A Novel Fractional Domain LFM Signal Parameter Estimation Method and its Application

Yong Guo, Xiao-Wei Zhang, Li-Dong Yang

***Abstract*—LFM signal contains target location information in its parameters, so the parameter estimation of linear frequency modulation (LFM) signal is a common concern in radar, communication and sonar fields. In this paper, a novel LFM signal parameter estimation method is proposed by using the fractional Fourier transform and Rényi entropy. An optimization model of LFM signal parameter estimation is established on the Rényi entropy distribution characteristic of LFM signal, and the HHO algorithm is applied to search for the optimal angle. The simulation results show that the proposed method has better parameter estimation stability and noise robustness than some existing methods. Finally, the Rényi entropy method is successfully applied to realize the high-precision separation of mixed interference fringe pattern.**

***Index Terms*—LFM signal, parameter estimation, fractional Fourier transform, Rényi entropy, Harris Hawks optimization, interference fringe**

I. INTRODUCTION

AS a typical non-stationary signal, the linear frequency modulation (LFM) signal is very suitable for the location, tracking and other target location perception applications because LFM signal contains target location information in its parameters. Since the signal will inevitably be mixed with noise during the actual acquisition and transmission process, the parameter estimation of LFM signal under noise environment has enormous theoretical and practical significance for many fields. The commonly used tools for LFM signal parameter estimation include maximum likelihood estimation (MLE), short-time Fourier transform (STFT), Wigner-Vier distribution (WVD) and fractional Fourier transform (FRFT) [5], [6]. MLE has an ideal asymptotic unbiased estimation effect, but it has high computational complexity owing to the two-dimensional search [7]. STFT uses a fixed time-frequency window to analyze the signal, so it cannot achieve the optimal resolution in both time and frequency domains [8], [9]. In addition, WVD has high computational complexity and cross-term interference for multi-component LFM signal processing [10].

FRFT uses the basis formed by orthogonal linear frequency modulation functions to decompose the signal, which is especially suitable for the fine feature extraction of LFM signal. Moreover, FRFT can give a lower time-bandwidth product, which means that FRFT can represent LFM signals more sparsely [11]. Hence, FRFT has been widely applied to LFM signal processing [12]–[16]. However, the existing parameter estimation methods based on FRFT have two problems: 1) the parameter estimation model is established on the peak of the fractional amplitude spectrum of LFM signal. For example, in [13], the Gaussian fitting method is used to approach the peak point gradually based on the energy concentration characteristics of LFM signal in an optimal fractional domain. In [14], the FRFT of LFM signal at different angles are conducted to form a two-dimensional spectrum distribution, and the coarse estimation is realized by searching the peak on the (α, u) plane. However, if the peak point is annihilated by noise, the model can not detect the LFM signal effectively; 2) a traversal search or two-level search strategy from coarse to fine is used to search the optimal angle. In [15], a coarse estimation is performed by searching for the maximum FRFT amplitude of LFM signal, then the accuracy is further improved by interpolation. In [16], the search region of angle corresponding to the coarse estimation is determined firstly, then FRFT is applied to complete the accurate estimation of LFM signal parameters within the search region. However, this kind of search algorithm is easily affected by the coarse search step.

In view of these shortcomings, a novel fractional domain LFM parameter estimation method is proposed in this paper. Firstly, the Rényi entropy distribution characteristic of LFM signal is analyzed, and then a LFM signal parameter estimation method is proposed based on this characteristic. The main innovations of this paper include: 1) an optimization model of LFM signal parameter estimation is established on FRFT and Rényi entropy, which improves the noise robustness of the parameter estimation method; 2) Harris Hawks optimization (HHO) algorithm is used to search the optimal angle, which effectively improves the stability of the parameter estimation method; 3) the proposed parameter estimation method is applied to the separation of mixed interference fringe.

The remaining sections of this paper are organized as follows: Section II gives a brief introduction to FRFT. In Section III, the principle and algorithm of the LFM signal parameter estimation method are given. The simulation experiment is performed in Section IV. An application of the parameter estimation method is shown in Section V. Section VI gives the conclusion.

II. FRACTIONAL FOURIER TRANSFORM

Fractional Fourier transform (FRFT) is a generalized form of Fourier transform, which can be regarded as a linear projection of the signal in the fractional-frequency space

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Yong Guo is a lecturer in the School of Science, Inner Mongolia University of Science and Technology, Baotou, 014010, China (e-mail: guo_yong@imust.edu.cn).

Xiao-Wei Zhang is a master in the School of information engineering, Inner Mongolia University of science and technology, Baotou, 014010, China (e-mail: 1506591334@qq.com).

Li-Dong Yang is a professor in the School of Information Engineering, Inner Mongolia University of Science and Technology, Baotou, 014010, China (e-mail: yld_nkd@imust.cn).

[11]. The FRFT of a signal $f(t)$ is defined as

$$F_f^\alpha(u) = \mathscr{F}^\alpha[f(t)](u) = \int_{\mathbb{R}} f(t) K_\alpha(t,u)\,\mathrm{d}t \tag{1}$$

The kernel $K_\alpha(t,u)$ is given by

$$K_\alpha(t,u) = \begin{cases} A_\alpha \mathrm{e}^{\frac{i}{2}[(t^2+u^2)\cot\alpha - 2tu\csc\alpha]}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k-1)\pi \end{cases} \tag{2}$$

where $A_\alpha = \sqrt{(1-\mathrm{i}\cot\alpha)/2\pi}$. The u axis is called as FRFT domain (abbreviated as fractional domain), the corresponding variable u is called as fractional frequency.

Suppose signal is approximately constrained within a time-width of $[-T/2, T/2]$ and bandwidth of $[-F/2, F/2]$, the discrete time FRFT can be given by [25]

$$F_f^\alpha(u) = \frac{A_\alpha}{2\Delta x} \sum_{n=-N}^{N} f\left(\frac{n}{2\Delta x}\right) \mathrm{e}^{\frac{i}{2}[((\frac{n}{2\Delta x})^2+u^2)\cot\alpha - \frac{n}{\Delta x}u\csc\alpha]} \tag{3}$$

Furthermore, the discrete FRFT can be obtained by quantizing the variable u in the fractional domain, i.e.,

$$F_f^\alpha\left(\frac{m}{2\Delta x}\right) = \frac{A_\alpha}{2\Delta x} \sum_{n=-N}^{N} f\left(\frac{n}{2\Delta x}\right) \mathrm{e}^{\frac{i}{2}[\frac{(m^2+n^2)\cot\alpha - 2mn\csc\alpha}{(2\Delta x)^2}]} \tag{4}$$

where $\Delta x = \sqrt{TF}$ denotes the normalized width.

From Eq. (2), the basis of FRFT is composed by orthogonal LFM functions, so FRFT is very suitable for analyzing LFM signal. The fractional domain signal representation combines both time domain and frequency domain information of the signal, so FRFT can be regarded as a time-frequency analysis method [19]. When α changes from 0 to $\pi/2$, FRFT can give the multi-domain characteristics of the signal from time domain to frequency domain, which effectively expand the visual and available dimensions of the signal [11].

III. Parameter Estimation Method

A. Principle

Rényi entropy (RE) is a generalization of Hartley entropy, Shannon entropy and minimum entropy, which can be used to measure the energy concentration degree of the signal time-frequency distribution. The definition of Rényi entropy is [20]

$$H_p(X) = \frac{1}{1-p}\log_2 \sum_{m=1}^{M}\sum_{n=1}^{N} x_{mn}^p \tag{5}$$

where $X = (x_{mn})_{M\times N}$ denotes the signal time-frequency distribution, p is the order of Rényi entropy, satisfying $p \geq 0$ and $p \neq 1$. With the decrease of Rényi entropy, the energy concentration of the signal time-frequency distribution becomes better.

According to the property of Rényi entropy, the fractional spectrum of LFM signal becomes tighter with the decrease of Rényi entropy. Next, a LFM signal is selected to verify this characteristic. Without loss of generality, a LFM signal is given by

$$f(t) = A_0 \mathrm{e}^{j2\pi f_0 t + \pi k t^2} \tag{6}$$

where $A_0 = 1$, $f_0 = 200$, $k = 100$, $T = 13$s, $f_s = 500$Hz. The Rényi entropy of $F_f^\alpha(u)$ is calculated with $\alpha = m\pi/2 (m \in [0,2])$, and the results are shown in Fig. 1. It can be seen from Fig. 1 that the Rényi entropy of $F_f^\alpha(u)$ has a global minimum with a certain angle.

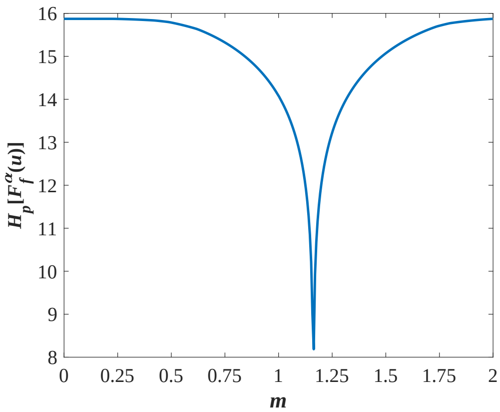

Fig. 1: The Rényi entropy distribution at different angles

The optimal angle can be determined by finding the minimum Rényi entropy, and the parameters of the LFM signal can be further estimated by the optimal angle. Therefore, according to this characteristic, the LFM signal parameter estimation model can be established as follows:

$$\begin{cases} \hat{\alpha} = \arg\min_\alpha H_p[F_f^\alpha(u)] \\ \hat{u} = \arg\max_u |F_f^{\hat{\alpha}}(u)| \\ \hat{k} = -\cot\hat{\alpha} \\ \hat{f}_0 = \hat{u}\csc\hat{\alpha} \end{cases} \tag{7}$$

where $\hat{\alpha}$ denotes the optimal angle, $\hat{k}$ and $\hat{f}_0$ represent the estimated frequency modulation rate and center frequency. In this method, Harris hawks optimization (HHO) algorithm is used to search for the optimal angle. HHO algorithm has the advantages of simple initialization parameter setting, strong global search ability, high search efficiency and no gradient limitation [17]. In particular, HHO is not affected by the search step, which can improve the stability of the parameter estimation method.

B. Algorithm

Step 1: The original discrete signal f_n is obtained by uniform sampling and dimensional normalization method.

Step 2: The cost function $H_p[F_f^\alpha(n)]$ is constructed by Eq. (1) and Eq. (5), and HHO is applied to search the optimal angle $\hat{\alpha}$.

Step 3: The estimated frequency modulation rate is given by $\hat{k} = -\cot\hat{\alpha}$.

Step 4: The discrete time FRFT of f_n is calculated with the optimal rotation angle $\hat{\alpha}$, denoted as $F_f^{\hat{\alpha}}(u)$.

Step 5: The cost function is set as $J(u) = |F_f^{\hat{\alpha}}(u)|$, and HHO is applied to search $\hat{u}$.

Step 6: The center frequency is estimated by $\hat{u}$ and $\hat{\alpha}$, i.e., $\hat{f}_0 = \hat{u}\csc\hat{\alpha}$.

IV. Simulation Experiment

Firstly, the Rényi entropy distribution characteristic for noisy LFM signal is analyzed. The noisy LFM signal is generated by adding Gaussian white noise to the LFM signal, where the range of SNR is $[-10, 5]$. Under different SNR, the

TABLE I: Parameter estimation results of Rényi entropy method

Results	-10	-5	0	5	10	$+\infty$
$\hat{k}$	99.9660	99.9645	99.9747	99.9954	99.9796	99.9967
$\hat{f}_0$	200.0713	200.0718	199.9990	199.9989	199.9922	200.0033
MSE$(k,\hat{k})$	0.0012	0.0013	6.38×10^{-4}	2.14×10^{-5}	4.16×10^{-4}	1.09×10^{-5}
MSE$(f_0,\hat{f}_0)$	0.0051	0.0052	1.00×10^{-6}	1.23×10^{-6}	6.01×10^{-5}	1.10×10^{-5}

Rényi entropy distribution of noisy LFM signal are shown in Fig. 2. From Fig. 2, the trends of Rényi entropy distribution gradually becomes flat with the decrease of SNR, but the peak characteristics are still obvious. In short, when SNR ≥ -10dB, the Rényi entropy distribution characteristic for noisy LFM signal is basically unaffected by Gaussian white noise.

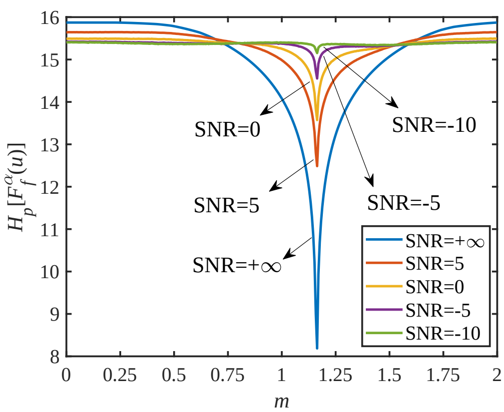

Fig. 2: The Rényi entropy distribution for noisy LFM signal

Next, the Rényi entropy distribution characteristic of LFM signal is used to establish the parameter estimation model and HHO algorithm is applied to search for the optimal angle. The parameters of HHO algorithm are

$$\{N,T,lb,ub,\text{dim}\} = \{200,9,1,2,1\}, \tag{8}$$

where N, T, lb, ub, dim denote the population number, maximum number of iterations, upper and lower bounds of variables and number of variables. For noiseless and noisy LFM signals, the estimated modulation frequency rate and center frequency are listed in Table I. From Table I, the parameter estimation accuracy of Rényi entropy method is basically unaffected by noise, so Rényi entropy method has good stability and noise robustness.

Finally, the Rényi entropy method is compared with the peak search method, concise FRFT (CFRFT) method and quasi-Newton method. For the search of the optimal angle, the equal step search is used in the peak search and CFRFT methods, and a two level search strategy from coarse-to-fine is used in the quasi-Newton method. The step of the peak search and CFRFT methods is set to 0.001, and the coarse search steps of the quasi-Newton method is set to 0.002. For noiseless and noisy LFM signal, the estimated parameters are listed in Table II. By comparing the data, it follows:

1) For noiseless LFM signal, the estimation accuracy of Rényi entropy method and quasi-Newton method with step 0.002 are higher than that of peak search and CFRFT methods. However, when the step size increases to 0.0046 in the quasi-Newton method, the $\hat{k}$ and $\hat{f}_0$ are 100.9513 and 203.3122, the corresponding MSE are 0.9049 and 10.9704. It shows that the estimation accuracy of the quasi-Newton method is easily affected by the coarse search step, especially for $\hat{f}_0$. Different from the quasi-Newton method, HHO is used to search the optimal angle in the Rényi entropy method, which is not affected by the search step. Therefore, the Rényi entropy method has better stability than that of quasi-Newton method.

2) For noisy LFM signal, the MSE$(k,\hat{k})$ and MSE$(f_0,\hat{f}_0)$ for different methods are shown in Fig. 3 and Fig. 4. From these figures, the Rényi entropy and quasi-Newton methods have better noise robustness than the other two methods. Moreover, the estimated parameters by quasi-Newton method with step sizes of 0.0046 and 0.002 are listed in table III. From Table III, the estimation accuracy of the quasi-Newton method decreases sharply when the step increases slightly to 0.0046. Compared with the quasi-Newton method, the Rényi entropy method has a more stable parameter estimation performance for the noisy LFM signal.

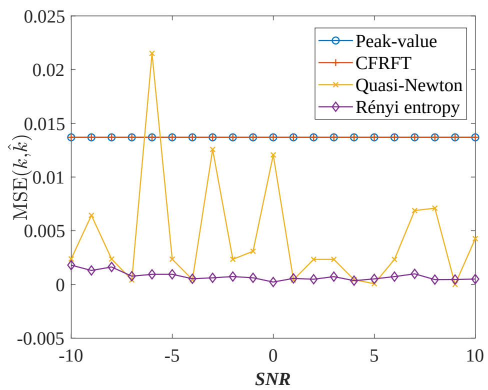

Fig. 3: MSE of estimated frequency modulation rate

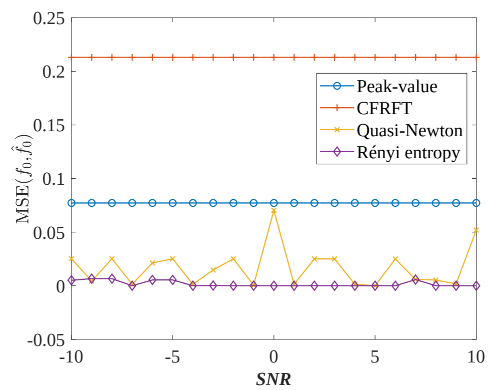

Fig. 4: MSE of estimated center frequency

In brief, the estimation accuracy and noise robustness of Rényi entropy method are higher than the peak search and CFRFT methods, and the stability is better than the quasi-Newton method.

TABLE II: Parameter estimation results for the noiseless case

Method	Step	$\hat{\alpha}$	$\hat{u}$	$\hat{k}$	$\hat{f}_0$	MSE$(k,\hat{k})$	MSE$(f_0,\hat{f}_0)$
Peak search	0.001	1.7660	11.5724	99.8829	199.7221	0.0137	0.0772
CFRFT	0.001	1.7660	32.1746	99.8829	199.5384	0.0137	0.2130
Quasi-Newton	0.002	1.7663	11.5737	100.0252	199.9912	6.36×10^{-4}	7.68×10^{-5}
Rényi entropy		1.7662	11.5771	99.9967	200.0033	1.09×10^{-5}	1.10×10^{-5}

TABLE III: The estimated parameters of quasi-Newton

Step	SNR	$\hat{k}$	$\hat{f}_0$
	0	99.8902	199.7347
0.002	-5	99.9515	199.8415
	-10	99.9512	199.8411
	0	100.6038	202.6997
0.0046	-5	100.6621	202.8035
	-10	100.6005	202.6958

V. Application

Interferometry is a widely used method in the optical measurement field, such as holographic interferometry, electronic speckle interferometry and fringe projection et al. [21]–[23]. As a common quadratic phase interference fringe, Newton's rings is frequently used to judge the convexity and concavity of lens surface, test the surface quality of optical elements and measure the curvature radius of lens. Generally, an interference fringe pattern contains multiple interference fringe families, each fringe family needs to be separated for subsequent optical parameter measurement. In the traditional methods, the difference in the amplitude of each fringe family is used to separate the mixed interference fringe pattern. However, it is difficult to separate if there is an intersection between the fringe families. The Rényi entropy method is built on the Rényi entropy distribution characteristic of the signal, which is unaffected by the intersection of fringe families. Therefore, the Rényi entropy method is used to separate the mixed interference fringe patterns with cross fringe families.

According to the optical imaging principle of Newton's rings, its mathematical model is given by

$$f(x,y) = I_0 - \frac{I_1}{2}\cos[\pi\mu[(x-x_0)^2+(y-y_0)^2]] \tag{9}$$

where I_0 and I_1 denote the background intensity and modulation amplitude, (x_0,y_0) and μ denote the center and width of Newton's rings. Next, a mixed interference fringe pattern containing two Newton's rings (see Fig. 4 (a)) is given by

$$F(x,y) = f_A(x,y) + f_B(x,y) \tag{10}$$

where $f_A(x,y)$ and $f_B(x,y)$ denote the Newton's rings containing a fringe family. Newton's ring can be uniquely determined by its width and center position, the pixel of central positions are set as (246,246) and (106,106). The width is set as $\mu = 2/(\lambda_0 R) = 3.9463$, where $R = 0.86$m denotes the curvature radius of the convex lens, $\lambda_0 = 589.3$nm denotes the wavelength of the sodium yellow lamp.

From Eq. (9), a row or column of Newton's rings is a multi-component LFM signal, so a column of Newton's rings is selected as the original signal. Then, the physical parameters of Newton's rings are estimated by Rényi entropy method. Finally, the estimated physical parameters are substituted into Eq. (9) to reconstruct Newton's rings. The separated Newtons rings are shown in Fig. 5 (b) and (c). From Fig. 5, a pure mixed interference fringe pattern with cross fringe families can be separated effectively by Rényi entropy method.

Next, for the noisy mixed fringe patterns, the estimated center and width are listed in Tables IV and V. Moreover, the absolute error and structural similarity (SSIM) are used to measure the difference between the reconstructed and original Newton's rings. From Tables IV and V, the absolute errors of the center are less than one pixel, and the absolute errors of the width are controlled within 0.0064. In addition, the SSIM between the reconstructed and original Newton's rings are larger than 0.9682. These data indicate that the separated fringe family is highly similar to the original fringe family, even for noisy cases. To sum up, when the mixed interference fringe pattern contains multiple Newton's rings, the Rényi entropy method can effectively separate each Newton's rings, which has the advantages of high precision and strong noise robustness.

VI. Conclusion

In this paper, a novel fractional domain LFM signal parameter estimation method is proposed based on the Rényi entropy distribution characteristic of LFM signal. Simulation results show that the estimation accuracy and noise robustness of Rényi entropy method are higher than that of the peak search and CFRFT methods, and the stability is better than that of the quasi-Newton method. Finally, the Rényi entropy method is successfully applied to the separation of mixed interference fringe patterns with cross fringe families, which has the advantages of high precision and strong noise robustness.

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TABLE IV: The estimated parameters for Newton's rings A

SNR(dB)	Center	Width	Center error	Width error	SSIM
-5	(245.8882246.0178)	3.9401	(0.1118,0.0178)	0.0062	0.9682
0	(245.9200245.9872)	3.9410	(0.0800,0.0128)	0.0053	0.9779
5	(245.9401245.9715)	3.9415	(0.0599,0.0285)	0.0048	0.9813
$+\infty$	(245.9664245.9489)	3.9402	(0.0336,0.0511)	0.0061	0.9705

TABLE V: The estimated parameters for Newton's rings B

SNR(dB)	Center	Width	Center error	Width error	SSIM
-5	(105.5345, 105.7948)	3.9401	(0.4655, 0.2025)	0.0062	0.9878
0	(105.6379, 105.8329)	3.9413	(0.3621, 0.1671)	0.0050	0.9925
5	(105.6435, 105.8230)	3.9412	(0.3565, 0.1770)	0.0051	0.9926
$+\infty$	(105.7080, 105.7745)	3.9399	(0.2920, 0.2255)	0.0064	0.9888

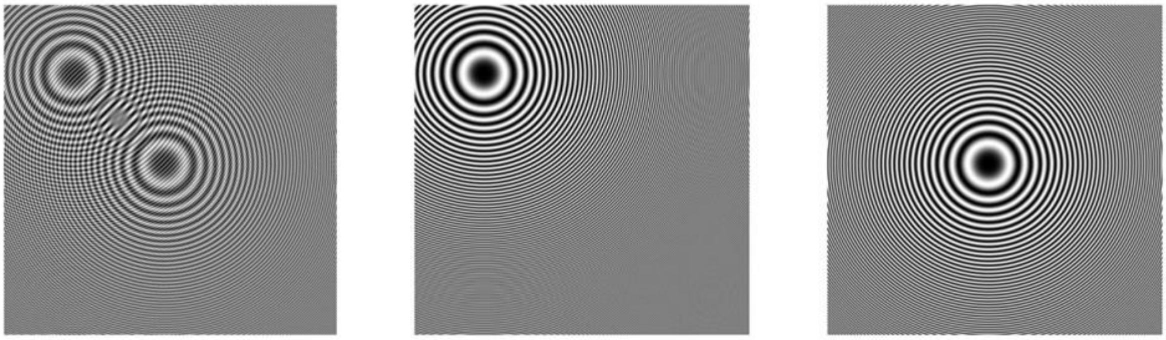

Fig. 5: The separation of mixed interference fringes, (a) Mixed interference fringe pattern, (b) Newton's rings A, (c) Newton's rings B

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