Third-Fourth Derivative Three-Step Block Method for Direct Solution of Second-Order Fuzzy Ordinary Differential Equations

Kashif Hussain, Oluwaseun Adeyeye, Nazihah Ahmad, and Rukhsana Bibi

Abstract—**Fuzzy differential equation models are applicable when real-word situations are uncertain. Numerical methods provide an approximate solution to certain problems when the exact solution of these equations does not exist. Previous research developed various numerical methods for solving second-order fuzzy ordinary differential equations (FODEs) with the obtained results having low accuracy in terms of absolute error which can be improved. As a result, to improve the accuracy in terms of the absolute error of these equations, this article developed a third and fourth derivative self-starting block method for the direct solution of second-order FODEs. For the derivation of the method, linear block approach is adopted with Taylor series expansion. The basic characteristics of the proposed technique are demonstrated using definitions of stability and consistency. Comparing the results of considered examples with exact solutions indicates that the proposed method outperforms previous numerical methods in terms of absolute error accuracy. Therefore, it shows that the proposed method is suitable for solving fuzzy initial and boundary value problems for second-order FODEs.**

Index Terms—**block method, fuzzy boundary value problem, fuzzy initial value problems, second-order**

I. Introduction

SECOND-ORDER differential equations have many applications, including engineering, biology, chemistry, electronics, and physics. Unfortunately, unanticipated events may arise, introducing the concept of uncertainty and the use of the FDEs to address these issues; the fuzzy derivative was first introduced in [1]. This article considers second-order FODE of the form

$$y''(x) = f(x, y(x), y(x)), \forall x \in [a,b], \quad (1)$$

where $y''(x)$ is an H-derivative and y is a fuzzy function of a crisp variable x. Since the function is fuzzy, there exist solutions known as lower and upper solutions based on the parametric form of the α -level as

$$\begin{aligned} \underline{\overline{y}}''(x,\alpha) &= \underline{\overline{f}}(x, y(x,\alpha), y'(x,\alpha)) \\ \underline{f} &= \min\left\{ f(x, y(x,\alpha), y'(x,\alpha)) \right\} . \\ \overline{f} &= \max\left\{ f(x, y(x,\alpha), y'(x,\alpha)) \right\} \end{aligned}$$

Since exact solutions aren't always possible, and direct solutions to the problem in Equation (1) may be difficult to obtain, researchers employ various numerical methods to reach an approximate solution. Numerous researchers have developed a variety of numerical algorithms for solving second-order FODEs with initial and boundary conditions [2]-[10]. The biggest drawback of these approaches is the application of the methods on the first-order FODEs system which a reduction from the second-order FODEs. This leads to a computational burden and impacts the solution's accuracy. Thus, to bypass the rigor of reduction, block methods were developed in studies [11]-[13], but the accuracy of the obtained solution in terms of absolute error can still be improved. Therefore, this article develops a block method with third and fourth derivative terms to improve accuracy. Compared to existing approaches, the proposed method has the advantage of better accuracy, self-starting, and easy implementation of the block method.

The article is structured as follows: The essential definitions for fuzzy set theory are presented in Section II, and the methodology is presented in Section III. Section IV highlights the basic properties of the block technique, Section V considers the numerical examples, and Section VI concludes the article.

II. Preliminaries

This section recalls some basic definitions which will be adopted in this article.

Definition 1: Triangular Fuzzy Number [14]

Consider that $(u,v,w) \in \mathbb{R}^3, u \le v \le w$. Then the triangular fuzzy number, $M(x)$ is given as

$$M(x,u,v,w) = \begin{cases} 0, x < u \\ \frac{x-u}{v-u}, u \le x \le v \\ \frac{w-x}{w-v}, v \le x \le w \\ 0, x > w \end{cases} \quad (2)$$

with α -level set denoted as

$$M_{\varepsilon} = [u + \alpha(v-u), w - \alpha(w-v)], \alpha \in [0,1] . \quad (3)$$

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Definition 2: Trapezoidal Fuzzy Numbers [14]

Consider that $(u,v,w,\delta)\in\mathbb{R}^4, u\le v\le w\le\delta$. Then the trapezoidal fuzzy number $M(x)$ is given as

$$M(x,u,v,w,\delta)=\begin{cases}0, x<u\\ \frac{x-u}{v-u}, u\le x<v\\ 1, v\le x\le w\\ \frac{w-x}{w-v}, w<x\le\delta\\ 0, x>\delta\end{cases} \quad (4)$$

with α-level set denoted as

$$M_{\varepsilon}=[u+\alpha(v-u),\delta-\alpha(\delta-w)],\alpha\in[0,1]. \quad (5)$$

Definition 3: Fuzzy Set Support [15]

Consider a fuzzy set A with universal set X. The support of set A is defined as,

$$Supp(A)=\left\{x\in X \mid \mu_A(x)>0\right\}. \quad (6)$$

It contains all elements in X, and fuzzy elements' degree of membership is greater than zero.

Definition 4: α-Level Set [15]

Consider that $M\in R_f$, the α-level set is defined as

$$M_a=\begin{cases}\{x\in\mathbb{R}\mid M(x)>\alpha\},\alpha\in[0,1]\\ cl(\operatorname{supp}M),\alpha=0\end{cases} \quad (7)$$

with its closed bounded interval $[\underline{M}(x),\overline{M}(x)]$. $\underline{M}(x),\overline{M}(x)$ are the lower and upper bound, respectively.

Definition 5: Hukuhara Differential [16]

A function $f(x)$ is called Hukuhara differentiable if for h>0 sufficiently small, then H-difference exist $f(x+h)-f(x), f(x)-f(x-h)$ and there exist an element

$$f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)-f(x-h)}{h}.$$

The fuzzy function $f'(x)$ is called H-derivative.

III. Methodology

Given that the second-order FODE of the form defined in Equation (1) is a mapping $y_0\in R_f$ with α-level set $y_0=\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)_{\underline{\alpha}}^{\overline{\alpha}}$. The partition of [a, b] has the set of grid points $a=x_0<x_1<x_2<,\dots,<x_{n-1}<x_n=X$ with an approximate solution denoted as

$$y(x_n,\alpha)=\left(\underline{y}(x_n,\alpha),\overline{y}(x_n,\alpha)\right)_{\underline{\alpha}}^{\overline{\alpha}} \quad (8)$$

at points, $h=\frac{b-a}{n}, x_n=x_0+nh, 0\le n\le N.$

The three-step linear block method with the presence of third and fourth derivatives in second-order form is stated below,

$$y_{n+\eta}=\left(\sum_{v=0}^{1}\frac{(\eta h)^v}{v!}y_n^{(v)}+\sum_{d=0}^{2}\left[\sum_{v=0}^{3}\psi_{dv\eta}\,f_{n+\eta}^{(d)}\right]\right)_{\underline{\alpha}}^{\overline{\alpha}} \quad (9)$$

$\eta=1,2,3$

with first derivative expression for the block method form given as

$$y'_{n+\eta}=\left(y'_n+\sum_{d=0}^{2}\left[\sum_{v=0}^{3}\phi_{dv\eta}\,f_{n+\eta}^{(d)}\right]\right)_{\underline{\alpha}}^{\overline{\alpha}}. \quad (10)$$

$\eta=1,2,3$

Expand Equation (9) to obtain

$$y_{n+1}=\begin{pmatrix}y_n+hy'_n+\\ \begin{bmatrix}\psi_{001}f_n+\psi_{011}f_{n+1}+\psi_{021}f_{n+2}+\psi_{031}f_{n+3}+\\ \psi_{101}f'_n+\psi_{111}f'_{n+1}+\psi_{121}f'_{n+2}+\psi_{131}f'_{n+3}+\\ \psi_{201}f''_n+\psi_{211}f''_{n+1}+\psi_{221}f''_{n+2}+\psi_{231}f''_{n+3}\end{bmatrix}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+2}=\begin{pmatrix}y_n+2hy'_n+\\ \begin{bmatrix}\psi_{002}f_n+\psi_{012}f_{n+1}+\psi_{022}f_{n+2}+\psi_{032}f_{n+3}+\\ \psi_{102}f'_n+\psi_{112}f'_{n+1}+\psi_{122}f'_{n+2}+\psi_{132}f'_{n+3}+\\ \psi_{202}f''_n+\psi_{212}f''_{n+1}+\psi_{222}f''_{n+2}+\psi_{232}f''_{n+3}\end{bmatrix}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+3}=\begin{pmatrix}y_n+3hy'_n+\\ \begin{bmatrix}\psi_{003}f_n+\psi_{013}f_{n+1}+\psi_{023}f_{n+2}+\psi_{033}f_{n+3}+\\ \psi_{103}f'_n+\psi_{113}f'_{n+1}+\psi_{123}f'_{n+2}+\psi_{133}f'_{n+3}+\\ \psi_{203}f''_n+\psi_{213}f''_{n+1}+\psi_{223}f''_{n+2}+\psi_{233}f''_{n+3}\end{bmatrix}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}. \quad (11)$$

The expansion for Equation (10) follows as

$$y'_{n+1}=\left(y'_n+\begin{bmatrix}\phi_{001}f_n+\phi_{011}f_{n+1}+\phi_{021}f_{n+2}+\phi_{031}f_{n+3}+\\ \phi_{101}f'_n+\phi_{111}f'_{n+1}+\phi_{121}f'_{n+2}+\phi_{131}f'_{n+3}+\\ \phi_{201}f''_n+\phi_{211}f''_{n+1}+\phi_{221}f''_{n+2}+\phi_{231}f''_{n+3}\end{bmatrix}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y'_{n+2}=\left(y'_n+\begin{bmatrix}\phi_{002}f_n+\phi_{012}f_{n+1}+\phi_{022}f_{n+2}+\phi_{032}f_{n+3}+\\ \phi_{102}f'_n+\phi_{112}f'_{n+1}+\phi_{122}f'_{n+2}+\phi_{132}f'_{n+3}+\\ \phi_{202}f''_n+\phi_{212}f''_{n+1}+\phi_{222}f''_{n+2}+\phi_{232}f''_{n+3}\end{bmatrix}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y'_{n+3}=\left(y'_n+\begin{bmatrix}\phi_{003}f_n+\phi_{013}f_{n+1}+\phi_{023}f_{n+2}+\phi_{033}f_{n+3}+\\ \phi_{103}f'_n+\phi_{113}f'_{n+1}+\phi_{123}f'_{n+2}+\phi_{133}f'_{n+3}+\\ \phi_{203}f''_n+\phi_{213}f''_{n+1}+\phi_{223}f''_{n+2}+\phi_{233}f''_{n+3}\end{bmatrix}\right)_{\underline{\alpha}}^{\overline{\alpha}}. \quad (12)$$

By applying Taylor series expansions

$$y(x+h;\alpha)=\left(\sum_{i=0}^{n}\frac{h^i}{i!}f^{(i)}(x,\alpha)\right)_{\underline{\alpha}}^{\overline{\alpha}} \quad (13)$$

defined in [17] to expand each term in Equations (11) and (12) adopts the expressions

$$y_{n+\eta}=y(x+\eta h;\alpha)=\left(\sum_{i=0}^{n}\frac{(\eta h)^i}{i!}f^{(i)}(x_n,\alpha)\right)_{\underline{\alpha}}^{\overline{\alpha}},\eta=0,1.2.3$$

$$y_{n+\eta} = \begin{pmatrix} y(x_n;\alpha) + \eta h y'(x_n;\alpha) + \frac{(\eta h)^2}{2!} y''(x_n;\alpha) \\ + \frac{(\eta h)^3}{3!} y'''(x_n;\alpha) + \ldots + + \frac{(\eta h)^n}{n!} y^n(x_n;\alpha) \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} .$$

After that, the unknown coefficients $\psi_{dv\eta}$ and $\phi_{dv\eta}$ are obtained using the matrix inverse method $\psi_{dv\eta} = A^{-1}B$ and $\phi_{dv\eta} = A^{-1}D$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 2h & 3h & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & 0 & h & 2h & 3h & 1 & 1 & 1 & 1 \\ 0 & \frac{h^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} & 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & 0 & h & 2h & 3h \\ 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & 0 & \frac{h^3}{3!} & \frac{(2h)3}{3!} & \frac{(3h)^3}{3!} & 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} \\ 0 & \frac{h^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} & 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & 0 & \frac{h^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} \\ 0 & \frac{h^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} & 0 & \frac{h^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} & 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} \\ 0 & \frac{h^7}{7!} & \frac{(2h)^7}{7!} & \frac{(3h)^7}{7!} & 0 & \frac{h^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} & 0 & \frac{h^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} \\ 0 & \frac{h^8}{8!} & \frac{(2h)^8}{8!} & \frac{(3h)^8}{8!} & 0 & \frac{h^7}{7!} & \frac{(2h)^7}{7!} & \frac{(3h)^7}{7!} & 0 & \frac{h^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} \\ 0 & \frac{h^9}{9!} & \frac{(2h)^9}{9!} & \frac{(3h)^9}{9!} & 0 & \frac{h^8}{8!} & \frac{(2h)^8}{8!} & \frac{(3h)^8}{8!} & 0 & \frac{h^7}{7!} & \frac{(2h)^7}{7!} & \frac{(3h)^7}{7!} \\ 0 & \frac{h^{10}}{10!} & \frac{(2h)^{10}}{10!} & \frac{(3h)^{10}}{10!} & 0 & \frac{h^9}{9!} & \frac{(2h)^9}{9!} & \frac{(3h)^9}{9!} & 0 & \frac{h^8}{8!} & \frac{(2h)^8}{8!} & \frac{(3h)^8}{8!} \\ 0 & \frac{h^{11}}{11!} & \frac{(2h)^{11}}{11!} & \frac{(3h)^{11}}{11!} & 0 & \frac{h^{10}}{10!} & \frac{(2h)^{10}}{10!} & \frac{(3h)^{10}}{10!} & 0 & \frac{h^9}{9!} & \frac{(2h)^9}{9!} & \frac{(3h)^9}{9!} \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} ,$$

$$B = \begin{pmatrix} \frac{(\eta h)^2}{2!} \\ \frac{(\eta h)^3}{3!} \\ \frac{(\eta h)^4}{4!} \\ \frac{(\eta h)^5}{5!} \\ \frac{(\eta h)^6}{6!} \\ \frac{(\eta h)^7}{7!} \\ \frac{(\eta h)^8}{8!} \\ \frac{(\eta h)^9}{9!} \\ \frac{(\eta h)^{10}}{10!} \\ \frac{(\eta h)^{11}}{11!} \\ \frac{(\eta h)^{12}}{12!} \\ \frac{(\eta h)^{13}}{13!} \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} \text{ and } D = \begin{pmatrix} \eta h \\ \frac{(\eta h)^2}{2!} \\ \frac{(\eta h)^3}{3!} \\ \frac{(\eta h)^4}{4!} \\ \frac{(\eta h)^5}{5!} \\ \frac{(\eta h)^6}{6!} \\ \frac{(\eta h)^7}{7!} \\ \frac{(\eta h)^8}{8!} \\ \frac{(\eta h)^9}{9!} \\ \frac{(\eta h)^{10}}{10!} \\ \frac{(\eta h)^{11}}{11!} \\ \frac{(\eta h)^{12}}{12!} \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} . \text{Therefore,}$$

$$\begin{pmatrix} \psi_{001} \\ \psi_{011} \\ \psi_{021} \\ \psi_{031} \\ \psi_{101} \\ \psi_{111} \\ \psi_{121} \\ \psi_{131} \\ \psi_{201} \\ \psi_{211} \\ \psi_{221} \\ \psi_{231} \end{pmatrix} = \begin{pmatrix} \frac{2857219}{9729720} \\ \frac{594283}{1921920} \\ \frac{-13373}{120120} \\ \frac{1316741}{155675520} \\ \frac{1941647}{51891840} \\ \frac{-7453}{360360} \\ \frac{233897}{5765760} \\ \frac{-9497}{3706560} \\ \frac{97159}{51891840} \\ \frac{3617}{137280} \\ \frac{-11005}{1153152} \\ \frac{565}{2594592} \end{pmatrix}, \begin{pmatrix} \psi_{002} \\ \psi_{012} \\ \psi_{022} \\ \psi_{032} \\ \psi_{102} \\ \psi_{112} \\ \psi_{122} \\ \psi_{132} \\ \psi_{202} \\ \psi_{212} \\ \psi_{222} \\ \psi_{232} \end{pmatrix} = \begin{pmatrix} \frac{817216}{1216215} \\ \frac{22112}{15015} \\ \frac{-2482}{15015} \\ \frac{25184}{1216215} \\ \frac{1048}{11583} \\ \frac{-1366}{45045} \\ \frac{3392}{45045} \\ \frac{-2554}{405405} \\ \frac{1888}{405405} \\ \frac{3722}{45045} \\ \frac{-46}{2145} \\ \frac{218}{405405} \end{pmatrix}, \begin{pmatrix} \psi_{003} \\ \psi_{013} \\ \psi_{023} \\ \psi_{033} \\ \psi_{103} \\ \psi_{113} \\ \psi_{123} \\ \psi_{133} \\ \psi_{203} \\ \psi_{213} \\ \psi_{223} \\ \psi_{233} \end{pmatrix} = \begin{pmatrix} \frac{48231}{45760} \\ \frac{1725543}{640640} \\ \frac{203391}{320320} \\ \frac{75321}{640640} \\ \frac{92709}{640640} \\ \frac{-2187}{64064} \\ \frac{63243}{640640} \\ \frac{-4113}{160160} \\ \frac{4833}{640640} \\ \frac{2187}{16016} \\ \frac{-2187}{640640} \\ \frac{81}{45760} \end{pmatrix}$$

and

$$\begin{pmatrix} \phi_{001} \\ \phi_{011} \\ \phi_{021} \\ \phi_{031} \\ \phi_{101} \\ \phi_{111} \\ \phi_{121} \\ \phi_{131} \\ \phi_{201} \\ \phi_{211} \\ \phi_{221} \\ \phi_{231} \end{pmatrix} = \begin{pmatrix} \frac{912523}{2395008} \\ \frac{23717}{29568} \\ \frac{-5851}{29568} \\ \frac{35339}{2395008} \\ \frac{214943}{3991680} \\ \frac{-10657}{147840} \\ \frac{10657}{147840} \\ \frac{-5941}{1330560} \\ \frac{11369}{3991680} \\ \frac{4423}{88704} \\ \frac{-7453}{443520} \\ \frac{1513}{3991680} \end{pmatrix}, \begin{pmatrix} \phi_{002} \\ \phi_{012} \\ \phi_{022} \\ \phi_{032} \\ \phi_{102} \\ \phi_{112} \\ \phi_{122} \\ \phi_{132} \\ \phi_{202} \\ \phi_{212} \\ \phi_{222} \\ \phi_{232} \end{pmatrix} = \begin{pmatrix} \frac{7031}{18711} \\ \frac{302}{231} \\ \frac{71}{231} \\ \frac{178}{18711} \\ \frac{544}{10395} \\ \frac{32}{1155} \\ \frac{-32}{1155} \\ \frac{-92}{31185} \\ \frac{17}{6237} \\ \frac{212}{3465} \\ \frac{-19}{3465} \\ \frac{8}{31185} \end{pmatrix}, \begin{pmatrix} \phi_{003} \\ \phi_{013} \\ \phi_{023} \\ \phi_{033} \\ \phi_{103} \\ \phi_{113} \\ \phi_{123} \\ \phi_{133} \\ \phi_{203} \\ \phi_{213} \\ \phi_{223} \\ \phi_{233} \end{pmatrix} = \begin{pmatrix} \frac{3849}{9856} \\ \frac{10935}{9856} \\ \frac{10935}{9856} \\ \frac{3849}{9856} \\ \frac{2799}{49280} \\ \frac{-2187}{49280} \\ \frac{2187}{49280} \\ \frac{-2799}{49280} \\ \frac{153}{49280} \\ \frac{2187}{49280} \\ \frac{2187}{49280} \\ \frac{153}{49280} \end{pmatrix} .$$

The values of the coefficients put in Equations (11) and (12) are the required three-step block method with third and fourth derivatives.

$$y_{n+1} = \begin{pmatrix} y_n + hy'_n + h^2(\frac{2857219}{9729720} f_n + \frac{594283}{1921920} f_{n+1} - \frac{13373}{120120} f_{n+2} \\ + \frac{1316741}{155675520} f_{n+3}) + h^3(\frac{1941647}{51891840} g_n - \frac{7453}{360360} g_{n+1} \\ + \frac{233897}{5765760} g_{n+2} - \frac{9497}{3706560} g_{n+3}) + h^4(\frac{97159}{51891840} m_n \\ + \frac{3617}{137280} m_{n+1} - \frac{11005}{1153152} m_{n+2} + \frac{565}{2594592} m_{n+3}) \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+2} = \begin{pmatrix} y_n + 2hy'_n + h^2(\frac{817216}{1216215} f_n + \frac{22112}{15015} f_{n+1} - \frac{2482}{15015} f_{n+2} \\ + \frac{25184}{1216215} f_{n+3}) + h^3(\frac{1048}{11583} g_n - \frac{1366}{45045} g_{n+1} + \frac{3392}{45045} g_{n+2} \\ - \frac{2554}{405405} g_{n+3}) + h^4(\frac{1888}{405405} m_n + \frac{3722}{45045} m_{n+1} - \frac{46}{2145} m_{n+2} \\ + \frac{218}{405405} m_{n+3}) \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+3} = \begin{pmatrix} y_n + 3hy'_n + h^2(\frac{48231}{45760} f_n + \frac{1725543}{640640} f_{n+1} + \frac{203391}{320320} f_{n+2} \\ + \frac{75321}{640640} f_{n+3}) + h^3(\frac{92709}{640640} g_n - \frac{2187}{64064} g_{n+1} + \frac{63423}{640640} g_{n+2} \\ - \frac{4113}{160160} g_{n+3}) + h^4(\frac{4833}{640640} m_n + \frac{2187}{16016} m_{n+1} - \frac{2187}{640640} m_{n+2} \\ + \frac{81}{45760} m_{n+3}) \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} \quad (14)$$

$$y'_{n+1} = \begin{pmatrix} y'_n + h(\frac{912523}{2395008} f_n + \frac{23717}{29568} f_{n+1} - \frac{5851}{29568} f_{n+2} \\ + \frac{35339}{2395008} f_{n+3} + h^2(\frac{214943}{3991680} g_n - \frac{10657}{147840} g_{n+1} \\ + \frac{10657}{147840} g_{n+2} - \frac{5941}{1330560} g_{n+3}) + h^3(\frac{11369}{3991680} m_n \\ + \frac{4423}{88704} m_{n+1} - \frac{7453}{443520} m_{n+2} + \frac{1513}{3991680} m_{n+3}) \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y'_{n+2} = \begin{pmatrix} y'_n + h(\frac{7031}{18711} f_n + \frac{302}{231} f_{n+1} + \frac{71}{231} f_{n+2} + \frac{178}{18711} f_{n+3}) \\ + h^2(\frac{544}{10395} g_n + \frac{32}{1155} g_{n+1} - \frac{32}{1155} g_{n+2} - \frac{92}{31185} g_{n+3}) \\ + h^3(\frac{17}{6237} m_n + \frac{212}{3465} m_{n+1} - \frac{19}{3465} m_{n+2} + \frac{8}{31185} m_{n+3}) \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y'_{n+3} = \begin{pmatrix} y'_n + h(\frac{3849}{9856} f_n + \frac{10935}{9856} f_{n+1} + \frac{10935}{9856} f_{n+2} + \frac{3849}{9856} f_{n+3}) \\ + h^2(\frac{2799}{49280} g_n - \frac{2187}{49280} g_{n+1} + \frac{2187}{49280} g_{n+2} - \frac{2799}{49280} g_{n+3}) \\ + h^3(\frac{153}{49280} m_n + \frac{2187}{49280} m_{n+1} + \frac{2187}{49280} m_{n+2} + \frac{153}{49187} m_{n+3}) \end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} \quad (15)$$

Hence, Equations (14) and (15) represents the proposed block method, and its correctors take the form,

$$\begin{pmatrix} A^0 Y_{n+k} = \\ A^1 Y_{n-k} + h\left(B^1 Y'_{n-k}\right) + h^2\left(C^0 F_{n+k} + C^1 F_{n-k}\right) \\ +h^3\left(D^0 G_{n+k} + D^1 G_{n-k}\right) + h^4\left(E^0 M_{n+k} + E^1 M_{n-k}\right) \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}}.$$

Here,

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}}, A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}}, B^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}},$$

$$C^0 = \begin{pmatrix} \frac{594283}{1921920} & \frac{-13373}{120120} & \frac{1316741}{155675520} \\ \frac{22112}{15015} & \frac{-2482}{15015} & \frac{25184}{1216215} \\ \frac{1725543}{640640} & \frac{203391}{320320} & \frac{75321}{640640} \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}}, C^1 = \begin{pmatrix} 0 & 0 & \frac{2857219}{9729720} \\ 0 & 0 & \frac{817216}{1216215} \\ 0 & 0 & \frac{48231}{45769} \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}},$$

$$D^0 = \begin{pmatrix} \frac{-7453}{360360} & \frac{233897}{5765760} & \frac{-9497}{3706560} \\ \frac{-1366}{45045} & \frac{3392}{45045} & \frac{-2554}{405405} \\ \frac{-2187}{64064} & \frac{63423}{640640} & \frac{-4113}{160160} \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}}, D^1 = \begin{pmatrix} 0 & 0 & \frac{1941647}{51891840} \\ 0 & 0 & \frac{1048}{11583} \\ 0 & 0 & \frac{92709}{640640} \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}},$$

$$E^0 = \begin{pmatrix} \frac{3617}{137280} & \frac{-11005}{1153152} & \frac{565}{2594592} \\ \frac{3722}{45045} & \frac{-46}{2145} & \frac{218}{405405} \\ \frac{2187}{16016} & \frac{-2187}{640640} & \frac{81}{45760} \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}}, E^1 = \begin{pmatrix} 0 & 0 & \frac{97159}{51891840} \\ 0 & 0 & \frac{1888}{405405} \\ 0 & 0 & \frac{4833}{640640} \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}},$$

$$Y_{n+k} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix}_{\underline{\alpha}}^{\alpha}, Y_{n-k} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}_{\underline{\alpha}}^{\alpha}, Y'_{n-k} = \begin{pmatrix} y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}_{\underline{\alpha}}^{\alpha},$$

$$F_{n+k} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}_{\underline{\alpha}}^{\alpha}, F_{n-k} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}_{\underline{\alpha}}^{\alpha}, G'_{n+k} = \begin{pmatrix} f'_{n+1} \\ f'_{n+2} \\ f'_{n+3} \end{pmatrix}_{\underline{\alpha}}^{\alpha},$$

$$G'_{n-k} = \begin{pmatrix} f'_{n-2} \\ f'_{n-1} \\ f'_n \end{pmatrix}_{\underline{\alpha}}^{\alpha}, M'_{n+k} = \begin{pmatrix} f''_{n+1} \\ f''_{n+2} \\ f''_{n+3} \end{pmatrix}_{\underline{\alpha}}^{\alpha}, M'_{n-k} = \begin{pmatrix} f''_{n-2} \\ f''_{n-1} \\ f''_n \end{pmatrix}_{\underline{\alpha}}^{\alpha}.$$

IV. PROPERTIES OF PROPOSED METHOD

This section will detail the basic properties of the developed three-step third-fourth scheme, following the given theorem and definitions.

Theorem 1: Convergence [18]

A block method is convergent if it is consistent and zero-stable.

Definition 6: Consistency [18]

A block method is consistent if it has order $p \geq 1$.

Definition 7: Zero-Stability [19]

Block with matrix difference equation in the following form

$$A^0 Y_{n+k} = A^1 Y_{n-k} + B^1 Y'_{n-k} + B^2 Y''_{n-k} + \ldots + B^1 Y^{(m-1)}_{n-k}$$
$$+h^m\left(C^0 Y^{(m)}_{n+k} + C^1 Y^{(m)}_{n-k}\right)_{\underline{\alpha}}^{\bar{\alpha}} + h^{(m+1)}\left(D^0 Y^{(m+1)}_{n+k} + D^1 Y^{(m+1)}_{n-k}\right)_{\underline{\alpha}}^{\bar{\alpha}}, \quad (16)$$
$$+h^{(m+2)}\left(E^0 Y^{(m+2)}_{n+k} + E^1 Y^{(m+2)}_{n-k}\right)_{\underline{\alpha}}^{\bar{\alpha}}$$

where $Y^{(d)}_{n+k} = \left(y^{(d)}_{n+1}, y^{(d)}_{n+2}, \ldots, y^{(d)}_{n+k}\right)^T$ and $Y^{(d)}_{n-k} = \left(y^{(d)}_{n1(k-1)}, y^{(d)}_{n-(k-2)}, \ldots, y^{(d)}_{n}\right)^T$, is zero-stable if the first characteristic polynomial takes the form

$$P(\varphi) = \det(\varphi_v A^0 - A^1) \quad (17)$$

and the root of $P(\varphi) = 0$ satisfies $|\varphi_v| \leq 1, v = 1, \ldots, k$.

Definition 8: Region of Absolute Stability [19]

The absolute stability region is determined by obtaining the polynomial of the form

$$\left(\det\begin{bmatrix} -(w)^k + A^1 + q\left[\sum_{j=0}^{k} B^j w^{k-j}\right] + q^2\left[\sum_{j=0}^{k} C^j w^{k-j}\right] \\ +q^3\left[\sum_{j=0}^{k} D^j w^{k-j}\right] + q^4\left[\sum_{j=0}^{k} E^j w^{k-j}\right] \end{bmatrix}\right)_{\underline{\alpha}}^{\bar{\alpha}}$$

$q = \lambda h.$

Then the region of absolute stability is then determined by plotting the polynomial roots using the boundary locus approach. If obtained roots of the polynomial lie in the unit circle, then the block method is absolute stable, and its region is called the region of absolute stability.

These definitions for block methods in the crisp form are adopted to the proposed method for the fuzzy version to prove the convergence properties of the proposed method.

Order and Error constant

The linear operator is defined, which is associated with Equation (9), as:

$$L(y(x),h) = \left(y_{n+\eta} - \sum_{v=0}^{1}\frac{(\eta h)^v}{v!} y^{(v)}_n + \sum_{d=0}^{2}\left[\sum_{v=0}^{3}\psi_{dv\eta} f^{(d)}_{n+v}\right]\right)_{\underline{\alpha}}^{\bar{\alpha}} \quad (21)$$

$\eta = 1, 2$

with

$$L(y(x),h) = \begin{pmatrix} C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) \\ +,\ldots,+C_{z+1} h^{z+1} y^{z+1}(x_n) + C_{z+2} h^{z+2} y^{z+2}(x_n) \end{pmatrix}_{\underline{\alpha}}^{\bar{\alpha}}.$$

The method is said to be of order z if $C_0 = C_1 =, \ldots, = C_Z = C_{Z+1} = 0, C_{Z+2} \neq 0$ and C_{Z+2} is error constant. The order of the proposed method by using this definition is nine with an error constant $\left[\frac{-0.00029593}{5753767219}, \frac{-0.0059}{449513064}, \frac{-0.00027}{11275264}\right]$.

Zero-Stability

Applying the definition of zero-stability in fuzzy form for the proposed method gives

$$P(\varphi)=\left|\begin{pmatrix}\varphi & 0 & 0\\ 0 & \varphi & 0\\ 0 & 0 & \varphi\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\right|_{\underline{\alpha}}^{\overline{\alpha}}$$

with $P(\varphi)=\varphi^2(\varphi-1)=0$.

The obtained roots satisfy the above condition. Hence, the proposed block method is zero-stable.

Convergence

The proposed method is convergent because it is consistent and zero stable.

Region of Absolute Stability

The polynomial of the proposed block method for stability is obtained by using Definition 8 as:

$$R(w)=$$

$$\left|\begin{pmatrix}w & 0 & 0\\ 0 & w^2 & 0\\ 0 & 0 & w^3\end{pmatrix}+\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}+q\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 2\\ 0 & 0 & 3\end{pmatrix}\right.$$
$$+q^2\left[\begin{pmatrix}\frac{594283w}{1921920} & \frac{-13373w^2}{120120} & \frac{1316741w^3}{155675520}\\ \frac{22112w}{15015} & \frac{-2482w^2}{15015} & \frac{25184w^3}{1216215}\\ \frac{1725543w}{640640} & \frac{203391w^2}{320320} & \frac{75321w^3}{640640}\end{pmatrix}+\begin{pmatrix}0 & 0 & \frac{2857219}{9729720}\\ 0 & 0 & \frac{817216}{1216215}\\ 0 & 0 & \frac{48231}{45769}\end{pmatrix}\right]$$
$$+q^3\left[\begin{pmatrix}\frac{-7453w}{360360} & \frac{233897w^2}{5765760} & \frac{-9497w^3}{3706560}\\ \frac{-1366w}{45045} & \frac{3392w^2}{45045} & \frac{-2554w^3}{405405}\\ \frac{-2187w}{64064} & \frac{63423w^2}{640640} & \frac{-4113w^3}{160160}\end{pmatrix}+\begin{pmatrix}0 & 0 & \frac{1941647}{51891840}\\ 0 & 0 & \frac{1048}{11583}\\ 0 & 0 & \frac{92709}{640640}\end{pmatrix}\right]$$
$$\left.+q^4\left[\begin{pmatrix}\frac{3617w}{137280} & \frac{-11005w^2}{1153152} & \frac{565w^3}{2594592}\\ \frac{3722w}{45045} & \frac{-46w^2}{2145} & \frac{218w^3}{405405}\\ \frac{2187w}{16016} & \frac{-2187w^2}{640640} & \frac{81w^3}{45760}\end{pmatrix}+\begin{pmatrix}0 & 0 & \frac{97159}{51891840}\\ 0 & 0 & \frac{1888}{405405}\\ 0 & 0 & \frac{4833}{640640}\end{pmatrix}\right]\right|_{\underline{\alpha}}^{\overline{\alpha}}$$

$$=[\frac{97q^{12}}{303564800}-\frac{26227q^{11}}{4439635200}+\frac{1073183q^{10}}{17758540800}-\frac{608731q^9}{1479878400}+\frac{3461377q^8}{18265927680}-\frac{75303q^7}{14094080}+\frac{3634039q^6}{761080320}+\frac{7863q^5}{320320}-\frac{15953q^4}{183040}+\frac{927q^4}{32032}+\frac{50255q^2}{192192}+1]w^6+$$
$$[\frac{361q^{12}}{564277760}+\frac{39359q^{11}}{1775854080}+\frac{2450893q^{10}}{8879270400}+\frac{511502297q^9}{213102489600}+\frac{286829923q^8}{18265927680}+\frac{20171051q^7}{253693440}+\frac{238635989q^6}{761080320}+\frac{1223643q^5}{1281280}+\frac{402443q^4}{183040}+\frac{236179q^3}{64064}+\frac{814609q^2}{192192}+3q+1]w^3.$$

The region of absolute stability is plotted in Figure 1.

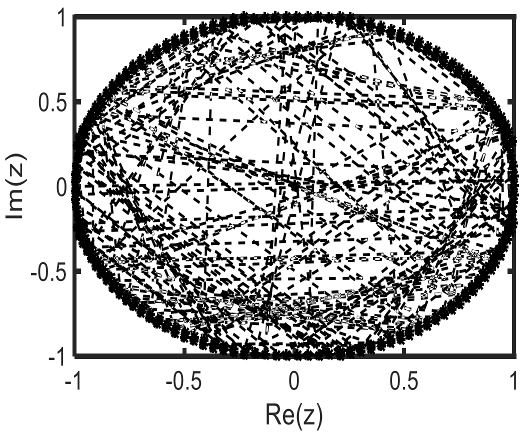

Fig. 1. Absolute stability region of the proposed method

V. RESULTS AND DISCUSSION

This section details the application of the developed three-step block method for the numerical solution of second-order FODEs (FIVPs and FBVPs). The results obtained are compared with the exact solution and existing methods as shown in tables and graphs.

x-axis shows the value of the approximation solution,
y-axis show the α -level values,

$\underline{Y},\overline{Y}$ are the exact solution of lower and upper bound, respectively,

$\underline{y},\overline{y}$ are the approximate solution of lower and upper bound, respectively,

$\underline{E}=\left|\underline{Y}-\underline{y}\right|$ is the absolute error of the lower bound solution,

$\overrightarrow{E}=\left|\overrightarrow{Y}-\overrightarrow{y}\right|$ is the absolute error of the upper bound solution,

and h is the step size.

The following notations are used in the tables:

TFDTBM	Third Fourth Derivative Three-step Block Method
EBHDEF	Extended Block Hybrid Backward Differentiation Formula [12]
OOMB	Optimization of One-Step Block Method [13]
STHWS	Single-Term Haar Wavelet Series Method [20]
HPM	Homotopy Perturbation Method [22]
VIM	Variational Iteration Method [22]

Example 1. [13]. Consider the second-order FIVP

$$y''(x)=x^2y''(x)+2xy(x)+\vartheta x$$

$$y(0,\alpha)=(1+\alpha,3-\alpha),\ y'(0,\alpha)=0,\ \vartheta=(2+2\alpha,6-2\alpha)$$

with exact solution

$$Y(x,\alpha)=\left[(2\bullet e^{\frac{x^2}{3}}-1)\bullet(1-\alpha,3-\alpha)\right]$$

computed at $Y(1,\alpha)=\left[\underline{Y}(1,\alpha),\overline{Y}(1,\alpha)\right]$.

The results obtained for Example 1 are shown in Table I, II, and Figure 2 displays the exact and approximate solution graph with step size h=0.1 partition of the time interval x ∈ [0,1].

TABLE I
LOWER AND UPPER SOLUTIONS FOR EXAMPLE 1

α	$\underline{y}$	$\overline{y}$
0	1.791224850172261000	5.373674550516782700
0.2	2.149469820206713100	5.015429580482330800
0.4	2.507714790241165400	4.657184610447878000
0.6	2.865959760275617300	4.298939640413426100
0.8	3.224204730310069600	3.940694670378974300
1	3.582449700344521900	3.582449700344521900

TABLE II
COMPARISON OF ABSOLUTE ERROR FOR EXAMPLE 1

α	TFDTBM $\underline{E}$	TFDTBM $\overline{E}$	OOBM $\underline{E}$	OOBM $\overline{E}$
0	8.14E-14	2.44E-13	1.20E-11	2.20E-11
0.2	9.76E-14	2.28E-13	2.30E-10	3.30E-10
0.4	1.14E-13	2.11E-13	2.60E-10	3.60E-10
0.6	1.30E-13	1.95E-13	3.40E-09	4.40E-10
0.8	1.46E-13	1.78E-13	3.60E-09	5.60E-09
1	1.62E-13	1.62E-13	4.20E-09	5.70E-09

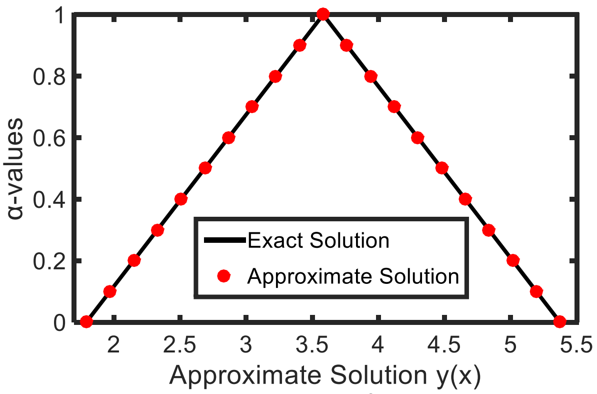

Fig. 2. Solution graph for Example 1

Table II shows that the approximate solution achieved by the developed block method is highly impressive in terms of absolute error compared to the exact solution. Figure 2 illustrates the results graphically, and the approximate solution overlaps the exact solution, indicating that the suggested method is highly accurate.

Example 2. [20] Consider the second-order FIVP

$$y''(x) = y(x) + x, x \geq 0$$

$$y(0,\alpha) = (0.9 + 0.1\alpha, 1.1 - 0.1\alpha)$$

$$y(0,\alpha) = (1.8 + 0.2\alpha, 2.2 - 0.2\alpha)$$

with exact solution

$$\underline{Y}(x,\alpha) = \left[(\tfrac{4}{5} + \tfrac{1}{5}\alpha)\bullet \sin x + (\tfrac{9}{10} + \tfrac{1}{10}\alpha)\bullet \cos x + x \right]$$

$$\overline{Y}(x,\alpha) = \left[(\tfrac{6}{5} - \tfrac{1}{5}\alpha)\bullet \sin x + (\tfrac{11}{10} + \tfrac{1}{10}\alpha)\bullet \cos x + x \right]$$

computed at $Y(1,\alpha) = \left[\underline{Y}(1,\alpha), \overline{Y}(1,\alpha) \right]$.

The results obtained for Example 2 are shown in Table III, IV, and Figure 3 displays the exact and approximate solution graph with step size h=0.1 partition of the time interval x ∈ [0,1].

TABLE III
LOWER AND UPPER SOLUTIONS FOR EXAMPLE 2

α	$\underline{y}$	$\overline{y}$
0	2.159448863127642900	2.604097718224429600
0.2	2.203913748637321500	2.559632832714750600
0.4	2.248378634147000500	2.515167947205072100
0.6	2.292843519656679000	2.470703061695393600
0.8	2.337308405166357500	2.426238176185715000
1	2.381773290676036000	2.381773290676036000

TABLE IV
COMPARISON OF ABSOLUTE ERROR FOR EXAMPLE 2

α	TFDTBM $\underline{E}$	TFDTBM $\overline{E}$	STHWS $\underline{E}$	STHWS $\overline{E}$
0	0	0	2.19E-10	3.06E-10
0.2	4.44E-16	0	2.20E-10	3.04E-10
0.4	0	0	2.22E-10	3.02E-10
0.6	0	0	2.24E-10	2.99E-10
0.8	0	0	2.26E-10	2.97E-10
1	0	0	2.28E-10	2.95E-10

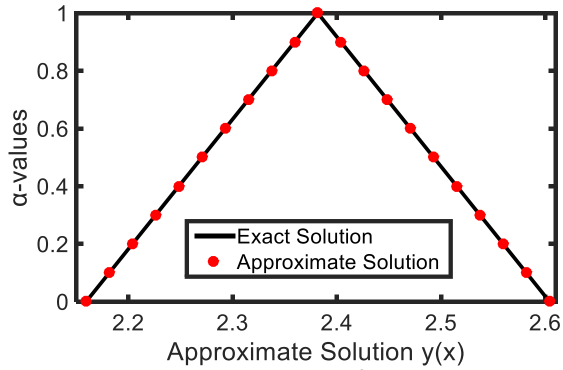

Fig. 3. Solution graph for Example 2

Table IV shows that the approximate solution achieved by the developed block method is highly impressive in terms of absolute error compared to the exact solution and gives the same results as the exact solution at certain points. Figure 3 illustrates the results graphically. The approximate solution entirely overlaps the exact solution, indicating that the suggested method is highly accurate.

Example 3. [21] Consider the second-order nonlinear crisp ordinary differential equation. From section 2, the fuzzy version of the second-order nonlinear ordinary differential equation with FIVP is written as follows

$$y''(x) - x(y'(x))^2 = 0, y(0) = 1, y'(0) = 0.5.$$

According to [2], one can defuzzify the initial condition in this problem. According to the definition of fuzzy number in section 2, let $[\alpha]_{\underline{\alpha}}^{\overline{\alpha}}$ be a triangular fuzzy number such that α ∈ [0,1]. Then the differential equation is defuzzified as

$$y''(x) - x\bullet y'^2(x) = 0, y(0) = (0.75 + 0.25\alpha, 1,25 - 0.25\alpha),$$

$$y'(0) = 0.5$$

with the exact solution as follows

$$\begin{cases} \underline{Y} = (0.75 + 0.25\alpha)\ln(\tfrac{2+x}{2-x})^{0.5} \\ \overline{Y} = (1.25 - 0.25\alpha)\ln(\tfrac{2+x}{2-x})^{0.5} \end{cases}.$$

The proposed method directly solves the second-order nonlinear FIVP with improved accuracy in terms of absolute error when comparing the approximate solution with the exact solution.

The results obtained for Example 3 are shown in Table V, VI, and Figure 4 displays the exact and approximate solution graph overlapping each other with step size h=0.1 partition of the time interval x ∈ [0,1].

TABLE V
LOWER AND UPPER SOLUTIONS FOR EXAMPLE 3

α	$\underline{y}$	$\overline{y}$
0	1.299306144334054800	1.799306144334054800
0.2	1.349306144334054800	1.749306144334054700
0.4	1.399306144334054600	1.699306144334054700
0.6	1.449306144334054900	1.649306144334054900
0.8	1.499306144334054700	1.599306144334054800
1	1.549306144334054800	1.549306144334054800

TABLE VI
ABSOLUTE ERROR OF EXAMPLE 3

α	TFDTBM $\underline{E}$	TFDTBM $\overline{E}$
0	1.634248e-13	1.838529e-13
0.2	1.643130e-13	1.816324e-13
0.4	1.647570e-13	1.798561e-13
0.6	1.676436e-13	1.774136e-13
0.8	1.696420e-13	1.743050e-13
1	1.718625e-13	1.718625e-13

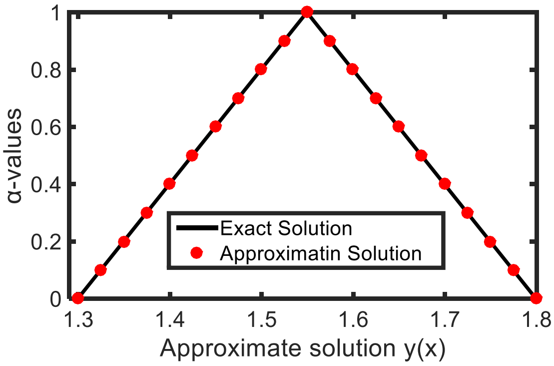

Fig. 4. Solution graph for Example 3

Example 4. [12] Consider the second-order FBVP

$$y''(x) = y(x) + 1$$

$$y(0,\alpha) = (\alpha - 1, 1 - \alpha),\ y(1,\alpha) = (\alpha + (e-2), e - \alpha),$$

with exact solution

$$\underline{Y}(x,\alpha) = \left[-1 + e^{x} + (\alpha - 1)\bullet\cos x + (\alpha - 1)\bullet\sin x\bullet\tan(0.5)\right]$$

$$\overline{Y}(x,\alpha) = \left[-1 + e^{x} + (1-\alpha)\bullet\cos x + (1-\alpha)\bullet\sin x\bullet\tan(0.5)\right]$$

computed at $Y(1,\alpha) = \left[\underline{Y}(1,\alpha), \overline{Y}(1,\alpha)\right]$.

The results obtained for Example 4 are shown in Table VII, VIII, and Figure 5 displays the exact and approximate solution graph with the step-size h=0.1 partition of the time interval x∈[0,1].

TABLE VII
LOWER AND UPPER SOLUTIONS FOR EXAMPLE 4

α	$\underline{y}$	$\overline{y}$
0	0.718281828459046200	2.718281828459045100
0.2	0.918281828459045490	2.518281828459044900
0.4	1.118281828459038300	2.318281828459046100
0.6	1.318281828459040700	2.118281828459045000
0.8	1.518281828459043100	1.918281828459045300
1	1.718281828459045300	1.718281828459045300

TABLE VIII
COMPARISON OF ABSOLUTE ERROR FOR EXAMPLE 4

α	TFDTBM $\underline{E}$	TFDTBM $\overline{E}$	EBHDEF $\underline{E}$	EBHDEF $\underline{E}$
0	0.00E+00	8.88E-16	4.13E-07	9.09E-03
0.2	6.66E-16	0.00E+00	4.25E-07	9.23E-03
0.4	7.54E-15	0.00E+00	4.26E-07	9.44E-03
0.6	5.32E-15	8.88E-16	4.27E-07	9.65E-03
0.8	2.66E-15	8.88E-16	4.28E-07	9.82E-03
1	6.66E-16	6.66E-16	4.20E-07	9.12E-03

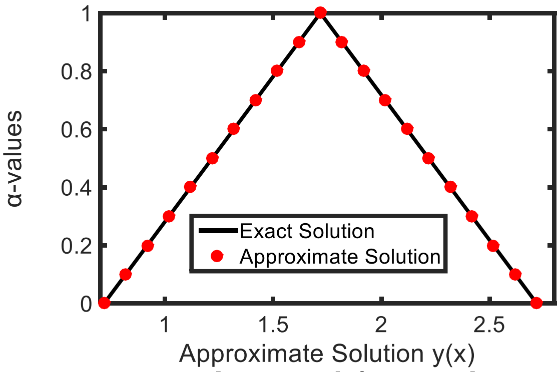

Fig. 5. Solution graph for Example 4

Table VII shows that the approximate solution achieved by the developed block method is highly impressive in terms of absolute error compared to the exact solution and gives the same results as the exact solution at certain points. Figure 5 illustrates the results graphically. The approximate solution entirely overlaps the exact solution, indicating that the suggested method is highly accurate.

Example 5. [22] Consider the non-linear second-order FBVP

$$y''(x) + y^{2}(x) = x^{4} + 2$$

$$y(0,\alpha) = (0.1\alpha - 0.1, 0.1 - 0.1\alpha)$$

$$y(1,\alpha) = (0.9 + 0.1\alpha, 1.1 - 0.1\alpha)$$

Since the considered Example 5 cannot be solved analytically, this example is solved by the proposed method in this study and compared to the obtained solutions using VIM and HPM. The approximate results obtained for Example 5 are shown in Table IX and X with the step size h=0.1 partition of the time interval x ∈ [0,1]. The initial guess for lower and upper solutions $C_1^{\alpha}, C_2^{\alpha}$, respectively, are given in the Table XI.

TABLE IX
LOWER SOLUTION OF EXAMPLE 5

α	TFDTBM $\underline{y}$	VIM $\underline{y}$	HPM $\underline{y}$
0	0.14326284380737	0.1436637372360	0.143660683335
0.25	0.169573368631593	0.1699759575944	0.169972596070
0.5	0.196062602033889	0.1964673284207	0.196463226242
0.75	0.222733404157411	0.2231409541657	0.223135428530
1	0.249588625319180	0.2500000293762	0.249992049854

TABLE X
UPPER SOLUTION OF EXAMPLE 5

α	TFDTBM $\overline{y}$	VIM $\overline{y}$	HPM $\overline{y}$
0	0.35891005049999	0.35935808190205	0.359319084359
0.25	0.33128904362649	0.33172332776916	0.331696667276
0.5	0.30386364452059	0.30428777924960	0.304269866579
0.75	0.27663110070175	0.27704784239391	0.277035924144
1	0.24958862531918	0.25000002937629	0.249992049854

TABLE XI
INITIAL GUESS FOR LOWER AND UPPER SOLUTIONS OF EXAMPLE 5

α	C_1^α	C_2^α
0	-0.012233517979781578	0.023538969063634813
0.25	-0.010209834659421881	0.016554919244886100
0.5	-0.007505097713710612	0.010307281584441760
0.75	-0.004110208069923338	0.004810786651353893
1	-0.000016001865820840	-0.00001600186582084

VI. Conclusion

The main goal of this study is to develop a numerical technique for solving second-order FODEs (FIVPs and FBVPs) with improved accuracy of the solution in terms of absolute error. As a result, for second-odder FODEs, this paper developed a three-step block technique with third and fourth derivatives. As indicated in the tables and graphs of the numerical results obtained, the suggested method surpasses previous methods discovered in the literature. Furthermore, the method eliminates the requirement for complicated subroutines in conventional methods that require starting values or predictors. Therefore, the proposed block method is viable for solving FIVPs and FBVPs with higher accuracy. The technique employed a linear block approach with minimal computational complexity and fulfilled all convergence conditions. Hence, the proposed method in this article is more suitable for obtaining the approximate solutions of second-order FIVPs and FBVPs.

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