Orbit Spaces and Pre-Bundles of CNF-Base Hypergraphs

Stefan Porschen

***Abstract*—In this paper an abstract discrete pre-bundle hierarchy is introduced and discussed. Numerous pre-bundles are identified for classes of CNF base hypergraphs over positive integer sets. Thereby the question whether such instances admit a given number of orbits of fibre-transversals provided by the CNF complementation group is attacked. Extending a preliminary version several more pre-bundles and properties of the orbit spaces are established.**

***Index Terms*—satisfiability, orbit, bundle, hypergraph, Mersenne-number**

I. Introduction

THE genuine and one of the most important NP-complete problems is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas [5]. SAT offers various applications due to the fact that instances of numerous computational problems can be encoded as equivalent instances of CNF-SAT via reduction [7]. From a theoretical perspective, on the one hand one is interested in classes for which SAT can be solved efficiently. Meanwhile there are known several such classes, namely quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested, co-nested formulas, and exact linear formulas etc. [1], [3], [4], [8], [9], [10], [11], [17], [20]. On the other hand, a general study of the CNF structure might yield new approaches to the SAT problem itself. Here we discuss a discrete (pre-)bundle concept and identify numerous discrete pre-bundles for subclasses of base hypergraphs of the CNF theory. First an abstract pre-bundle hierarchy is established and some of its properties are proven. Moreover the enlargement towards discrete fibre bundles is described briefly. In [15] another hierarchy, namely of diagonal base hypergraphs has been constructed. It remained open there whether instances on every level of this hierarchy exist at all. E.g. $\hat{\mathfrak{H}}_i$ contains all base hypergraphs such that there are exactly i orbits of unsatisfiable fibre-transversals determined by the action of the CNF complementation group. The question of determining the number of orbits of satisfiable fibre-transversals also is addressed here; however both problems are not resolved completely so far. Defining orbit maps assigning the number of orbits of its fibre-transversals to a base hypergraph enables one to interprete the mentioned existence questions within the pre-bundle structure: Every $\hat{\mathfrak{H}}_i$ then yields a fibre of the bundle. In this manner, extending the results of a previous, more preliminary paper [16], here we provide further discrete pre-bundles. So specifically for the class of exact linear base hypergraphs. Moreover, effective constructions are presented showing the existence of a base hypergraph even as a connected instance such that the number of orbits of its satisfiable fibre-transversals matches any given finite product of arbitrary Mersenne powers. Moreover, additional properties of the orbit maps are proven. Finally, we consider pre-bundles and sections of total clause sets admitting a fibre-stable action of the complementation group.

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Stefan Porschen, member of the Mathematics Group at Department 4, HTW Berlin, D-10313 Berlin, Germany (e-mail: porschen@htw-berlin.de).

II. Notation and Preliminaries

A Boolean or propositional variable, for short variable, x taking values from $\{0, 1\}$ can appear as a positive literal which is x or as a negative literal which is the negated variable $\overline{x}$ also called the *flipped* or *complemented* variable. Setting a literal to 1 means to set the corresponding variable accordingly. A clause c is a finite non-empty disjunction of different literals and it is represented as a set $c = \{l_1, \ldots, l_k\}$. A conjunctive normal form formula, for short formula, C is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. Let CNF be the collection of all formulas. For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). Let CNF_+ denote that part of CNF containing only clauses with no negated variables. Given $C \in \mathrm{CNF}$, SAT asks whether there is a truth assignment $t : V(C) \to \{0, 1\}$ such that there is no $c \in C$ all literals of which are set to 0. If such an assignment exists it is called a *model* of C, and let $\mathcal{M}(C)$ be the space of all models of C. Let $\mathrm{SAT} \subseteq \mathrm{CNF}$ denote the collection of all clause sets for which there is a model, and $\mathrm{UNSAT} := \mathrm{CNF} \setminus \mathrm{SAT}$. For a (not necessarily finite) set M, let 2^M be its power set. The set of all positive integers is denoted by $\mathbb{N}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$, and $[0] := \emptyset$. Let $\mathbb{P}$ denote the set of all prime numbers, and let $\mathbb{M}$ be the collection of all Mersenne numbers. For a given (partial) mapping f, let $\mathrm{dom}(f)$ denote its domain, and $\mathrm{im}(f)$ its image. Further denote the (proper) restriction of f to a subset $A \subset \mathrm{dom}(f)$ by $f|_A$. As usual a *total* map is defined on the whole pre-image set. For a group G acting on a space M, meaning the existence of a map $G \times M \to M$: $(g, m) \mapsto m^g$, let $\mathcal{O}(m) := \{m^g : g \in G\}$ denote the orbit of $m \in M$ (under G) (cf. e.g. [18]). Membership to orbits clearly yields an equivalence relation on M. Its classes, i.e., the orbits are collected in the *orbit space* M/G which in case of a finite group and a finite space has the finite cardinality $|M|/|G|$. Given a non-empty set $A \subseteq M$ and $g \in G$, we set $A^g := \{m^g : m \in A\}$, and by convention $\emptyset^g := \emptyset$, for all $g \in G$. For $m \in M$, respectively $A \subseteq M$, let $G(m) := \{g \in G : m^g = m\}$, $G(A) := \{g \in G : A^g = A\}$ denote the isotropy group of m, respectively A. If $G(m) = G$, respectively $G(A) = G$, m respectively A, is a fixed point [18] of the corresponding action. Finally, 'iff' is used as an abbreviation for 'if and only if'.

III. A Hierarchy of Discrete Pre-Bundles

In this section we provide the basics of a discrete pre-bundle concept. (Continuous) fibre bundles are an important concept of topology and geometry (e.g. [6], [19]) and the concept restricted to discrete structures might be fruitful as well. Let I be any non-empty discrete (index) set. If I is infinite it is bijective to $\mathbb{N}$. In case I is finite it is bijective to $[n]$ where $n := |I|$. Note that I may have a discrete structure so that in general (even in the finite case) I needs not to be isomorphic to $\mathbb{N}$ (respectively $[n]$). Consider a disjoint union $K := \bigcup_{k\in I} K_k$, called the *total space*, of certain non-empty, mutually disjoint and (not necessarily finite) discrete spaces K_k, for every $k \in I$, called the *fibres*. Let $\pi : K \to I$ be a total map such that for every $\kappa \in K$ one requires $\pi(\kappa) := k$ iff $\kappa \in K_k$ ensuring the identity $K_k = \pi^{-1}(k)$. Hence π is surjective and is called the *(discrete bundle) projection* onto the *discrete base* I. The triple (K, I, π) is called a *discrete pre-bundle over* I. A *(partial) section* of the discrete pre-bundle is a (partial) mapping $s : I \to K$ such that $\pi|_{\mathrm{im}(s)} \circ s = \mathrm{id}_{\mathrm{dom}(s)}$. A total section s, we shall also call a *(discrete) fibre-transversal*, because $\mathrm{im}(s)$ contains exactly one member from every fibre. Let $\mathcal{S}(I, K)$ denote the set of all total sections of (K, I, π). Let $\mathcal{K} := \{\hat{\kappa} \in 2^K : \exists k \in I \forall \kappa \in \hat{\kappa}, \pi(\kappa) = k\}$ and define the total map $\hat{\pi} : \mathcal{K} \to I$ induced by π such that for every $\hat{\kappa} \in \mathcal{K}$ one has $\hat{\pi}(\hat{\kappa}) := k$ iff $\hat{\kappa} \subseteq K_k$ and $\hat{\kappa} \neq \emptyset$. Set $(\mathcal{K})_k := \hat{\pi}^{-1}(k)$. A (partial) *(multi-)section* is a (partial) mapping $\hat{s} : I \to \mathcal{K}$ such that $\hat{\pi}|_{\mathrm{im}(\hat{s})} \circ \hat{s} = \mathrm{id}_{\mathrm{dom}(\hat{s})}$. Hence for every $k \in \mathrm{dom}(\hat{s})$ one has $\hat{s}(k) \subseteq K_k$. As is explained next, a multi-section is no distinct concept. We set $\mathcal{K}_0 := K$, $\pi_0 := \pi$, and $\mathcal{K}_1 := \mathcal{K}$, $\hat{\pi} := \pi_1$. Furthermore, for any integer $\nu \geq 2$, defining

$$\mathcal{K}_\nu := \left\{\hat{\kappa} \in 2^{\mathcal{K}_{\nu-1}} : \exists k \in I \forall \kappa \in \hat{\kappa}, \pi_{\nu-1}(\kappa) = k\right\}$$

and $\pi_\nu : \mathcal{K}_\nu \to I$ such that for every $\hat{\kappa} \in \mathcal{K}_\nu$ one sets $\pi_\nu(\hat{\kappa}) := k$ iff $\emptyset \neq \hat{\kappa} \subseteq (\mathcal{K}_{\nu-1})_k$, where $(\mathcal{K}_{\nu-1})_k := \pi_{\nu-1}^{-1}(k)$, we obtain a hierarchy of discrete pre-bundles as follows:

Lemma 1: If $(\mathcal{K}_0, I, \pi_0)$ is a discrete pre-bundle over I then also $(\mathcal{K}_\nu, I, \pi_\nu)$ is a discrete pre-bundle over I, for every $\nu \in \mathbb{N}$.

Proof. The proof proceeds by induction on ν, where the base is clear. For fixed $\nu > 0$ assume that $(\mathcal{K}_{\nu-1}, I, \pi_{\nu-1})$ is a discrete pre-bundle over I. By definition of $\pi_{\nu-1}$, K_ν does not contain the emptyset. Now suppose there are $k_1, k_2 \in I$ such that $(\emptyset \neq)\hat{\kappa} \in \pi_\nu^{-1}(k_1) \cap \pi_\nu^{-1}(k_2)$ then $\hat{\kappa} \subseteq (\mathcal{K}_{\nu-1})_{k_1}$, $\hat{\kappa} \subseteq (\mathcal{K}_{\nu-1})_{k_2}$, by definition meaning that for all $\kappa \in \hat{\kappa}$ we have $\pi_{\nu-1}(\kappa) = k_1$ and also $\pi_{\nu-1}(\kappa) = k_2$. As $\hat{\kappa}$ contains at least one member one has $k_1 = k_2$ ensuring that π_ν is a well-defined (partial) map which of course is total by construction. Finally, let $k \in I$ then there is $\kappa \in \mathcal{K}_{\nu-1}$ with $k = \pi_{\nu-1}(\kappa)$ by its surjectivity therefore $\{\kappa\} \in \mathcal{K}_\nu$ by its definition and so π_ν also is surjective. □

In view of the preceding discussion a (partial) multi-section of $(\mathcal{K}_{\nu-1}, I, \pi_{\nu-1})$ appears as a (partial) section of $(\mathcal{K}_\nu, I, \pi_\nu)$, for every fixed $\nu \in \mathbb{N}$. Let G be a group acting on the total space K of the pre-bundle (K, I, π) such that each fibre remains invariant, hence one requires $\kappa^g \in K_k$ iff $\kappa \in K_k$ for every $k \in I$ and $g \in G$. Let us call this action *fibre-stable*. Using the notation as above one obtains:

Proposition 1: If G acts fibre-stable on $(\mathcal{K}_0, I, \pi_0)$ then a fibre-stable G-action on $(\mathcal{K}_\nu, I, \pi_\nu)$, $\nu \in \mathbb{N}$, is induced.

Proof. Proceeding by induction on ν, fix $\nu > 0$ and assume that G acts fibre-stable on the discrete pre-bundle $(\mathcal{K}_{\nu-1}, I, \pi_{\nu-1})$. For any $\hat{\kappa} \in \mathcal{K}_\nu$ and $g \in G$ we set by induction $\hat{\kappa}^g := \{\kappa^g : \kappa \in \hat{\kappa}\}$. As K_ν does not contain the empty set, on basis of Lemma 1 one then has $\hat{\kappa}^g \in (\mathcal{K}_\nu)_k$ iff $k = \pi_\nu(\hat{\kappa}^g)$ iff $k = \pi_{\nu-1}(\kappa^g)$, for all $\kappa^g \in \hat{\kappa}^g$ iff, by the induction hypothesis, $k = \pi_{\nu-1}(\kappa)$, for all $\kappa \in \hat{\kappa}$ iff $k = \pi_\nu(\hat{\kappa})$ iff $\hat{\kappa} \in (\mathcal{K}_\nu)_k$, from which the claim follows. □

Given a fibre-stable action of G, let $\varphi_\nu := \{\kappa \in \mathcal{K}_\nu : \forall g \in G, \kappa^g = \kappa\}$ denote the set of fixed points in $\mathcal{K}_\nu$, $\nu \in \mathbb{N}_0$.

Proposition 2: Let G act fibre-stable on $(\mathcal{K}_0, I, \pi_0)$, and let $\kappa \in \mathcal{K}_{\nu-1}$, for fixed $\nu \in \mathbb{N}$, then there is a unique $k \in I$ such that the G-orbit of κ satisfies $\mathcal{O}(\kappa) \in (\mathcal{K}_\nu)_k$. Moreover $2^{\varphi_{\nu-1}} \subseteq \varphi_\nu$.

Proof. Clearly by the pre-bundle property due to Lemma 1 there is a unique $k \in I : \pi_{\nu-1}(\kappa) = k$, for all $\nu > 0$. Thus $\kappa \in (\mathcal{K}_{\nu-1})_k$, and by the fibre-stable action of G one has $\{\kappa^g : g \in G\} = \mathcal{O}(\kappa) \subseteq (\mathcal{K}_{\nu-1})_k$, from which the first claim follows. Moreover if $\varphi_{\nu-1} = \emptyset$ then the second claim obviously holds true, as by convention $\emptyset^g = \emptyset$, $g \in G$. Otherwise let $\emptyset \neq \hat{\kappa} \in 2^{\varphi_{\nu-1}}$ then $\hat{\kappa}^g = \{\kappa^g : \kappa \in \hat{\kappa}\} = \hat{\kappa}$, for all $g \in G$. Thus $\hat{\kappa} \in \varphi_\nu$. □

Regarding the isotropy group of sections one has:

Theorem 1: Let $\nu \in \mathbb{N}_0$ and $s \in \mathcal{S}(I, \mathcal{K}_\nu)$ arbitrary. Then the isotropy group of $\mathrm{im}(s)$ is given by: $G(\mathrm{im}(s)) = \bigcap_{k\in I} G(s(k))$.

Proof. Let $g \in G(\mathrm{im}(s))$, then $\mathrm{im}(s) = \mathrm{im}(s)^g = \{s(k)^g : k \in I\}$. Now $s(k) \in (\mathcal{K}_\nu)_k$ is equivalent with $s(k)^g \in (\mathcal{K}_\nu)_k$, $k \in I$, because G acts fibre-stable. Since the spaces $(\mathcal{K}_\nu)_k$, $k \in I$, are mutually disjoint one obtains $s(k) = s(k)^g$ for all $k \in I$. Therefore $g \in \bigcap_{k\in I} G(s(k))$ which clearly is a subgroup of G. The reverse inclusion is obvious and the assertion is proven. □

Next we briefly describe how the pre-bundle notion can be extended to a discrete fibre-bundle structure. Let (K, I, π) be a discrete pre-bundle and let $U \neq \emptyset$ be such that there are bijections $\phi_k : U \to K_k$, for every $k \in I$. Clearly, if every K_k is equipped with a specific structure one would require that U carries the same structure and that these bijections are isomorphisms in the categorical sense. Now the tuple $(K, I, \pi, U, \mathrm{Aut}(U))$ is called a discrete (fibre) bundle, where $\mathrm{Aut}(U)$ is the group of all automorphisms of U. Let $\{I_\mu\}$ be an arbitrary family of *discrete local neighborhoods* where $\emptyset \neq I_\mu \subset I, |I_\mu| < \infty$, and the μ are taken from any suitable index set J. For every $\mu \in J$ there is a unique positive integer n_μ with $|I_\mu| = n_\mu$. Further assume $\bigcup_{\mu\in J} I_\mu = I$ and that there are bijections $\phi_\mu : I_\mu \times U \to \pi^{-1}(I_\mu)$ such that $\phi_\mu(k, u) \in K_k$, for every $(k, u) \in I_\mu \times U$. Every pair (I_μ, ϕ_μ), $\mu \in J$, is called a *local trivialisation* of the discrete bundle. For fixed $k \in I_\mu$ let

$$\phi_{\mu,k} := \phi_\mu(k, \cdot) : U \to K_k$$

we then have for any $k \in I_{\mu_1} \cap I_{\mu_2} \neq \emptyset$,

$$\phi_{\mu_2,k}^{-1} \circ \phi_{\mu_1,k} : U \to U$$

which obviously is a member of $\mathrm{Aut}(U)$. The collection of all pairs $\{(I_\mu, \phi_\mu)_{\mu\in J}\}$ plays the role of a *discrete bundle atlas*. If specifically U is finite with $N := |U|$, then also $|K_k| = N$, for all $k \in I$, and $\mathrm{Aut}(U) = S_N$ becomes the finite symmetric group of degree N and of order $N!$. Conversely, i.e., no pre-bundle is given, suppose that there is a (structure-preserving) bijection $\phi_k : U \to K_k$ mapping a space U to every member of a collection of mutually disjoint spaces K_k, $k \in I$, where I is a discrete (possibly infinite) index set. Then the following mapping is induced for every finite $I_n = \{k_1, \ldots, k_n\} \subset I$:

$$\phi_n : I_n \times U \ni (k_j, u) \mapsto \phi_{k_j}(u) \in \bigcup_{j=1}^{n} K_{k_j}$$

Now we have $J = \mathbb{N}$ for the index set. The latter mappings are (structure-preserving) bijections, because each ϕ_{k_j}, for fixed $k_j \in I_n$ is such a bijection on the whole of K_{k_j}. Further one has

$$\pi^{-1}(I_n) := \bigcup_{j=1}^{n} K_{k_j}$$

as a disjoint union, and for fixed $\kappa \in \pi^{-1}(I_n)$, there is a unique $j \in [n]$ such that $\kappa \in K_j$. Now $\phi_{n,k} = \phi_k, \forall k \in I_n$, and every positive integer n. It follows that the map

$$\pi : \bigcup_{k \in I} K_k \to I$$

thereby defined via $\pi(\kappa) = k$ iff $\kappa \in K_k$ is total and surjective. Note that the fibre-bundle constructed in this manor is trivial in the sense that K is bijective to $I \times U$. Here one typically would have $I \in \{\mathbb{N}, \mathbb{Z}\}$, but also other possibly structured discrete index sets might be useful.

IV. Properties of the Orbit Maps

Recall that the hyperedge set $B(C)$ of the base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ of a formula $\emptyset \neq C \in$ CNF is $B(C) := \{V(c) : c \in C\} \in \text{CNF}_+$. Also a given hypergraph $\mathcal{H} = (V, B)$ serves as a base hypergraph if its vertex set V is a finite non-empty set of Boolean variables such that for every $x \in V$ there is a $b \in B$ containing x. Thus ensuring $B \neq \emptyset$, which is assumed throughout. Recall that a loop is a hyperedge containing exactly one vertex [2]. Let $\mathfrak{H}$ denote the space of all (finite) base hypergraphs of non-empty formulas, and $\mathfrak{H}^c$ the fraction of connected instances. A hypergraph $\mathcal{H} = (V, B)$ is called connected if its intersection graph is connected in the usual sense. Here the intersection graph of $\mathcal{H} = (V, B)$ gets a vertex for each $b \in B$ and there is exactly one edge joining a pair of vertices $b \neq b'$ iff $b \cap b' \neq \emptyset$. A hypergraph $\mathcal{H} = (V, B)$ is *linear* if $|b \cap b'| \leq 1$, for all distinct $b, b' \in B$. Further $\mathcal{H}$ is *exact linear* if $\leq$ above is replaced with $=$. Let $\mathfrak{H}_{\text{lin}}$ denote the subclass of all loopless, linear base hypergraphs, and let $\mathfrak{H}^c_{\text{lin}}$ be its subclass of connected instances. Analogously, we define $\mathfrak{H}_{\text{xlin}} \subseteq \mathfrak{H}^c_{\text{lin}}$ as the proper subclass of exact linear instances. Observe that the base hypergraph $\mathcal{H}(C)$ is (exact) linear if the formula C is (exact) linear. Moreover $\mathcal{H}(C)$ is loopless if C is free of unit clauses. Recall that a hypergraph is called Sperner if none of its hyperedges contains another hyperedge [2]. Let $\mathfrak{H}_{\text{sper}}$ be the collection of those instances. Obviously, $\mathfrak{H}_{\text{xlin}} \subseteq \mathfrak{H}_{\text{lin}} \subseteq \mathfrak{H}_{\text{sper}}$. For simplicity let $\mathcal{H} \cup \{b\}$ be the hypergraph having the same vertex set as $\mathcal{H} = (V, B)$ and the edge set $B \cup \{b\}$. By $W_b := \{c : V(c) = b\}$ denote the collection of all clauses over a fixed $b \in B$. As usual $C_b = C \cap W_b$ is the fibre over b of a formula $C \in$ CNF [12]. The set of all clauses over $\mathcal{H} = (V, B)$ is the total clause set $K_{\mathcal{H}} := \bigcup_{b \in B} W_b$. A *fibre-transversal* F of $K_{\mathcal{H}}$ contains exactly one member of every W_b, $b \in B$, namely $F(b)$ and $\mathcal{F}(K_{\mathcal{H}})$ is the set of all fibre-transversals [12]. Note that F can be viewed as a map or as a formula which formally results as the image of the map F. It shall become clear which view of F is meant in either context. A *compatible* fibre-transversal satisfies $\bigcup_{b \in B} F(b) \in W_V$, let them be collected in $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$. Observe that the model space $\mathcal{M}(C) \subseteq \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$, if $\mathcal{H}(C) = \mathcal{H}$. To that end, a truth assignment $t : V \to \{0, 1\}$ is identified with the clauses $t(b) := \{x : x \in b \wedge t(x) = 1\} \cup \{\bar{x} : x \in b \wedge t(x) = 0\}$, for all $b \in B$. A *diagonal* fibre-transversal has a non-empty intersection with every member of $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$, they are collected in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. A base hypergraph $\mathcal{H}$ is called diagonal [13] iff $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \emptyset$. Further $\mathcal{H}$ is minimal diagonal if it does not contain another diagonal base hypergraph [14]. Set $\mathfrak{H}_{\text{mdiag}} \subseteq \mathfrak{H}^c$ for the class of all minimal diagonal instances. Clause c^X results from c via complementing all variables in $X \cap V(c)$, for $X \subseteq V$. As considered in [14] let $G_V := (2^V, \oplus)$ denote the finite complementation group with neutral element $\emptyset$ inducing this flipping action on CNF by observing that $\{c\} \in$ CNF. By $\mathcal{O}(C) := \{C^X : X \in G_V\}$ denote the (G_V-)orbit of C in CNF. Given $\mathcal{H} = (V, B) \in \mathfrak{H}$, as defined in [14], $\omega(\mathcal{H})$ denotes the number of all such orbits in $\mathcal{F} := \mathcal{F}(K_{\mathcal{H}})$, i.e. $\omega(\mathcal{H}) = |\mathcal{F}/G_V|$, and $\delta(\mathcal{H})$ is the cardinality of the orbit space $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})/G_V$. As also defined in [14], let $\beta : \mathfrak{H} \to \mathbb{N}_0$ where $\beta(\mathcal{H}) = \sum_{b \in B} |b| - |V|$, for every $\mathcal{H} \in \mathfrak{H}$. One has $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$ which also directly follows from the (so-called) Burnside orbit Lemma for finite groups: The number of orbits of a group action is determined by the sum of the cardinalities of all its fixed point sets. Here, every fixed point set $\mathcal{F}_X := \{F \in \mathcal{F} : F^X = F\} = \emptyset$, for $X \in G_V$ with $X \neq \emptyset$, and $\mathcal{F}_\emptyset = \mathcal{F}$. Hence $\sum_{X \in G_V} |\mathcal{F}_X| = |\mathcal{F}|$, and therefore $\omega(\mathcal{H}) = |\mathcal{F}|/2^{|V|}$. In [15] a further map on $\mathfrak{H}$ is defined, namely $\rho : \mathfrak{H} \to \mathbb{N}_0$, where for any $\mathcal{H} \in \mathfrak{H}$, $\rho(\mathcal{H})$ denotes the number of orbits with respect to the G_V-action of all fibre-transversals in $\mathcal{F}(K_{\mathcal{H}})$ which are neither compatible nor diagonal. Set $\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}}) := \mathcal{F}(K_{\mathcal{H}}) \cap \text{SAT}$ $= \mathcal{F}(K_{\mathcal{H}}) \setminus \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. For short, the functions $\beta, \omega, \delta, \rho$ on $\mathfrak{H}$ determined by the cardinalities of the corresponding orbit spaces are refered to as the *orbit maps* (notice that $\beta = \log_2 \omega$). For convenience, a fixed fibre-transversal $F \in \mathcal{F}(K_{\mathcal{H}})$ may be refered to as the *orbit base* of its orbit $\mathcal{O}(F) \in \mathcal{F}(K_{\mathcal{H}})/G_V$.

Proposition 3: Let $\mathcal{H} = (V, B) \in \mathfrak{H}$.

(1) It is $\rho(\mathcal{H}) = 0$ iff $\mathcal{F}(K_{\mathcal{H}}) = \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ iff $\omega(\mathcal{H}) = 1$. In this case also $\delta(\mathcal{H}) = 0$.

(2) Let $\delta(\mathcal{H}) = 0$, $b \subseteq V$ be arbitrary such that $b \notin B$, and $\mathcal{H}' := \mathcal{H} \cup \{b\}$. If there is $F \in \mathcal{F}(K_{\mathcal{H}})$ with $|\mathcal{M}(F)| = 1$ then $\delta(\mathcal{H}') > 0$. Moreover, $\mathcal{O}(F) \in \mathcal{F}(K_{\mathcal{H}})/G_V$ determines exactly one orbit in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}'})/G_V$.

Proof. Let $\mathcal{H} \in \mathfrak{H}$ then $\omega(\mathcal{H}) = 1 + \rho(\mathcal{H}) + \delta(\mathcal{H})$ [15]. Therefore, $\omega(\mathcal{H}) = 1$ iff $\rho(\mathcal{H}) + \delta(\mathcal{H}) = 0$ iff $\rho(\mathcal{H}) = 0 = \delta(\mathcal{H})$ as both are non-negative. The last statement is equivalent with $\mathcal{F}(K_{\mathcal{H}}) = \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$. Finally, if $\rho(\mathcal{H}) = 0$ then $\omega(\mathcal{H}) = 1 + \delta(\mathcal{H})$. Assume $\delta(\mathcal{H}) \geq 1$ then there is $F \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ containing a minimal unsatisfiable subformula $\hat{F} \subseteq F$. Since $\delta(\mathcal{H}(\hat{F})) \geq 1$, $B(\hat{F})$ cannot consist of loops only. So, select any $b \in B(\hat{F})$ with $|b| \geq 2$. According to [15], Lemma 6, $\hat{F}_c := (\hat{F} \setminus \{\hat{F}(b)\}) \cup \{c\}$ is satisfiable, for any $c \in W_b$ with $c \neq \hat{F}(b)$. Hence choose c such that $\hat{F}_c$ is non-compatible. This is always possible: let t be a model of $\hat{F} \setminus \{\hat{F}(b)\}$ then every literal in $\hat{F}(b)$ is set by t to 0, and must occur outside $\hat{F}(b)$ as a complemented literal. As $|b| \geq 2$, one can choose c as required. Now let t_c be a model of $\hat{F}_c$ and extend $\hat{F}_c$ over the remaining part of B compatible with t_c yielding $F_1 \in \mathcal{F}(K_{\mathcal{H}}) \cap \mathrm{SAT}$. Thus we obtain $\rho(\mathcal{H}) > 0$ yielding a contradiction implying $\delta(\mathcal{H}) = 0$ thus $\omega(\mathcal{H}) = 1$ finishing (1). Assume that t is the unique model of $F \in \mathcal{F}(K_{\mathcal{H}})$. Then $F' := F \cup \{t(b)^b\}$ is a diagonal fibre-transversal of $\mathcal{H}'$, for any fixed $b \subseteq V$, $b \notin B$, because $V = V(\mathcal{H}')$. Let $F_0 \in \mathcal{O}(F)$ then there is $X \in G_V$: $F^X = F_0$ and clearly t^X is the unique model of F_0 otherwise F had further models. The resulting diagonal fibre-transversal $F'_0 := F_0 \cup \{t^X(b)\}^b = F_0 \cup \{t(b)^b\}^X \in \mathcal{O}(F')$. Hence the orbit $\mathcal{O}(F) \in \mathcal{F}(K_{\mathcal{H}})/G_V$ determines at least one orbit $\mathcal{O}(F') \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})/G_V$. Assume there is $\mathcal{O}(F'_1) \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})/G_V$ such that $\mathcal{O}(F'_1) \neq \mathcal{O}(F')$ but $F_1 := F'_1 \setminus F'_1(b) \in \mathcal{O}(F)$. Hence there is $X_1 \in G_V$: $F^{X_1} = F_1$ and t^{X_1} is the unique model of F_1. Clearly, $(t^{X_1}(b))^b \in W_b$ is the unique clause unsatisfied by t^{X_1}, thus $(t^{X_1}(b))^b = (t(b)^b)^{X_1} = F'_1(b)$ otherwise $F'_1 \in \mathrm{SAT}$. In summary, $F'^{X_1} = F'_1$ providing a contradiction, therefore $\mathcal{O}(F)$ determines exactly one orbit in $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})/G_V$. □

So, if $\omega(\mathcal{H}) = 1$ then $\beta(\mathcal{H}) = 0$ and also $\rho(\mathcal{H}) = 0 = \delta(\mathcal{H})$. Call such an instance $\mathcal{H}$ a *trivial* base hypergraph. The next result states under which circumstances it might be possible, by adding one hyperedge, to jump from $\delta = 0$ to a possibly arbitrary value:

Proposition 4: Let $\mathcal{H} = (V, B)$ with $\delta(\mathcal{H}) = 0$. Let $b \subseteq V$, $b \notin B$, and $\mathcal{H}' = \mathcal{H} \cup \{b\}$. If there are exactly r distinct orbits $\mathcal{O}(F) \in \mathcal{F}(K_{\mathcal{H}})/G_V$ such that for each orbit base F there is $c(F) \in W_b$ with $t(b) = c(F)$, for all $t \in \mathcal{M}(F)$, then $\delta(\mathcal{H}') = r$.

PROOF. For $r = 0$, assume that there is $F' \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})$ implying $F := F' \setminus \{F'(b)\} \in \mathrm{SAT}$. So, let $t \in \mathcal{M}(F) \neq \emptyset$ be such that $F'(b) \neq t(b)^b \in W_b$. Such a model must exist, otherwise all models would obey $t(b) = c(F) := F'(b)^b$. Hence there is $x \in b$ and w.l.o.g. $x \in t(b)^b$, $\bar{x} \in F'(b)$. Thus $t \in \mathcal{M}(F')$ because $\bar{x} \in t(b)$ satisfies $F'(b)$, also. Therefore $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'}) = \emptyset$ and $\delta(\mathcal{H}') = 0$ as asserted. For fixed integer $r > 0$, let $F \in \mathcal{F}(K_{\mathcal{H}})$ with $t(b) = c$, for all $t \in \mathcal{M}(F)$. Assume that $F' := F \cup \{c^b\} \in \mathrm{SAT}$ with model t' specifically satisfying c^b. Thus $t' \cap c^b \neq \emptyset$ meaning $t'(b) \neq c$ yielding a contradiction because also $t' \in \mathcal{M}(F)$. Therefore $\mathcal{O}(F') \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})/G_V$ as the satisfiability status clearly is invariant on orbits. Suppose there is $\mathcal{O}(\breve{F}) \in \mathcal{F}(K_{\mathcal{H}})/G_V$ with $\breve{t}(b) = \breve{c}$, for all $\breve{t} \in \mathcal{M}(\breve{F})$, and such that $\mathcal{O}(\breve{F}') = \mathcal{O}(F')$, where $\breve{F}' := \breve{F} \cup \{\breve{c}^b\} \in \mathrm{UNSAT}$. So, there is $X \in G_V$ such that $\breve{F}'^X = \breve{F}^X \cup \{\breve{c}^b\}^X = F' = F \cup \{c^b\}$. Hence, as all members of this equation are fibre-transversals, and only $\breve{c}^b, c^b \in W_b$, it follows that $\breve{F}^X = F$ meaning $\mathcal{O}(\breve{F}) = \mathcal{O}(F)$. Therefore $\delta(\mathcal{H}') \geq r$. Suppose there are distinct orbits $\mathcal{O}(F')$, $\mathcal{O}(\tilde{F}') \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})/G_V$ such that $\mathcal{O}(F) = \mathcal{O}(\tilde{F}) \in \mathcal{F}(K_{\mathcal{H}})/G_V$, where $F := F' \setminus F'(b)$ and $\tilde{F} := \tilde{F}' \setminus \tilde{F}'(b) \in \mathrm{SAT}$. Then there is $X \in G_V$ such that $F^X = \tilde{F}$. Moreover, $t(b) = F'(b)^b \in W_b$, for all $t \in \mathcal{M}(F)$, otherwise F' could be satisfied. Analogously, $\tilde{t}(b) = \tilde{F}'(b)^b \in W_b$, for all $\tilde{t} \in \mathcal{M}(\tilde{F})$. Thus $t^X(b) = (F'^X(b))^b \in W_b$, for all $t^X \in \mathcal{M}(F^X) = \mathcal{M}(\tilde{F})$ implying $\tilde{F}'(b)^b = F'^X(b)^b$. Hence, $\tilde{F}' = F'^X$ yielding a contradiction implying $\mathcal{O}(F') = \mathcal{O}(\tilde{F}')$. Therefore $\delta(\mathcal{H}') \leq r$, and in summary $\delta(\mathcal{H}') = r$. □

The converse statement also holds true in the following sense:

Proposition 5: Let $\mathcal{H}' = (V, B)$ and $b \in B$ with $\delta(\mathcal{H}) = 0$ for $\mathcal{H} = (V, B \setminus \{b\})$. If $\delta(\mathcal{H}') = r$ then there are exactly r distinct orbits $\mathcal{O}(F) \in \mathcal{F}(K_{\mathcal{H}})/G_V$ such that for each orbit base F there is $c(F) \in W_b$ with $t(b) = c(F)$, for all $t \in \mathcal{M}(F)$.

PROOF. First assume $r = 0$ and suppose there is $b \in B$ with $\delta(\mathcal{H}) = 0$ where $\mathcal{H} = (V, B \setminus \{b\})$ and let $F \in \mathcal{F}(K_{\mathcal{H}})/G_V$ with $t(b) = c(F) \in W_b$, for all $t \in \mathcal{M}(F)$. Then $F \cup \{c(F)\}^b = F' \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})$ providing a contradiction: Any model t' of F' also satisfies F, thus $t' \in \mathcal{M}(F)$ thus $t'(b) = c(F)$ meaning $t'(b) \cap c(F)^b = \emptyset$. For $r > 0$, let $F' \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})$ then $F := F' \setminus F'(b) \in \mathcal{F}(K_{\mathcal{H}}) \cap \mathrm{SAT}$ because $\delta(\mathcal{H}) = 0$. Suppose there is $t \in \mathcal{M}(F)$ with $t(b) \neq F'(b)^b =: c(F)$ then t satisfies $F'(b)$ and $t \in \mathcal{M}(F')$, a contradiction. So for all $t \in \mathcal{M}(F)$ one has $t(b) = c(F)$. For $\tilde{F}' \in \mathcal{O}(F')$ there is $X \in G_V$: $F'^X = \tilde{F}'$, so $\tilde{F} := \tilde{F}' \setminus \tilde{F}'(b) = F'^X \setminus F'^X(b) = (F' \setminus F'(b))^X = F^X$. yielding $\tilde{F} \in \mathcal{O}(F)$. So every orbit in $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})/G_V$ determines at least one orbit in $\mathcal{F}(K_{\mathcal{H}})/G_V$ with the asserted property. Finally suppose there is $\mathcal{O}(\breve{F}) \neq \mathcal{O}(F)$ with the asserted property but there is $c \in W_b$ such that $\breve{F}' := \breve{F} \cup \{c\} \in \mathcal{O}(F')$. Again there is $Y \in G_V$ with $F'^Y = F^Y \cup \{F'(b)\}^Y = \breve{F}' = \breve{F} \cup \{c\} = \breve{F} \cup \{\breve{F}'(b)\}$. Here the last equality is valid as $t(b) = \breve{F}'(b)^b$ for every $t \in \mathcal{M}(\breve{F})$, thus $c = \breve{F}'(b)$, otherwise $\breve{F}' \in \mathrm{SAT}$. Thus $F^Y = \breve{F}$, a contradiction. So every orbit in $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}'})/G_V$ determines at most one orbit in $\mathcal{F}(K_{\mathcal{H}})/G_V$ with the asserted property finishing the proof. □

The combination of both the previous Propositions yield:

Theorem 2: Let $\mathcal{H} = (V, B)$ with $\delta(\mathcal{H}) = 0$ and $b \subseteq V$, $b \notin B$, with $\mathcal{H}' = \mathcal{H} \cup \{b\}$. Then $\delta(\mathcal{H}') = r$ iff there are exactly r distinct orbits $\mathcal{O}(F) \in \mathcal{F}(K_{\mathcal{H}})/G_V$ such that for each orbit base F there is $c(F) \in W_b$ with $t(b) = c(F)$, for all $t \in \mathcal{M}(F)$.

For any $\mathcal{H}_0 = (V_0, B_0)$, $\mathcal{H} = (V, B) \in \mathfrak{H}$, as usual one sets $\mathcal{H}_0 \subseteq \mathcal{H}$ iff $V_0 \subseteq V$ and $B_0 \subseteq B$. Furthermore, let $\mathcal{H}_0 \preceq \mathcal{H}$ be defined by $V_0 \subseteq V$ and for every $b_0 \in B_0$ there is $b \in B$ such that $b_0 \subseteq b$.

Lemma 2: For the binary relations $\subseteq, \preceq$ on $\mathfrak{H}$ one has:

(1) $\subseteq\, \subset\, \preceq$ as a proper inclusion,
(2) $\preceq$ is reflexive, transitive, but not antisymmetric,
(3) $\preceq$ is a partial order restricted to $\mathfrak{H}_{\mathrm{sper}}$.

PROOF. Obviously $\subseteq\, \subseteq\, \preceq$. Let $V_0 = V = \{x_i : i \in [4]\}$, $B_0 = \{b_1, b_2, b_3\}$ with $b_1 = \{x_1\}$, $b_2 = \{x_2\}$, $b_3 = \{x_3, x_4\}$, and $B = \{b_2, b_3, b\}$, where $b = \{x_1, x_3, x_4\}$. Then $\mathcal{H}_0 \not\subseteq \mathcal{H}$, but $\mathcal{H}_0 \preceq \mathcal{H}$, so (1) is true. Clearly, $\preceq$ is reflexive. Let $\mathcal{H}_0 \preceq \mathcal{H}' = (V', B')$ and $\mathcal{H}' \preceq \mathcal{H}$ then clearly $V_0 \subseteq V$. For $b \in B_0$ there is $b' \in B'$ containing it, and there is $b \in B$ such that $b_0 \subseteq b' \subseteq b$, hence the transitivity of $\preceq$ follows. Let $V = V_0 = \{x_1, x_2\}$, $b = \{x_1, x_2\}$, $B = \{b\}$ and $B_0 = B \cup \{\{x_i\} : i \in [2]\}$, then $\mathcal{H}_0 \preceq \mathcal{H}$ and $\mathcal{H} \preceq \mathcal{H}_0$ but $\mathcal{H}_0 \neq \mathcal{H}$ implying (2). For verifying (3) it therefore suffices to establish antisymmetry of $\preceq$ restricted to arbitrary instances $\mathcal{H}_0, \mathcal{H} \in \mathfrak{H}_{\mathrm{sper}}$: Assume $\mathcal{H}_0 \preceq \mathcal{H}$ and $\mathcal{H} \preceq \mathcal{H}_0$ implying $V_0 = V$. Let $b_0 \in B_0$ then there is $b \in B$ with

$b_0 \subseteq b$, in turn there is $b_0' \in B_0$ containing b meaning $b_0 \subseteq b_0'$ hence $b_0 = b_0'$ as $\mathcal{H}_0$ is Sperner. Thus $b \subseteq b_0$ implying $b_0 = b \in B$ yielding $B_0 \subseteq B$. Exchanging the roles of B, B_0, analogously implies the reverse inclusion, hence $B_0 = B$ finishing the proof. □

Regarding the monotony of the orbit maps on these partially ordered spaces one has:

Proposition 6: Let $\mathfrak{H}^{\subseteq} = (\mathfrak{H}, \subseteq)$, $\mathfrak{H}^{\preceq}_{\text{sper}} = (\mathfrak{H}_{\text{sper}}, \preceq)$.

(1) $\alpha : \mathfrak{H}^{\subseteq} \to \mathbb{N}_0$ is monotone, for $\alpha \in \{\beta, \omega, \delta\}$.

(2) $\gamma : \mathfrak{H}^{\preceq}_{\text{sper}} \to \mathbb{N}_0$ is non-monotone, for $\gamma \in \{\beta, \omega, \delta, \rho\}$.

PROOF. Suppose $\mathcal{H}_0 \subseteq \mathcal{H}$. If $V = V_0$ then clearly $\beta(\mathcal{H}) \geq \beta(\mathcal{H}_0)$. In the remaining case, every $x \in V \setminus V_0$ occurs in at least one $b \in B$ by the definition of base hypergraphs. Hence it is $\beta(\mathcal{H}) \geq \beta(\mathcal{H}_0)$ also in this case, directly implying $\omega(\mathcal{H}) \geq \omega(\mathcal{H}_0)$ yielding that β, ω are monotone on $\mathfrak{H}^{\subseteq}$. Assume there is $F_0 \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$, otherwise $\delta(\mathcal{H}) \geq \delta(\mathcal{H}_0) = 0$. As $B_0 \subseteq B$, possibly extending F_0 to F over B cannot yield a satisfiable transversal. Therefore $|\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})| \leq |\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})|$. Moreover, let $\mathcal{O}(F_0), \mathcal{O}(F_0')$ be distinct orbits in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})/G_{V_0}$, and assume there are extensions of F_0, F_0' to F, F' over B, respectively, such that $\mathcal{O}(F) = \mathcal{O}(F')$ in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})/G_V$. Thus there is $X \in G_V$ such that $F^X = F'$ equivalent to $F_0^X \cup (F \setminus F_0)^X = F_0' \cup F \setminus F_0'$ as disjoint unions on both sides of the equation. No clause $c \in F \setminus F_0$ can be mapped via X to a clause in F_0' as $b(c) \in B \setminus B_0$, and for the analogous reason no clause from F_0 can be transformed to a clause in $F' \setminus F_0'$. Thus $F_0^{X \cap V_0} = F_0'$ yielding a contradiction. One concludes that for every orbit in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})/G_{V_0}$ there is an orbit in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})/G_V$ so δ is monotone on $\mathfrak{H}^{\subseteq}$. For verifying (2), let $V_0 = \{x, x_1, x_2\} = V$ $B_0 = \{b_1, b_2\}$, $B = \{b\}$ where $b_1 = \{x, x_1\}, b_2 = \{x, x_2\}, b = \{x, x_1, x_2\}$. Then clearly $\mathcal{H}_0 \preceq \mathcal{H}$ and $\mathcal{H}_0, \mathcal{H} \in \mathfrak{H}_{\text{sper}} \cap \mathfrak{H}_{\text{xlin}}$, hence $\delta(\mathcal{H}_0) = \delta(\mathcal{H}) = 0$. Further one has $\beta(\mathcal{H}_0) = 1, \beta(\mathcal{H}) = 0$ and $\rho(\mathcal{H}_0) = 2 - 1 = 1, \rho(\mathcal{H}) = 1 - 1 = 0$ proving that neither of β, ω, ρ in general is monotone on $\mathfrak{H}^{\preceq}_{\text{sper}}$. Next recall that every unsatisfiable loopless linear formula C is a diagonal fibre transversal. Hence $\mathcal{H}_0 := \mathcal{H}(C) \in \mathfrak{H}_{\text{lin}}$ and is therefore Sperner, moreover $\delta(\mathcal{H}_0) > 0$, where the existence of C is ensured according to [17] (a concrete small example is provided e.g. in the proof of Thm. 8, [15]). Let $\mathcal{H} = (V, B)$ be obtained from $\mathcal{H}_0$ by introducing a new variable $x(b_0)$, for every $b_0 \in B_0$. Then setting $V := V_0 \cup \{x(b_0) : b_0 \in B_0\}$ and $B := \{b_0 \cup \{x(b_0)\} : b_0 \in B_0\}$ yields a linear thus Sperner instance with $\mathcal{H}_0 \preceq \mathcal{H}$. Let $F \in \mathcal{F}(K_{\mathcal{H}})$ be arbitrary. As the clause $F(b)$ contains a unique literal from $\{x(b_0), \bar{x}(b_0)\}$, for all $b \in B$, one has $F \in \text{SAT}$. Therefore $\delta(\mathcal{H}) = 0 < \delta(\mathcal{H}_0)$, so δ is non-monotone on $\mathfrak{H}^{\preceq}_{\text{sper}}$. □

Let $\bar{\mathfrak{H}}$ denote the class of all base hypergraphs free of trivial components.

Theorem 3: For $\mathcal{H} \in \bar{\mathfrak{H}}$ one has:

(1) If $\delta(\mathcal{H}) = 1$ then $\mathcal{H} \in \mathfrak{H}^c$.

(2) There is $\mathcal{H} \notin \mathfrak{H}^c$ with $\rho(\mathcal{H}) < \delta(\mathcal{H})$, $\delta(\mathcal{H}) \equiv 1 \bmod 2$.

PROOF. For $\mathcal{H}$ ($\mathcal{H}_i$) set $\alpha(\mathcal{H}) := \alpha$ ($\alpha(\mathcal{H}_i) := \alpha_i$), $\alpha \in \{\beta, \delta, \rho, \omega\}$. Assume that $\mathcal{H}$ is composed of two disjoint components $\mathcal{H}_i$, $i = 1, 2$ such that at least $\mathcal{H}_1$ has $\delta_1 \geq 1$, then $\delta \geq 1$ as $\mathcal{H} \supseteq \mathcal{H}_1$ due to its monotony. In general due to [15], Lemma 1, one has for such a disjoint union $\delta(\mathcal{H}) = \delta_1\omega_2 + \delta_2\omega_1 - \delta_1\delta_2$ and $\rho(\mathcal{H}) = \rho_1 + \rho_2 + \rho_1\rho_2$. So, if $\delta_2 = 0$ one has $\delta > \delta_1$ as $\beta_2 > 0$. Otherwise, as $\omega_1 > \delta_1$, $\delta > \delta_1\omega_2 \geq \delta_1$. It also follows that if $\delta_i = 0$, $i = 1, 2$ then $\delta = 0$. Hence for a disconnected $\mathcal{H} \in \bar{\mathfrak{H}}$ either $\delta = 0$ or $\delta > 1$ thus (1). Regarding (2) consider $\mathcal{H}' = (V', B')$ with $V' = \{x_1, x_2\}$, $B' = \{b_1, b_2, b_3\}$ and $b_1 = \{x_1\}$, $b_2 = \{x_2\}$, $b_3 = \{x_1, x_2\}$, for which $\omega' = 4$, $\delta' = 1$ (as is easy to see) and $\rho' = 2$. Let $\mathcal{H}_1$ be two disjoint copies of $\mathcal{H}'$. Again using the combination formulas as above one obtains $\omega_1 = \omega'^2 = 16$, $\delta_1 = 7$ and $\rho_1 = 8$. Adding another disjoint copy of $\mathcal{H}'$ to $\mathcal{H}_1$ yields a disconnected $\mathcal{H} \in \bar{\mathfrak{H}}$ with $\omega = 64$, and odd $\delta = 37 > \rho = 26$. □

Definition 1: Let $\mathcal{H}_0 = (V_0, B_0)$, non-empty $b \notin B_0$ with $\mathcal{H} := (V, B)$, where $V := V_0 \cup b$, $B := B_0 \cup \{b\}$. Then the *fluctuation* f_b is the number of all orbits in $\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})/G_{V_0}$ which become orbits in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})/G_V$. Set $f_b := 0$ iff $V_0 \cap b = \emptyset$.

If $b \cap V_0 = \emptyset$ then $\mathcal{H}$ specifically is no bifurcation augmentation of $\mathcal{H}_0$ (cf. Def. 1, [15]). Hence, adapting Corollary 1 in [15], it is $\delta = \delta_0$ thus $f_b = 0$ which is in accordance with the setting above. For $\alpha(\mathcal{H}) =: \alpha, \alpha \in \{\rho, \delta\}$ then one obtains:

Theorem 4: Let b and $\mathcal{H}_0, \mathcal{H} \in \mathfrak{H}$ as in Definition 1.

(1) $\delta = 2^{|b \cap V_0|}\delta_0 + f_b$.

(2) If $b \setminus V_0 \neq \emptyset$ then $f_b = 0$.

(3) If for every $x \in b$ there is $\{x\} \in B_0$ then $f_b = \rho_0 + 1$.

(4) The condition in (3), and $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})$ contributes exactly 1 to f_b, are equivalent.

PROOF. (1) Let $j := |b \cap V_0| \in \mathbb{N}_0$. If $j = 0$ then $f_b = 0$, $\delta = \delta_0$ by the remark above. If $\delta_0 = 0$ then clearly $\delta = f_b$ by definition. Assume that $j, \delta_0 > 0$. Fixing an orbit base $F_k \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$ of every orbit in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})/G_{V_0}$, $k \in [\delta_0]$, yields mutually distinct fibre-transversals because their orbits are mutually disjoint. Let $b =: b' \cup \tilde{b}$ as disjoint union where $b' := V_0 \cap b$. Defining $F_k[c] := F_k \cup \{c\} \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ where $c := d \cup \tilde{b}$, for every $d \in W_{b'}$, yields the collection of 2^j mutually disjoint G_V-orbit bases in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ because all variables of b' occur as constant fixed literals in F_k. The negation of any member of $\tilde{b}$, if non-empty, results in an orbit member, hence yields no additional orbit base. Suppose there are $i, j \in [\delta_0]$, $i \neq j$, $c, c' \in W_b$ such that $\mathcal{O}(F_i[c]) = \mathcal{O}(F_j[c'])$, then there is $X \subseteq V$, $X_0 := X \cap V_0$ with $F_i[c]^X = F_i^{X_0} \cup \{c\}^X = F_j \cup \{c'\} = F_j[c']$. Hence $F_i^{X_0} = F_j$ providing a contradiction. So, running through all members of $\{F_k : k \in [\delta_0]\}$ as above yields exactly $2^j\delta_0$ further distinct orbits in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})/G_V$. In summary we obtain $\delta = 2^j\delta_0 + f_b$. (2) If $|b \setminus V_0| > 0$ then any $F \in \mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ yields $F \cup \{c\} \in \mathcal{F}_{\text{SAT}}(K_{\mathcal{H}})$, as every $c \in W_b$ can be satisfied independently of all clauses in F. Thus $f_b = 0$. (3) Since by definition no member of $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$ contributes to f_b, w.l.o.g. assume $\delta_0 = 0$ and let $F \in \mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ be arbitrary. Then for every $t \in \mathcal{M}(F)$ all loop variables are fixed. By assumption the new edge b satisfies $b \subseteq V_0$ and consists of loop variables only. Thus $t(b) = \{F(\{x\}) : x \in b\} \in W_b$, for all $t \in \mathcal{M}(F)$. According to Thm. 2 therefore it is $r = f_b = |\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})/G_{V_0}| = \rho_0 + 1$. (4) Since (3) means that any orbit in $\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})/G_{V_0}$ increases f_b of exactly 1 it specifically is sufficient for the unique orbit in $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})/G_{V_0}$. Regarding the necessity, let $F \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})$ be a fixed orbit base, and $c \in W_b$ be arbitrary. Let t_0 be the model of F setting every literal in F to 1. Assume there is $x \in b$ but $\{x\} \notin B_0$ then every edge in B_0 contains a variable distinct to x. So, modifying t_0 such that the literal over x in c is set to 1 yields $F \cup \{c\} \in \mathcal{F}_{\text{SAT}}(K_{\mathcal{H}})$

and the proof is finished by contraposition. □

So, it always is $0 \leq f_b \leq \rho_0 + 1$, and there are instances for which the boundary values are valid. Further, one immediately concludes in combination with Thm. 2:

Corollary 1: Let b and $\mathcal{H}_0, \mathcal{H} \in \mathfrak{H}$ as in Definition 1. Then $\delta(\mathcal{H}) = 2^{|b \cap V_0|}\delta_0 + f_b$ iff there are exactly f_b orbits $\mathcal{O}(F) \in \mathcal{F}_{\mathrm{SAT}}(K_{\mathcal{H}_0})/G_{V_0}$ such that for each orbit base F there is $c(F) \in W_b$ with $t(b) = c(F)$, for all $t \in \mathcal{M}(F)$.

Consider the disconnected base hypergraph $\mathcal{H} = (V, B)$ with $\omega = 64$, $\delta = 37$, and $\rho = 26$ as constructed in the proof of Theorem 3, (2). Recall that $\mathcal{H}$ is the union of three disjoint copies of $\mathcal{H}' = (V', B')$ with $V' = \{x_1, x_2\}$, $B' = \{\{x_1\}, \{x_2\}, \{x_1, x_2\}\}$. So one may assume that $V = \{x_i : i \in [6]\}$, and $B = \{\{x_i\} : i \in [6]\} \cup \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$. Setting $\tilde{B} := B \cup \{b\}$ with $b := V$ yields a connected base hypergraph $\tilde{\mathcal{H}} = (V, \tilde{B})$, for which the condition (3) of Theorem 4 is valid. Thus $f_b = \rho + 1 = 27$, $\tilde{\omega} = 2^{12}$, and $\tilde{\delta} = 2^6 \cdot 37 + 27 = 2395 > \tilde{\rho} = 2^{12} - 2395 - 1 = 1700$; yielding:

Theorem 5: There is $\mathcal{H} \in \mathfrak{H}^c$ such that $\delta(\mathcal{H}) > \rho(\mathcal{H})$.□

V. Discrete Pre-Bundles of Base Hypergraphs

Next we consider the question of the existence of base hypergraphs for specific values of the orbit maps. Hence subclasses of base hypergraphs are identified admitting discrete pre-bundles over $\mathbb{N}_0$ where those functions appear as the projections.

A. Pre-Bundles With Projections β Or ρ

Let $\beta_{\mathrm{lin}} := \beta|_{\mathfrak{H}_{\mathrm{lin}}}$, and $\beta_{\mathrm{lin}}^c := \beta|_{\mathfrak{H}_{\mathrm{lin}}^c}$.

Proposition 7: $(\mathfrak{H}, \mathbb{N}_0, \beta)$ is a discrete pre-bundle. Also $(\mathfrak{H}_{\mathrm{lin}}, \mathbb{N}_0, \beta_{\mathrm{lin}})$, $(\mathfrak{H}_{\mathrm{lin}}^c, \mathbb{N}_0, \beta_{\mathrm{lin}}^c)$ are discrete pre-bundles.

Proof. It suffices to verify the last claim which implies the remaining. So let $i = 0$, then $\mathcal{H}_0 := (\{x_1, x_2\}, \{\{x_1, x_2\}\})$ satisfies $\beta(\mathcal{H}_0) = 0$ and $\mathcal{H}_0 \in \mathfrak{H}_{\mathrm{lin}}^c$. For $i \in \mathbb{N}$ take $V_i = \{x_j : j \in [i+2]\}$, and $B_i = \{b_l : l \in [i+1]\}$ such that $|b| = 2$, for every $b \in B_i$. Setting $b_l = \{x_l, x_{l+1}\}$, for $l \in [i+1]$, obviously yields $V(B_i) = V_i$ and $\sum_{l \in [i+1]} |b_l| = 2i + 2$ hence $\beta(\mathcal{H}_i) = i$ where $\mathcal{H}_i := (V_i, B_i)$ and $\mathcal{H}_i \in \mathfrak{H}_{\mathrm{lin}}^c$. Thus β_{lin}^c is a projection onto $\mathbb{N}_0$. □

We identify $\mathfrak{H} =: \mathfrak{H}_0$, respectively $\mathfrak{H}_{\mathrm{lin}} =: \mathfrak{H}_{\mathrm{lin}0}$ and $\mathfrak{H}_{\mathrm{lin}} =: \mathfrak{H}_{\mathrm{lin}0}$ with $\mathcal{K}_0$, and also $\beta =: \beta_0$ respectively $\beta_{\mathrm{lin}} =: \beta_{\mathrm{lin}0}$, and $\beta_{\mathrm{lin}}^c =: \beta_{\mathrm{lin}0}^c$ with π_0. Further let $\mathfrak{H}_\nu$, respectively $\mathfrak{H}_{\mathrm{lin}\nu}$, $\mathfrak{H}_{\mathrm{lin}\nu}^c$ be identified by $\mathcal{K}_\nu$, and β_ν respectively $\beta_{\mathrm{lin}\nu}$, $\beta_{\mathrm{lin}\nu}^c$ by π_ν, for every $\nu > 0$. On the basis of Lemma 1 and the previous result one concludes:

Corollary 2: $(\mathfrak{H}_\nu, \mathbb{N}_0, \beta_\nu)$, $(\mathfrak{H}_{\mathrm{lin}\nu}, \mathbb{N}_0, \beta_{\mathrm{lin}\nu})$, as well as $(\mathfrak{H}_{\mathrm{lin}\nu}^c, \mathbb{N}_0, \beta_{\mathrm{lin}\nu}^c)$, $\nu \in \mathbb{N}_0$, are discrete pre-bundles.

Specifically notice that $\beta^{-1}(0) = \beta_{\mathrm{lin}}^{-1}(0)$, and that $\mathfrak{H}_{\mathrm{xlin}} \subseteq \delta^{-1}(0)$ [17] meaning that there exist non-trivial non-diagonal instances. Defining $\beta_{\mathrm{xlin}} := \beta|_{\mathfrak{H}_{\mathrm{xlin}}}$, in this context one even has:

Proposition 8: $(\mathfrak{H}_{\mathrm{xlin}}, \mathbb{N}_0, \beta_{\mathrm{xlin}})$, $(\mathfrak{H}_{\mathrm{xlin}\nu}, \mathbb{N}_0, \beta_{\mathrm{xlin}\nu})$, for $\nu \in \mathbb{N}_0$, are discrete pre-bundles.

Proof. Let $i \in \mathbb{N}_0$ be arbitrarily fixed. Specifically using $[0] := \emptyset$, set $V_i = \{x\} \cup \{x_{l,j} : l \in [i+1], j \in [i]\}$, and $B_i = \{b_l : l \in [i+1]\}$ such that $|b| = i + 1$, for every $b \in B_i$. Namely, setting $b_l = \{x\} \cup \{x_{l,j} : j \in [i]\}$, for every $l \in [i+1]$, yields $|V(B_i)| = |V_i| = 1 + i(i+1)$ and $\sum_{l \in [i+1]} |b_l| = (i+1)^2$ hence $\beta(\mathcal{H}_i) = i$ where $\mathcal{H}_i := (V_i, B_i)$. Observe that the intersection graph of B_i forms a clique K_{i+1}. Moreover, by construction, x is the only variable occurring in the pairwise intersections of the members of B_i hence $\mathcal{H}_i \in \mathfrak{H}_{\mathrm{xlin}}$ yielding the first claim. Together with Lemma 1 and the usual correspondences, the second claim is verified. □

Recall that $\mathbb{M}$ contains all Mersenne numbers, and set $\rho_{\mathrm{xlin}} := \rho|_{\mathfrak{H}_{\mathrm{xlin}}}$. Due to the previous result one obtains:

Corollary 3: (1) It is $\delta(\mathcal{H}) = \rho(\mathcal{H})$ iff $\mathcal{H} \in \beta^{-1}(0)$.
(2) $(\mathfrak{H}_{\mathrm{xlin}}, \mathbb{M}, \rho_{\mathrm{xlin}})$ is a discrete pre-bundle.

Proof. Let $\mathcal{H} \in \mathfrak{H}$. If $\delta(\mathcal{H}) = \rho(\mathcal{H}) =: c$ then $\omega(\mathcal{H}) - 1 = 2c$. As $\omega(\mathcal{H}) - 1 =: M \in \mathbb{M}$ it is $2|M$ only if $M = 0$ meaning $\omega(\mathcal{H}) = 1$, hence $\beta(\mathcal{H}) = 0$. Reversely, if $\beta(\mathcal{H}) = 0$ then $\omega(\mathcal{H}) = 1$ then $\delta(\mathcal{H}) = \rho(\mathcal{H}) = 0$, hence (1). As mentioned above $\delta|_{\mathfrak{H}_{\mathrm{xlin}}} = 0$, hence $\rho(\mathcal{H}) = \omega(\mathcal{H}) - 1 = 2^{\beta(\mathcal{H})} - 1 \in \mathbb{M}$, for every $\mathcal{H} \in \mathfrak{H}_{\mathrm{xlin}}$. The surjectivity of ρ_{xlin} is established by Prop. 8, thus (2). □

Lemma 3: Let $i \in \mathbb{N}_0$.

(1) There is $\mathcal{H} \in \mathfrak{H}_{\mathrm{lin}}$ with $\omega(\mathcal{H}) = 2^i$.
(2) If $\mathcal{H}_0$ exists with $\rho_0 = i$ then there is $\mathcal{H}_1$: $\rho_1 = 2i + 1$. Further $\mathcal{H}_1$ can be chosen both as diagonal or non-diagonal.

Proof. (1) directly follows from Proposition 7 due to $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$, $\mathcal{H} \in \mathfrak{H}$. For (2) consider $\mathcal{H}_1 = (V_1, B_1)$ with $V_1 = \{x_1, x_2\}$ and $B_1 = \{b_1, b_2\}$ with $b_1 = \{x_1\}$, $b_2 = \{x_1, x_2\}$ yielding $\omega_1 = 2$, $\delta_1 = 0$ and $\rho_1 = 1$. Next let $\mathcal{H}_2 = (V_2, B_2) \in \mathfrak{H}$ be arbitrary but disjoint to $\mathcal{H}_1$. Then for $\mathcal{H} := \mathcal{H}_2 \cup \mathcal{H}_1$ one obtains $\rho(\mathcal{H}) = 1 + 2\rho_2$. Moreover iff $\mathcal{H}_2$ is diagonal also $\mathcal{H} \supset \mathcal{H}_2$ is diagonal because of the monotony of δ. □

According to [14] a diagonal base hypergraph $\mathcal{H}$ is called simple if $\delta(\mathcal{H}) = 1$. Set $\mathfrak{H}_{\mathrm{simp}}$ for the class of all simple base hypergraphs. Recall that due to Theorem 3, (1) all members of $\bar{\mathfrak{H}}_{\mathrm{simp}}$ are connected. Further there is no upper bound on ρ in $\mathfrak{H}_{\mathrm{simp}}$. To state it more precisely, let $\hat{\mathbb{M}}_{-1} := \{M - 1 : M \in \hat{\mathbb{M}}\}$ where $\hat{\mathbb{M}}$ denotes the set of all Mersenne numbers excluding $0, 1$, and set $\rho_{\mathrm{simp}} := \rho|_{\mathfrak{H}_{\mathrm{simp}}}$, $\rho_{\mathrm{simp}}^c := \rho|_{\mathfrak{H}_{\mathrm{simp}}^c}$.

Theorem 6: For the classes $\mathfrak{H}_{\mathrm{simp}}, \mathfrak{H}_{\mathrm{simp}}^c$ one has:

(1) $(\mathfrak{H}_{\mathrm{simp}}, \hat{\mathbb{M}}_{-1}, \rho_{\mathrm{simp}})$ and $(\mathfrak{H}_{\mathrm{simp}}^c, \hat{\mathbb{M}}_{-1}, \rho_{\mathrm{simp}}^c)$ are discrete pre-bundles.
(2) For $\mathcal{H} \in \mathfrak{H} \setminus \mathfrak{H}_{\mathrm{simp}}$ with $\rho(\mathcal{H}) \in \hat{\mathbb{M}}_{-1}$ one has $\delta(\mathcal{H}) > \rho(\mathcal{H})$.

Proof. Regarding (1) we first show that ρ_{simp} cannot take values outside $\hat{\mathbb{M}}_{-1} = \{2(2^{k-1} - 1) : k \in \mathbb{N} \setminus \{1\}\}$. Clearly $\omega(\mathcal{H}) = 1 + 1 + \rho(\mathcal{H})$ for any simple base hypergraph. In general we have $\rho(\mathcal{H}) = \delta(\mathcal{H}) = 0$ according to Prop. 3 (1) only in case $\omega(\mathcal{H}) = 1$. Thus $\omega(\mathcal{H}) = 2$ is excluded for simple base hypergraphs. Hence $\rho(\mathcal{H}) = 2^k - 2$, for $k \geq 2$ is the only possible range of values, meaning that also ρ_{simp}^c cannot take values outside $\hat{\mathbb{M}}_{-1}$. It remains to prove that ρ_{simp}^c is a surjection. Take $\mathcal{H}' \in \mathfrak{H}_{\mathrm{simp}}^c$ as defined in the proof of Theorem 3 (2) with $\rho(\mathcal{H}') = 2$ being the smallest member in $\hat{\mathbb{M}}_{-1}$, for $k = 2$. For $k \geq 3$ take $\mathcal{H}_k = (V_k, B_k)$ such that $V_k := \{x_i : i \in [k]\}$ and $B_k := \{\{x_i\} : i \in [k]\}$. Then $\beta_k = 0$ therefore $\omega_k = 1$ and due to Prop. 3 (1) it follows that $\rho_k = \delta_k = 0$. Now for $\mathcal{H}'_k := (V_k, B'_k)$ with $B'_k := B_k \cup \{b\}$ and $b := V_k \notin B_k$, one has $\beta(\mathcal{H}'_k) = 2k - k = k$ thus $\omega(\mathcal{H}'_k) = 2^k$. Moreover for $\mathcal{H}'_k$ the condition in (3) of Theorem 4 is valid. Therefore $f_b = \rho_k + 1 = 1$ and due to (1) of Theorem 4, $\delta(\mathcal{H}'_k) = 2^k \delta_k + 1 = 1$, hence

$\rho(\mathcal{H}'_k) = 2^k - 2 \in \hat{\mathbb{M}}_{-1}$. Finally, it is $\mathcal{H}'_k \in \mathfrak{H}^c_{\mathrm{simp}}$ by construction. Concerning assertion (2) let $\mathcal{H} \in \mathfrak{H} \setminus \mathfrak{H}_{\mathrm{simp}}$ with $\delta(\mathcal{H}) = c \neq 1$, $\omega(\mathcal{H}) = 2^j$, and $\rho(\mathcal{H}) \in \hat{\mathbb{M}}_{-1}$. For $c = 0$ one has $\rho(\mathcal{H}) = 2^j - 1 \notin \hat{\mathbb{M}}_{-1}$ thus $c \geq 2$. Therefore there is a unique $k \in \mathbb{N}$, $k \geq 2$, with $\rho(\mathcal{H}) = 2^k - 2 = 2^j - 1 - c$. As $c \geq 2$ it follows $j > k$, hence

$$c = 2^j - 2^k + 1 \geq 2^k + 1 > 2^k - 2$$

implying $\delta(\mathcal{H}) > \rho(\mathcal{H})$. □

The previous proof directly yields:

Corollary 4: For $\beta_{\mathrm{simp}} := \beta|_{\mathfrak{H}_{\mathrm{simp}}}$, and $\beta^c_{\mathrm{simp}} := \beta|_{\mathfrak{H}^c_{\mathrm{simp}}}$ one has that $(\mathfrak{H}_{\mathrm{simp}}, \mathbb{N} \setminus \{1\}, \beta_{\mathrm{simp}})$, $(\mathfrak{H}^c_{\mathrm{simp}}, \mathbb{N} \setminus \{1\}, \beta^c_{\mathrm{simp}})$, respectively are discrete pre-bundles.

Defining $\beta_{\mathrm{mdiag}} := \beta|_{\mathfrak{H}_{\mathrm{mdiag}}}$, one has for the class of minimal diagonal base hypergraphs:

Corollary 5: $(\mathfrak{H}_{\mathrm{mdiag}}, \mathbb{N} \setminus \{1\}, \beta_{\mathrm{mdiag}})$ is a discrete pre-bundle. Moreover, $(\mathfrak{H}_{\mathrm{mdiag}_\nu}, \mathbb{N} \setminus \{1\}, \beta_{\mathrm{mdiag}_\nu})$ are discrete pre-bundles, for every integer $\nu \geq 0$.

PROOF. Cor. 3, [15] provides the inclusion $\mathfrak{H}^c_{\mathrm{simp}} \subseteq \mathfrak{H}_{\mathrm{mdiag}}$. □

Theorem 7: Let μ_r be a product of r Mersenne-powers, for fixed integer $r \geq 0$. Then there explicitly is $\mathcal{H} \in \mathfrak{H}$ such that $\rho(\mathcal{H}) = 2^t \mu_r - 1$, for every fixed integer $t \geq 0$.

PROOF. If $r = 0 = t$ any trivial $\mathcal{H} \in \mathfrak{H}$ obviously matches the assertion. For arbitrary $r \in \mathbb{N}$, $\varepsilon_i \in \mathbb{N}$, pairwise distinct integers $j_i \geq 2$, $i \in [r]$, let $M_{j_i} \in \mathbb{M}$. Set $\mu_r = \prod_{i \in [r]} M_{j_i}^{\varepsilon_i}$. Due to Lemma 1, (iii) in [15], a base hypergraph $\mathcal{H}$ consisting of s disjoint components $\mathcal{H}_i, i \in [s]$, satisfies $\rho(\mathcal{H}) + 1 = \prod_{i \in [s]}(1 + \rho(\mathcal{H}_i))$. For factor M_{j_i} in μ_r, there is $\mathcal{H}_i \in \mathfrak{H}_{\mathrm{simp}}$ with $\rho_i = M_{j_i} - 1$ according to Theorem 6. Take ε_i disjoint copies $\mathcal{H}_i(k)$ of $\mathcal{H}_i$ and set $\mathcal{H}(j_i) := \bigcup_{k \in [\varepsilon_i]} \mathcal{H}_i(k)$. Thus $\rho(\mathcal{H}(j_i)) + 1 = (\rho_i + 1)^{\varepsilon_i} = M_{j_i}^{\varepsilon_i}$. Hence for $\mathcal{H} := \bigcup_{i \in [r]} \mathcal{H}(j_i) \in \mathfrak{H}$ it is

$$\rho(\mathcal{H}) + 1 = \prod_{i \in [r]} (1 + \rho(\mathcal{H}(j_i))) = \prod_{i \in [r]} M_{j_i}^{\varepsilon_i} = 2^0 \mu_r$$

For any fixed $t > 0$ take $\mathcal{H}_0 \in \mathfrak{H}_{\mathrm{xlin}}$ such that $\rho_0 = M_t$ existing according to Corollary 3. Hence specifically $\rho_0 + 1 = 2^t$ which is the assertion for $r = 0$. Further, taking a copy of $\mathcal{H}$ as before and disjoint to $\mathcal{H}_0$ yields

$$\rho(\mathcal{H}_0 \cup \mathcal{H}) + 1 = (\rho(\mathcal{H}_0) + 1)(\rho(\mathcal{H}) + 1) = 2^t \mu_r$$

and $\mathcal{H}_0 \cup \mathcal{H} \in \mathfrak{H}$. □

More indirect and to some extent complementary to the previous explicit result one has the following characterization:

Theorem 8: $(\mathfrak{H}, \mathbb{N}_0, \rho)$ is a discrete pre-bundle iff for every $p \geq 5$ with $p \in \mathbb{P} \setminus \mathbb{M}$ there is $\mathcal{H} \in \mathfrak{H}$ with $\rho(\mathcal{H}) = p - 1$.

PROOF. The only-if part is clear. To reversely verify the surjectivity of ρ by induction on $i \in \mathbb{N}_0$, it is $\rho_0 = 0$, for $\mathcal{H}_0 \in \beta^{-1}(0)$. For $i = 1$ we refer to $\mathcal{H}_1$ in the proof of Lemma 3 with $\rho_1 = 1$. For $i = 2$, consider $\mathcal{H}'$ as defined in the proof of Theorem 3 (2) having $\rho' = 2$. For the induction step, let $i + 1 \geq 3$ be fixed and assume that the claim is verified for all integers $\leq i$. If $i + 1$ is odd then there is a unique integer $i > k \geq 1$ with $(i + 1) = 2k + 1$. By the induction hypothesis there is a $\mathcal{H}_k$ such that $\rho_k = k$. On behalf of Lemma 3, (2) then there also is a $\mathcal{H} \in \mathfrak{H}$ with $\rho(\mathcal{H}) = 2k + 1 = i + 1$. If $i + 1$ is even then $i + 2 \geq 5$ is odd. If $i + 2 \in \mathbb{P} \cap \mathbb{M}$ there is a (simple) $\mathcal{H}$ with $\rho(\mathcal{H}) = i + 1$ according to Theorem 6. If $i + 2 \in \mathbb{P}$ is a non-Mersenne prime we are done by the assumption. Else let $q_j \leq (i + 2)/3$, $j \in [r]$, for appropriate $r \in \mathbb{N}$, be all the (not necessarily distinct) prime factors of $i + 2$. Note that $q_j \leq i$, $j \in [r]$ as $i \geq 2$. Hence by the induction hypothesis there are instances $\mathcal{H}_j$ such that $\rho(\mathcal{H}_j) = q_j - 1$, $j \in [r]$. And we can assume that all these instances are chosen mutually disjoint. According to [15], Lemma 1 (iii) for their union $\mathcal{H}$ one has

$$\begin{aligned} \rho(\mathcal{H}) &= -1 + \Pi_{j=1}^r (1 + \rho(\mathcal{H}_j)) \\ &= -1 + \Pi_{j=1}^r q_j \\ &= -1 + (i + 2) \end{aligned}$$

and the assertion follows.□

B. Further Results Concerning δ, ρ

As introduced in [15], let $\mathfrak{H}_1 = \{\mathcal{H} \in \mathfrak{H} : \delta(\mathcal{H}) \in \{0, 1\}\}$. Again set $\alpha_i := \alpha(\mathcal{H}_i)$, $\alpha \in \{\omega, \beta, \delta\}$, whenever i varies in an index set.

Proposition 9: Let $k \in \mathbb{N}$, $\mathcal{H}_i \in \mathfrak{H}_1$, $i \in [k]$, be pairwise disjoint, and $\mathcal{H} := \bigcup_{i \in [k]} \mathcal{H}_i$. Let $I(\mathcal{H}) =: I \subseteq [k]$ such that $\delta_i = 1$ iff $i \in I$. Then

$$\begin{aligned} \delta(\mathcal{H}) &= 2^{\sum_{i \in [k] \setminus I} \beta_i} \left(\sum_{J \in 2^I \setminus \{I\}} (-1)^{|I| + |J| - 1} \prod_{j \in J} \omega_j \right) \\ \rho(\mathcal{H}) &= 2^{\sum_{i \in [k] \setminus I} \beta_i} \left(\sum_{J \in 2^I} (-1)^{|I| + |J|} \prod_{j \in J} \omega_j \right) - 1 \end{aligned}$$

PROOF. The first equation is verfied by induction. For $k = 1$, either $|I| = 0$ meaning $\delta(\mathcal{H}) = 0$ which is also provided by the assertion as the second factor becomes 0 (empty sum). Or it is $|I| = 1 = k$ meaning $\delta(\mathcal{H}) = 1$ which is in accordance with the assertion as both factors become 1 (empty product). Next, for fixed k assume the truth of the claim, and consider $\mathcal{H}' = \mathcal{H} \cup \mathcal{H}_{k+1}$ as disjoint union. It is $(*)$: $\delta(\mathcal{H}') = \delta(\mathcal{H})\omega_{k+1} + \delta_{k+1}\omega(\mathcal{H}) - \delta(\mathcal{H})\delta_{k+1}$. First case $\delta_{k+1} = 0$: Then $\delta(\mathcal{H}') = \delta(\mathcal{H})\omega_{k+1} = \delta(\mathcal{H})2^{\beta_{k+1}}$, and $I(\mathcal{H}') = I(\mathcal{H}) = I$, as $k + 1 \notin I$. Thus the assertion follows directly as the first factor is adapted correctly. Second case $\delta_{k+1} = 1$: According to $(*)$ then $\delta(\mathcal{H}') = \omega(\mathcal{H}) + \delta(\mathcal{H})(\omega_{k+1} - 1)$ and $I(\mathcal{H}') = \{k + 1\} \cup I =: I'$. It is

$$\begin{aligned} \omega(\mathcal{H}) &= \omega_{k+1}^{-1} \omega(\mathcal{H}') = \omega_{k+1}^{-1} \prod_{i \in [k+1]} \omega_i \\ &= 2^{\sum_{i \in [k+1] \setminus I'} \beta_i} \prod_{i \in I} \omega_i =: r \prod_{i \in I} \omega_i \end{aligned}$$

Since $[k + 1] \setminus I' = [k] \setminus I$ one has

$$\begin{aligned} (\omega_{k+1} - 1)\delta(\mathcal{H}) &= r \left(\sum_{J \in 2^I \setminus \{I\}} (-1)^{|I| + |J| - 1} \prod_{j \in J} \omega_j \omega_{k+1} \right. \\ &\qquad \left. + \sum_{J \in 2^I \setminus \{I\}} (-1)^{|I'| + |J| - 1} \prod_{j \in J} \omega_j \right) \end{aligned}$$

The first summand within the brackets equals

$$\sum_{J' \in \{J \cup \{k+1\} : J \in 2^I \setminus \{I\}\}} (-1)^{|I'| + |J'| - 1} \prod_{j \in J'} \omega_j$$

Rewriting $\omega(\mathcal{H}) = r(-1)^{|I'|+|I|-1}\prod_{j\in I}\omega_j$, i.e., $J = I$ one concudes that

$$\delta(\mathcal{H}') = r\left(\sum_{J\in 2^{I'}\setminus\{I'\}}(-1)^{|I'|+|J|-1}\prod_{j\in J}\omega_j\right)$$

which, by induction, is the first equation of the assertion, where the identity

$$2^{I'}\setminus\{I'\} = 2^I \cup \left\{J\cup\{k+1\} : J\in 2^I\setminus\{I\}\right\}$$

has been used. For the second claim it is $\rho(\mathcal{H}) = \omega(\mathcal{H}) - \delta(\mathcal{H}) - 1$. Using the first equation and

$$\omega(\mathcal{H}) = r\prod_{j\in I}\omega_j = -r(-1)^{|I|+|I|-1}\prod_{j\in I}\omega_j$$

that yields

$$\rho(\mathcal{H}) = -1 - r\left(\sum_{J\in 2^I}(-1)^{|I|+|J|-1}\prod_{j\in J}\omega_j\right)$$

finishing the proof. □

Clearly in Prop. 9 w.l.o.g. one can assume that $I = [k_1]$, for $|I| = k_1 \le k$, and reorganize the second factor by summing over all subsets of equal cardinality, yielding:

$$(*) : \delta(\mathcal{H}) = r\left(\sum_{i=0}^{k_1-1}(-1)^{k_1-1}\sum_{J\subset[k_1]:|J|=i}(-1)^i 2^{\sum_{j\in J}\beta_j}\right)$$

and

$$\rho(\mathcal{H}) = r\left(\sum_{i=0}^{k_1}(-1)^{k_1}\sum_{J\subset[k_1]:|J|=i}(-1)^i 2^{\sum_{j\in J}\beta_j}\right) - 1$$

with $r = 2^{\sum_{i\in[k]\setminus[k_1]}\beta_i}$. For the cases $k\le 4$ one therefore can derive the following list of existence results:

Corollary 6: Let i, j, l, m, n be integers. There is $\mathcal{H}\in\mathfrak{H}$ such that $\delta(\mathcal{H})$ equals:

(1) 0, respectively 2^m, for every $m\ge 0$,
(2) $2^j + M_n$, for every $j, n\ge 2$,
(3) M_l, for every $l\ge 3$,
(4) $(2^j + M_n)2^m$, for every $m\ge 0$,$j, n\ge 2$,
(5) $M_l 2^m$, for every $m\ge 0$,$l\ge 3$,
(6) $M_n 2^j + M_i 2^n + M_j 2^i + 1$, for every $i, j, n\ge 2$,
(7) $2^{2i} + M_n M_{i+1}$, for every $i, n\ge 2$,
(8) $2^{2i+m} + M_n M_{i+1}2^m$, for every $m\ge 0$, $i, n\ge 2$,
(9) $M_{i+j+n} + (M_n 2^j + M_i 2^n + M_j 2^i + 1)M_l + 1$, for every $i, j, l, n\ge 2$,
(10) $2^{2i}(2^j + M_n) + M_j M_n M_{i+1}$, for every $i, j, n\ge 2$.

Moreover, for each of the cases above, there are also parameter values such that $\mathcal{H}$ contains exactly q loops, for every $q\ge 2$.

PROOF. Let $\mathcal{H}$ consist of q loops only, then $\delta(\mathcal{H}) = 0$. Setting $k = 2, k_1 = 1$ in $(*)$ directly yields (1), if $\mathcal{H}_2\in\mathfrak{H}_{\text{xlin}}$, because then by Prop. 8 it is $\beta_2 = 2^m, m\in\mathbb{N}_0$. For $\mathcal{H}_1$, take a (disjoint) copy of $\mathcal{H}'_q$ as constructed in the proof of Thm. 6 (there replacing $k-1$ with $q\ge 2$). Then $\mathcal{H} = \mathcal{H}_1\cup\mathcal{H}_2$ contains exactly q loops if required. For $k = k_1 = 2$ one obtains $\delta(\mathcal{H}) = 2^{\beta_1} + M_{\beta_2}$, with arbitrary integers $\beta_1, \beta_2\ge 2$ due to Cor. 4, thus (2). Choosing $\mathcal{H}_1$ e.g. as $\mathcal{H}'_q$ as before yields the demanded amount of loops, however fixing $\beta_1 = q\ge 2$. (3) is implied by (2) for $j = n$. In the case $k_1 = 2, k = 3$, it follows (4) from (1) and (2), respectively, (5) from (1), (3); in either case the existence of q loops can be ensured as above thereby not restricting the ranges of m, j, n, l. Setting $k_1 = 3 = k$ in $(*)$ yields $\delta(\mathcal{H}) = 1 + 2^{\beta_1}M_{\beta_2} + 2^{\beta_2}M_{\beta_3} + 2^{\beta_3}M_{\beta_1}$, $\beta_i\ge 2, i\in[3]$, so (6). Fixing $\beta_1 = q$ and $\mathcal{H}_1 = \mathcal{H}'_q$ as above guarantees q loops in $\mathcal{H}$. It follows (7), from (6) specifically for $i = j$. For $k_1 = 3, k = 4$, (8) follows from (7) and (1). Finally for $k_1 = k = 4$ in $(*)$ a straightforward calculation provides (9). Here setting $i = j$ yields (10). As before the existence of q loops can be ensured in (7) - (10) by fixing the component $\mathcal{H}_1$ accordingly as above. □

Fixing $\beta_j = \beta$ in the formulas of Prop. 9 directly yields:

Corollary 7: For $\beta\ge 2$, let $\omega_i = 2^\beta$, $i\in[k]$, then

$$\delta(\mathcal{H}) = 2^{k\beta} - 2^{(k-k_1)\beta}M_\beta^{k_1}, \quad \rho(\mathcal{H}) = 2^{(k-k_1)\beta}M_\beta^{k_1} - 1$$

The next result uses a base hypergraph containing an appropriate number of loops. As above let $\alpha(\mathcal{H}_i) =: \alpha_i$, $\alpha\in\{\beta,\omega,\delta,\rho\}$, $i\in\mathbb{N}_0$.

Theorem 9: For fixed integer $q > 0$, let $\mathcal{H}_0 = (V_0, B_0)\in\mathfrak{H}$ contain exactly q loops collected in $B[q]\subseteq B_0$. For $i\ge 1$, define $\mathcal{H}_i := (V_0, B_i) = \mathcal{H}_{i-1}\cup\{b_i\}$, if there is $b_i\notin B_{i-1}$ such that $b_i\subset V(B[q])$, and $j_i := |b_i| > 1$. Then for $i\ge 1$, with $j_i > 1$ one has:

$$\delta(\mathcal{H}_i) = S_i\left(\delta_0 + \omega_0\sum_{l=0}^{i-1}2^{\sum_{k=1}^{l}j_k}\prod_{m=1}^{l+1}M_{j_m}^{-1}\right)$$

$$\rho(\mathcal{H}_i) = -1 - S_i\delta_0 - \omega_0\left(\sum_{l=0}^{i-1}(-1)^{\Delta_{i,l}}2^{\sum_{k=1}^{l}j_k}\prod_{m=1}^{i-l-1}M_{j_m}\right)$$

where $S_i := \prod_{l=1}^{i}M_{j_l}$, and $\Delta_{i,l}$ is the Kronecker-Delta.

PROOF. Observe that the additional edges b_i, $i\ge 1$, if existing, only contain variables of $V(B[q])\subseteq V_0$, none of them is a loop because $j_i > 1$. Further they are chosen as pairwise distinct. Thus specifically one has that $B[q]$ is the constant set of loops for every $\mathcal{H}_i$. Hence when $\mathcal{H}_i$ is constructed either there is no further b_i meaning $\mathcal{H}_i = \mathcal{H}_{i-1}\cup\emptyset = \mathcal{H}_{i-1}$ and therefore $\delta_i = \delta_{i-1}$, $\rho_i = \rho_{i-1}$. Or the condition of Thm. 4, (3) is valid for b_i, and $\mathcal{H}_{i-1}$. In this case, for $i = 1$ it is $\delta_1 = 2^{j_1}\delta_0 + \rho_0 + 1 = M_{j_1}\delta_0 + \omega_0$ because $\rho_0 = \omega_0 - \delta_0 - 1$. That is in accordance with the claim as $\delta_1 = S_1\delta_0 + \omega_0\sum_{l=0}^{0}2^0 S_1\prod_{m=1}^{1}M_{j_m}^{-1}$. Now fix $i\ge 1$ and assume the truth of the first assertion for all smaller integers. Assume that there is a next appropiate b_i that can be added to $\mathcal{H}_{i-1}$, otherwise we are done. According to Thm. 4, (3) then $\delta_i = M_{j_i}\delta_{i-1} + \omega_{i-1}$. As $b_i\subseteq V_0$ it is $\beta_i = \beta_{i-1} + j_i = \beta_0 + \sum_{l=1}^{i}j_l$. Thus $\omega_{i-1} = \omega_0\prod_{k=1}^{i-1}2^{j_k} = S_i\omega_0 2^{\sum_{k=1}^{i-1}j_k}S_i$. So, by the induction hypothesis one has

$$\begin{aligned}\delta_i &= M_{j_i}S_{i-1}\left(\delta_0 + \omega_0\sum_{l=0}^{i-2}2^{\sum_{k=1}^{l}j_k}\prod_{m=1}^{l+1}M_{j_m}^{-1}\right)\\ &\quad + S_i\omega_0 2^{\sum_{k=1}^{i-1}j_k}\prod_{m=1}^{i}M_{j_m}^{-1}\end{aligned}$$

integrating the last summand into the l-sum within the brackets yields the first equation of the theorem for i. Inserting $\omega_i = 2^{\sum_{l=1}^{i}2^{j_l}}\omega_0$ and $\rho_i = \omega_i - \delta_i - 1$ into the first equation yields

$$\rho_i = -1 - S_i\delta_0 + \omega_0\left(2^{\sum_{l=1}^{i}j_l} - \sum_{l=0}^{i-1}2^{\sum_{k=1}^{l}j_k}\prod_{m=1}^{i-l-1}M_{j_m}\right)$$

from which the second equation immediately follows, finishing the proof. □

Observe that according to Cor. 7, as long as $k_1 < k$, the possible values for δ are always even. Those for ρ then always remain odd. By the next statement derived from the previous result further values are provided.

Corollary 8: Let $j > 1$ such that for every $b \in B_0$ with $b \subseteq V(B[q])$ it is $|b| \neq j$. Fixing $|b_i| = j$ in Thm. 9, for every $i \leq \binom{q}{j}$, one obtains:

$$\delta(\mathcal{H}_i) = M_j^i \delta_0 + (2^{ji} - M_j^i)\omega_0, \quad \rho(\mathcal{H}_i) = M_j^i(\rho_0 + 1) - 1$$

Moreover $\rho(\mathcal{H}_i)$ is even iff ρ_0 is even. For $\omega_0 > 1$ it is $\delta(\mathcal{H}_i)$ even iff δ_0 is even. For $\omega_0 = 1$, $\delta(\mathcal{H}_i)$ is odd.

PROOF. By assumption there is no edge of length j in B_0 with loop variables. Thus b_i with $|b_i| = j$, for every $i \leq \binom{q}{j}$, $b_i \subset V(B[q])$ can be chosen as mutually distinct edges. With Thm. 9 one therefore has:

$$\begin{aligned} \delta_i &= M_j^i \delta_0 + \left(\sum_{l=0}^{i-1} 2^{jl} M_j^{i-1-l} \right) \omega_0 \\ &= M_j^i \delta_0 + (2^{ji} - M_j^i)\omega_0 \end{aligned}$$

because of $\sum_{l=0}^{i-1} a^l b^{i-1-l} = b^{i-1} \sum_{l=0}^{i-1} (a/b)^l = (a^i - b^i)/(a-b)$, for $a = 2^j, b = M_j$, and $a - b = 1$. Thus the first equation of the claim is justified. The second one directly follows from the first equation with $\omega_i = 2^{ij}\omega_0$ and $\rho_i = \omega_i - \delta_i - 1$. The statements regarding the parity are obvious, as $\delta_0 = 0$ if $\omega_0 = 1$ by Prop. 3 (1). □

Only in the case that $\delta_0 = 0$, $\omega_0 = 1$, in Cor. 8 and simultaneously $k_1 = k$ in Cor. 7, the values for both δ and ρ coincide, respectively. Clearly δ_0 can be assigned any of the values in Prop. 9 or Cor. 6 as the corresponding instances can be equipped with q loops. In combination with Thm. 9 or Cor. 8 correspondingly new values for δ are provided. Regarding ρ, iteratively applying the previous result, the following, however slightly weaker version of Theorem 7 can be concluded for the connected case.

Theorem 10: Let μ_r be a product of r Mersenne-powers, for fixed integer $r \geq 1$. Then $\mathcal{H} \in \mathfrak{H}^c$ can be constructed such that $\rho(\mathcal{H}) = \mu_r - 1$.

PROOF. For arbitrary $r \in \mathbb{N}$, $\varepsilon_i \in \mathbb{N}$, pairwise distinct integers $j_i \geq 2$, $i \in [r]$, let $M_{j_i} \in \mathbb{M}$, and $\mu_r = \prod_{i \in [r]} M_{j_i}^{\varepsilon_i}$ ordered by decreasing j_i. Let $\mathcal{H}_0 = (V_0, B_0)$ be a trivial base hypergraph consisting of q loops only, hence $\rho_0 = 0$. Choosing ε_1 times a new j_1-subset out of V_0 yields distinct edges b_l, with $|b_l| = j_1$, $l \in [\varepsilon_1]$. Applying Cor. 8 to $\mathcal{H}(j_1) := \mathcal{H}_0 \cup \bigcup_{l \in [\varepsilon_1]} \{b_l\}$ means $\rho_{j_1} + 1 = M_{j_1}^{\varepsilon_1}(\rho_0 + 1) = M_{j_1}^{\varepsilon_1}$. Assuming q is chosen appropriately, one is enabled to add further, pairwise distinct edges b_l, with $|b_l| = j_2$, $l \in [\varepsilon_2]$ yielding $\mathcal{H}(j_2) := \mathcal{H}(j_1) \cup \bigcup_{l \in [\varepsilon_2]} \{b_l\}$ and with Cor. 8: $\rho_{j_2} + 1 = M_{j_2}^{\varepsilon_2}(\rho_{j_1} + 1) = M_{j_1}^{\varepsilon_1} M_{j_2}^{\varepsilon_2}$. Iterating further in this manner, finally for $\mathcal{H} := \mathcal{H}(j_{r-1}) \cup \bigcup_{l \in [\varepsilon_r]} \{b_l\}$ it is: $\rho(\mathcal{H}) + 1 = M_{j_r}^{\varepsilon_r}(\rho_{j_{r-1}} + 1) = M_{j_r}^{\varepsilon_r} \prod_{i \in [r-1]} M_{j_i}^{\varepsilon_i} = \mu_r$. It remains to verify that q can be chosen accordingly and that the construction ensures that $\mathcal{H} \in \mathfrak{H}^c$. If $\varepsilon_i \leq j_i + 1$, for all $i \in [r]$, set $q := 1 + j_1$ where $j_1 = \max\{j_i : i \in [r]\}$. Then $\binom{q}{j_i} = \binom{j_1+1}{j_i} = \binom{j_i+n+1}{j_i}$ for an appropriate integer $n \geq 0$. Thus

$$\binom{q}{j_i} = \prod_{l \in [n+1]} \frac{j_i + l}{l} \geq (j_i+1)\left(1 + \frac{j_i}{1+n}\right)^n \geq j_i+1 \geq \varepsilon_i$$

$i \in [r]$. Moreover $\mathcal{H}$ surely is connected in this case as $\binom{q}{j_1} = j_1 + 1$, so the edges added first obviously have pairwise non-empty intersections. Clearly all sets of edges added in each of the further iterations are members of the power set of V_0, and join the same connected component. If there is $i \in [r]$ with $\varepsilon_i > j_i + 1$ clearly there is a unique $k_i > 1$ such that $\binom{j_i+k_i-1}{j_i} < \varepsilon_i \leq \binom{j_i+k_i}{j_i}$. Then set $q := j_1 + k$ where $k := \max\{k_i : i \in [r]\}$. Hence, for every $i : \varepsilon_i > j_i + 1$ it is

$$\binom{q}{j_i} = \binom{j_1 + k}{j_i} = \binom{j_i + k_i + n}{j_i} \geq \binom{j_i + k_i}{j_i} \geq \varepsilon_i$$

for appropriate $n \geq 0$. And for every $i : \varepsilon_i \leq j_i + 1$ one has $\binom{q}{j_i} = \binom{j_1+k}{j_i} > \binom{j_1+1}{j_i} \geq \varepsilon_i$ as above. All edges as subsets of V_0 then can be added during the next iterations such that the same connected component is enlarged. If there remained still unused, i.e., isolated loops of the original $\mathcal{H}_0$, then they are to be removed in a last step. □

VI. CLAUSE BUNDLES AND SECTIONS

Let $\mathcal{H} = (V, B)$ be a base hypergraph and identify V with the mapping $V : K_{\mathcal{H}} \to B$ which assigns to a clause its set of variables, then one obtains:

Proposition 10: $(K_{\mathcal{H}}, B, V)$ is a (finite) discrete pre-bundle on which the flipping group G_V acts fibre-stable.

PROOF. The first assertion is clear and for any $b \in B$, $V^{-1}(b) = W_b = (K_{\mathcal{H}})_b$. Let $c \in W_b$ and $g \in G_V$ then $c^g = c^{g \cap V(c)} \in W_b$ hence G_V acts fibre-stable. □

Observe here that the base B has a discrete structure. Note that any fibre-transversal $F \in \mathcal{F}(K_{\mathcal{H}})$ is a total section of $(K_{\mathcal{H}}, B, V)$, hence $\mathcal{F}(K_{\mathcal{H}}) = \mathcal{S}(B, K_{\mathcal{H}})$. Similarly, for $C \in$ CNF with $\mathcal{H}(C) = \mathcal{H}$, it is (C, B, V) a discrete pre-bundle, the *formula bundle*. Here one has a characterization of the space of models $\mathcal{M}(C)$ as the subset of $\mathcal{S}(B, C)$ defined through all total sections s of the formula bundle obeying the conditions

$$\text{(i)} : \bigcup_{b \in B} s(b) \in W_V \quad \text{(ii)} : \forall b \in B : s(b)^b \in W_b \setminus C_b$$

where, as usual, $s(b) \in C$ is a set of literals. Again setting $K_{\mathcal{H}} =: K_{\mathcal{H}0}$ with $\mathcal{K}_0$, $V =: V_0$ with π_0, we define $K_{\mathcal{H}\nu}$, V_ν corresponding to $\mathcal{K}_\nu$, π_ν for every integer $\nu > 0$. On basis of Lemma 1, Propositions 1 and 10 one obtains:

Corollary 9: $(K_{\mathcal{H}\nu}, B, V_\nu)$ is a discrete pre-bundle, on which G_V acts fibre-stable, for every $\nu \in \mathbb{N}_0$.

Given $K_{\mathcal{H}} = (V, B)$, any total section s of $(K_{\mathcal{H}1}, B, V_1)$ yields a collection $\mathrm{im}(s) = \{C_b : b \in B\}$ of fibre-formulas over B. For this setting by adapting Theorem 1 here we directly have $G_V(\mathrm{im}(s)) = \bigcap_{b \in B} G_V(s(b))$, $s(b) = C_b$, $b \in B$. Using the fibre-decomposition [12] one has $C = \bigcup_{b \in B(C)} C_b$. As these C_b are mutually disjoint objects in the total space $K_{\mathcal{H}1}$, one can identify C with the section $s \in \mathcal{S}(B, K_{\mathcal{H}1})$ such that $s(b) = C_b$, $b \in B$.

VII. CONCLUSION AND OPEN PROBLEMS

From Theorem 8, and Theorem 6 (1), respectively Corollary 3, Corollary 4, one directly concludes via accordingly adapting the settings prior to Lemma 1:

Corollary 10: For every integer $\nu \geq 0$ one has:

(1) If for every $p \geq 5$ with $p \in \mathbb{P} \setminus \mathbb{M}$ there can be constructed a base hypergraph $\mathcal{H}$ with $\rho(\mathcal{H}) = p - 1$ then $(\mathfrak{H}_\nu, \mathbb{N}_0, \rho_\nu)$ is a discrete pre-bundle.

(2) $(\mathfrak{H}_{\mathrm{xlin}_\nu}, \mathbb{M}, \rho_{\mathrm{xlin}_\nu})$, as well as $(\mathfrak{H}_{\mathrm{simp}_\nu}, \hat{\mathbb{M}}_{-1}, \rho_{\mathrm{simp}_\nu})$, and $(\mathfrak{H}^c_{\mathrm{simp}_\nu}, \hat{\mathbb{M}}_{-1}, \rho^c_{\mathrm{simp}_\nu})$ are discrete pre-bundles.
(3) $(\mathfrak{H}_{\mathrm{simp}_\nu}, \mathbb{N} \setminus \{1\}, \beta_{\mathrm{simp}_\nu})$ and $(\mathfrak{H}^c_{\mathrm{simp}_\nu}, \mathbb{N} \setminus \{1\}, \beta^c_{\mathrm{simp}_\nu})$ are discrete pre-bundles.

There remain several directions for future work. So, Theorem 8, respectively Corollary 10 (1), make use of a strong assumption which should be established. With Thms. 7, 10 there are explicit effective versions, however the lack of Thm. 8 cannot be overcome so far. Further it remains open whether also δ induces a pre-bundle with integer base. Here Cor. 6 provides several preliminary results. Specifically, notice that according to (5), δ covers every (clearly even) perfect number greater than 6. Also the properties of the fluctuation parameter in connection with Cor. 1 should be investigated in more detail. Specifically one should provide instances for which f_b neither is 0 nor $\rho_0 + 1$. Several results rest on the existence of loops in the base hypergraphs. It remains to investigate whether those results can be transfered to the loop-free case also. Similarly, some constructions so far were only possible for non-Sperner base hypergraphs such as in the proofs of Thms. 7 or 10. However, providing corresponding versions for the Sperner class also, might be helpful specifically to fill in the gaps in the range of ρ left therein, because ρ behaves non-monotone on those instances equipped with the partial order $\preceq$ (cf. Prop. 6, (2)). Finally, one might examinate whether it could be fruitful to exploit the enumeration method provided by Polyas theorem resting on the cycle indices of all members of the underlying permutation group. Via the regular representation that appears to be a subgroup of $S_{|G_V|}$ which unfortuntely tends to be quite large, even for moderate $|V|$.

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