011-1018, 2017.

[22] D. Sheng, Y. H. Wei, S. S. Cheng, and J. M. Shuai, "Adaptive backstepping control for fractional order systems with input saturation," Journal of the Franklin Institute, vol. 354, pp2245-2268, 2017.

[23] J. Sun, Y. Wu, G. Cui, and Y. Wang, "Finite-time real combination synchronization of three complex-variable chaotic systems with unknown parameters via sliding mode control," Nonlinear Dynamics, vol. 88, pp1677-1690, 2017.

[24] X. M. Tian and Z. Yang, "Adaptive Complex Modified Projective Synchronization of Two Fractional-order Complex-variable Chaotic Systems with Unknown Parameters," Engineering Letters, vol. 25, no.4, pp438-445, 2017.

[25] Z. C. Yang, Y. L. Zhao, and F. Z. Gao, "Fixed-Time Stabilization for A Wheeled Mobile Robot With Actuator Dead-Zones," IAENG International Journal of Computer Science, vol. 48, no.3, pp514-518, 2021.

[26] Z. Y. Feng, M. Z. Chen, L. H. Ye, and L. L. Wu, "The Distributional Solution of the Fractional-order Descriptor Linear Time-Invariant System and Its Application in Fractional Circuits," IAENG International Journal of Applied Mathematics, vol. 50, no.3, pp549-557, 2020.

[27] K. Kankhunthodl, V. Kongratana, A. Numsomran, and V. Tipsuwanporn, "Self-balancing Robot Control Using Fractional-Order PID Controller," Lecture Notes in Engineering and Computer Science: Proceedings of The International MultiConference of Engineers and Computer Scientists 2019, 13-15 March, 2019, Hong Kong, pp77-82.