Several Oscillatory Results for a Class of Fractional Differential Equations With the Fractional Derivative in Conformable Sense

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Abstract—**In this paper, we are concerned with oscillation of a class of fractional differential equations, where the fractional derivative is defined in the sense of the conformable fractional derivative. Based on the properties of conformable fractional calculus, Riccati transformation, inequality and integration average technique, some new oscillatory criteria for the fractional differential equations are established. We also present some examples for the established results.**

Index Terms—**oscillation, fractional equations, fractional derivative, Riccati transformation**

I. Introduction

Recently, many effective numerical and analytical methods have been proposed for various differential equations [1-9]. In this work, we focus on the research of oscillatory properties for differential equations. In [10-26], oscillation of solutions of various differential equations and systems as well as dynamic equations on time scales were researched, and a lot of new oscillation criteria for these equations have been established therein. In these investigations, we notice that relatively less attention has been paid to the research of oscillation of fractional differential equations [27-32], and the fractional derivative lying in the existing results are almost defined in the sense of the Riemann-Liouville derivative.

Recently, Khalil et al. proposed a new definition for fractional derivative named conformable fractional derivative [33]. The fractional derivative is defined as follows:

Definition 1. $D^\alpha f(t)=\lim\limits_{\varepsilon\to 0}\dfrac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$.

From the definition of the conformable fractional derivative, one can easily verify the following properties:

(i). $D_t^\alpha[af(t)+bg(t)]=aD_t^\alpha f(t)+bD_t^\alpha g(t)$.

(ii). $D_t^\alpha(t^\gamma)=\gamma t^{\gamma-\alpha}$.

(iii). $D_t^\alpha[f(t)g(t)]=f(t)D_t^\alpha g(t)+g(t)D_t^\alpha f(t)$.

(iv). $D^\alpha C=0$, where C is a constant.

(v). $D_t^\alpha f[g(t)]=f'_g[g(t)]D_t^\alpha g(t)$.

(vi). $D_t^\alpha(\dfrac{f}{g})(t)=\dfrac{g(t)D^\alpha f(t)-f(t)D^\alpha g(t)}{g^2(t)}$.

(vii). $D_t^\alpha f(t)=t^{1-\alpha}f'(t)$.

As one can see, the conformable fractional derivative is of fine characters, especially the chain rule can be satisfied here. Many authors investigated various applications of the conformable fractional derivative [34-39].

In this paper, we are concerned with oscillation of a class of fractional differential equations as follows:

$$D_t^\alpha(r(t)D_t^\alpha x(t))+q(t)f(x(t))=0,$$
$$t\geq t_0>0,\ 0<\alpha<1, \tag{1.1}$$

where $D_t^\alpha(.)$ denotes the conformable fractional derivative with respect to the variable t, the function $r\in C^\alpha([t_0,\infty),R_+)$, $q\in C([t_0,\infty),R_+)$, and C^α denotes continuous derivative of order α, the function f is continuous satisfying $f(x)/x\geq K$ for some positive constant K and $\forall x\neq 0$.

As usual, a solution $x(t)$ of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Eq. (1.1) is called oscillatory if all its solutions are oscillatory.

We organize the next of this paper as follows. In Section 2, using the properties of conformable fractional calculus, Riccati transformation, inequality and integration average technique, we establish some new oscillatory criteria for Eq. (1.1), while we present some applications for them in Section 3. Some conclusions are presented at the end of this paper.

For the sake of convenience, in the next of this paper, we denote $\xi=\dfrac{t^\alpha}{\alpha}$, $\xi_i=\dfrac{t_i^\alpha}{\alpha}$, $i=0,1,2,3$, $R_+=(0,\infty)$.

II. Oscillatory criteria for Eq. (1.1)

Lemma 1. Assume $x(t)$ is a eventually positive solution of Eq. (1.1), and

$$\int_{t_0}^\infty\frac{t^{\alpha-1}}{r(t)}dt=\infty. \tag{2.1}$$

Then there exists a sufficiently large T such that $D_t^\alpha x(t)>0$ for $t\in[T,\infty)$.

Proof. Let $r(t)=\widetilde{r}(\xi)$, $x(t)=\widetilde{x}(\xi)$, $q(t)=\widetilde{q}(\xi)$, where $\xi=\dfrac{t^\alpha}{\alpha}$. Then by use of the property (ii) we obtain

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$D_t^\alpha \xi(t) = 1$, and furthermore by use of the property (v) we have

$$D_t^\alpha r(t) = D_t^\alpha \widetilde{r}(\xi) = \widetilde{r}'(\xi) D_t^\alpha \xi(t) = \widetilde{r}'(\xi).$$

Similarly we have $D_t^\alpha x(t) = \widetilde{x}'(\xi)$. So Eq. (1.1) can be transformed into the following form:

$$(\widetilde{r}(\xi)\widetilde{x}'(\xi))' + \widetilde{q}(\xi) f(\widetilde{x}(\xi)) = 0,$$

$$\xi \geq \xi_0 \geq 0, \tag{2.2}$$

Since $x(t)$ is a eventually positive solution of (1.1), then $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (2.2), and there exists $\xi_1 > \xi_0$ such that $\widetilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Furthermore, we have

$$\begin{aligned}(\widetilde{r}(\xi)\widetilde{x}'(\xi))' &= -\widetilde{q}(\xi) f(\widetilde{x}(\xi)) \\ &\leq -K\widetilde{q}(\xi)\widetilde{x}(\xi) < 0,\ \xi \geq \xi_1. \end{aligned} \tag{2.3}$$

Then $\widetilde{r}(\xi)\widetilde{x}'(\xi)$ is strictly decreasing on $[\xi_1, \infty)$, and thus $\widetilde{x}'(\xi)$ is eventually of one sign. We claim $\widetilde{x}'(\xi) > 0$ on $[\xi_2, \infty)$, where $\xi_2 > \xi_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $\xi_3 > \xi_2$ such that $\widetilde{x}'(\xi) < 0$ on $[\xi_3, \infty)$. Then for $\xi \in [\xi_3, \infty)$ we have

$$\begin{aligned}\widetilde{x}(\xi) - \widetilde{x}(\xi_3) &= \int_{\xi_3}^{\xi} \widetilde{x}'(s)ds = \int_{\xi_3}^{\xi} \frac{\widetilde{r}(s)\widetilde{x}'(s)}{\widetilde{r}(s)} ds \\ &\leq \widetilde{r}(\xi_3)\widetilde{x}'(\xi_3) \int_{\xi_3}^{\xi} \frac{1}{\widetilde{r}(s)} ds = \widetilde{r}(\xi_3)\widetilde{x}'(\xi_3) \int_{t_3}^{\infty} \frac{t^{\alpha-1}}{r(t)} dt.\end{aligned}$$

By (2.1) we deduce that $\lim_{\xi\to\infty} \widetilde{x}(\xi) = -\infty$, which contradicts to the fact that $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (2.2). So $\widetilde{x}'(\xi) > 0$ on $[\xi_2, \infty)$, and furthermore $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$. The proof is complete by setting $T = t_2$.

Theorem 2. Assume (2.1) holds, and there exist two functions $\phi \in C^1([t_0, \infty), R_+)$ and $\varphi \in C^1([t_0, \infty), [0, \infty))$ such that

$$\int_{\xi_0}^{\infty} \{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\} ds = \infty, \tag{2.4}$$

where $\widetilde{\phi}(\xi) = \phi(t),\ \widetilde{q}(\xi) = q(t),\ \widetilde{\varphi}(\xi) = \varphi(t),\ \widetilde{r}(\xi) = r(t)$. Then every solution of Eq. (1.1) is oscillatory.

Proof. Assume (1.1) has a non-oscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ for some sufficiently large $t_2 > t_1$. Define the generalized Riccati transformation function:

$$\omega(t) = \phi(t)\{\frac{r(t)D_t^\alpha x(t)}{x(t)} + \varphi(t)\}.$$

Then for $t \in [t_2, \infty)$, by use of the property (v) and (vi) we have

$$\begin{aligned}D_t^\alpha \omega(t) &= D_t^\alpha \phi(t)\frac{r(t)D_t^\alpha x(t)}{x(t)} - \phi(t)\frac{r(t)(D_t^\alpha x(t))^2}{x^2(t)} \\ &\quad + \phi(t)\frac{D_t^\alpha(r(t)D_t^\alpha x(t))}{x(t)} + D_t^\alpha \phi(t)\varphi(t) + \phi(t)D_t^\alpha \varphi(t) \\ &= \frac{D_t^\alpha \phi(t)}{\phi(t)}\omega(t) - \frac{(\omega(t) - \phi(t)\varphi(t))^2}{\phi(t)r(t)} \\ &\quad - \phi(t)\frac{q(t)f(x(t))}{x(t)} + \phi(t)D_t^\alpha \varphi(t) \\ &= -\phi(t)\frac{q(t)f(x(t))}{x(t)} + \phi(t)D_t^\alpha \varphi(t) - \frac{\phi(t)\varphi^2(t)}{r(t)} \\ &\quad + \frac{2\phi(t)\varphi(t) + D_t^\alpha \phi(t)r(t)}{\phi(t)r(t)}\omega(t) - \frac{1}{\phi(t)r(t)}\omega^2(t) \\ &\leq -K\phi(t)q(t) + \phi(t)D_t^\alpha \varphi(t) - \frac{\phi(t)\varphi^2(t)}{r(t)} \\ &\quad + \frac{[2\phi(t)\varphi(t) + D_t^\alpha \phi(t)r(t)]^2}{4\phi(t)r(t)}.\end{aligned} \tag{2.5}$$

Let $\omega(t) = \widetilde{\omega}(\xi)$. Then $D_t^\alpha w(t) = \widetilde{w}'(\xi)$, $D_t^\alpha \phi(t) = \widetilde{\phi}'(\xi)$, $D_t^\alpha \varphi(t) = \widetilde{\varphi}'(\xi)$, and (2.5) is transformed into the following form

$$\widetilde{\omega}'(\xi) \leq -K\widetilde{\phi}(\xi)\widetilde{q}(\xi) + \widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi) - \frac{\widetilde{\phi}(\xi)\widetilde{\varphi}^2(\xi)}{\widetilde{r}(\xi)} + \frac{[2\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi) + \widetilde{\phi}'(\xi)\widetilde{r}(\xi)]^2}{4\widetilde{\phi}(\xi)\widetilde{r}(\xi)}, \xi \geq \xi_2. \tag{2.6}$$

Substituting ξ with s in (2.6), an integration for (2.6) with respect to s from ξ_2 to ξ yields

$$\begin{aligned}&\int_{\xi_2}^{\xi} \{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\} ds \\ &\leq \widetilde{\omega}(\xi_2) - \omega(\xi) \leq \omega(\xi_2) < \infty,\end{aligned}$$

which contradicts to (2.4). So the proof is complete.

Theorem 3. Assume (2.1) holds, and there exists a function $H \in C([\xi_0, \infty), \mathbf{R})$ such that $H(\xi, \xi) = 0,\ for\ \xi \geq \xi_0,\ \ H(\xi, s) > 0,\ for\ \xi > s \geq \xi_0$, and H has a nonpositive continuous partial derivative $H_s'(\xi, s)$. If

$$\lim_{\xi\to\infty} \sup \frac{1}{H(\xi, \xi_0)}\{\int_{\xi_0}^{\xi} H(\xi, s)\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\} ds\} = \infty, \tag{2.7}$$

where $\widetilde{\phi},\ \widetilde{\varphi},\ \widetilde{q},\ \widetilde{r}$ are defined as in Theorem 2, then every solution of Eq. (1.1) is oscillatory.

Proof. Assume (1.1) has a non-oscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ for some sufficiently large $t_2 > t_1$. Let $\omega(t)$ and $\widetilde{\omega}(\xi)$ be defined as in Theorem 2. By (2.6) we have

$$K\widetilde{\phi}(\xi)\widetilde{q}(\xi) - \widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi) + \frac{\widetilde{\phi}(\xi)\widetilde{\varphi}^2(\xi)}{\widetilde{r}(\xi)} - \frac{[2\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi) + \widetilde{\phi}'(\xi)\widetilde{r}(\xi)]^2}{4\widetilde{\phi}(\xi)\widetilde{r}(\xi)} \leq -\widetilde{\omega}'(\xi),\ \xi \geq \xi_2. \tag{2.8}$$

Substituting ξ with s in (2.8), multiplying both sides by

$H(\xi, s)$ and then integrating with respect to s from ξ_2 to ξ yields

$$\int_{\xi_2}^{\xi} H(\xi,s)\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds$$
$$\leq -\int_{\xi_2}^{\xi} H(\xi,s)\widetilde{\omega}'(s)ds$$
$$= H(\xi,\xi_2)\omega(\xi_2) + \int_{\xi_2}^{\xi} H_s'(\xi,s)\omega(s)\Delta s$$
$$\leq H(\xi,\xi_2)\omega(\xi_2) \leq H(\xi,\xi_0)\omega(\xi_2).$$

Then

$$\int_{\xi_0}^{\xi} H(\xi,s)\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds$$
$$= \int_{\xi_0}^{\xi_2} H(\xi,s)\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds$$
$$+ \int_{\xi_2}^{\xi} H(\xi,s)\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds$$
$$\leq H(\xi,\xi_0)\widetilde{\omega}(\xi_2) + H(\xi,\xi_0)\int_{\xi_0}^{\xi_2} |K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}|ds.$$

So

$$\lim_{\xi\to\infty}\sup \frac{1}{H(\xi,\xi_0)}\{\int_{\xi_0}^{\xi} H(\xi,s)\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\}$$
$$\leq \widetilde{\omega}(\xi_2) + \int_{\xi_0}^{\xi_2} |K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}|ds < \infty,$$

which contradicts to (2.7). So the proof is complete.

Corollary 4. Under the conditions of Theorem 3, if

$$\lim_{\xi\to\infty}\sup \frac{1}{(\xi-\xi_0)^\lambda}\{\int_{\xi_0}^{\xi} (\xi-s)^\lambda\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\} = \infty,$$

then every solution of Eq. (1.1) is oscillatory.

Corollary 5. Under the conditions of Theorem 3, if

$$\lim_{\xi\to\infty}\sup \frac{1}{(\ln\xi - \ln\xi_0)}\{\int_{\xi_0}^{\xi} (\ln\xi - \ln s)\{K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)} - \frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds\} = \infty,$$

then every solution of Eq. (1.1) is oscillatory.

The proof of Corollaries 4-5 can be completed by choosing $H(\xi,s) = (\xi-s)^\lambda,\ \lambda > 1$ or $H(\xi,s) = \ln\frac{\xi}{s}$ in Theorem 3.

Theorem 6. Let $h_1,\ h_2,\ \widehat{H} \in C([\xi_0,\infty),R)$ satisfying $\widehat{H}(\xi,\xi) = 0, \widehat{H}(\xi,s) > 0,\ \xi > s \geq \xi_0$, and H has continuous partial derivatives $\widehat{H}_\xi'(\xi,s)$ and $\widehat{H}_s'(\xi,s)$ on $[\xi_0,\infty)$ such that

$$\widehat{H}_\xi'(\xi,s) = -h_1(\xi,s)\sqrt{\widehat{H}(\xi,s)},$$
$$\widehat{H}_s'(\xi,s) = -h_2(\xi,s)\sqrt{\widehat{H}(\xi,s)}.$$

If for any sufficiently large $T \geq \xi_0$, there exist $a,\ b,\ c$ with $T \leq a < c < b$ satisfying

$$\frac{1}{\widehat{H}(c,a)}\int_a^c \widehat{H}(s,a)[K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds + \frac{1}{\widehat{H}(b,c)}\int_c^b \widehat{H}(b,s)[K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$> \frac{1}{4\widehat{H}(c,a)}\int_a^c \widetilde{\phi}(s)\widetilde{r}(s)Q_1^2(s,a)ds + \frac{1}{4\widehat{H}(b,c)}\int_c^b \widetilde{\phi}(s)\widetilde{r}(s)Q_2^2(b,s)ds, \tag{2.9}$$

where $\widetilde{\phi},\ \widetilde{\varphi},\ \widetilde{q},\ \widetilde{r}$ are defined as in Theorem 2, $Q_1(s,\xi) = h_1(s,\xi) - (\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})\sqrt{\widehat{H}(s,\xi)}$, $Q_2(\xi,s) = h_2(\xi,s) - (\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})\sqrt{\widehat{H}(\xi,s)}$, then Eq. (1.1) is oscillatory.

Proof. Assume (1.1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_2,\infty)$, where t_2 is sufficiently large. Let $\omega(t)$ and $\widetilde{\omega}(\xi)$ be defined as in Theorem 2. So for $t \in [t_2,\infty)$, we have

$$D_t^\alpha\omega(t) = -\phi(t)\frac{q(t)f(x(t))}{x(t)} + \phi(t)D_t^\alpha\varphi(t) - \frac{\phi(t)\varphi^2(t)}{r(t)} + \frac{2\phi(t)\varphi(t) + D_t^\alpha\phi(t)r(t)}{\phi(t)r(t)}\omega(t) - \frac{1}{\phi(t)r(t)}\omega^2(t)$$
$$\leq -K\phi(t)q(t) + \phi(t)D_t^\alpha\varphi(t) - \frac{\phi(t)\varphi^2(t)}{r(t)} + \frac{2\phi(t)\varphi(t) + D_t^\alpha\phi(t)r(t)}{\phi(t)r(t)}\omega(t) - \frac{1}{\phi(t)r(t)}\omega^2(t). \tag{2.10}$$

Furthermore, similar to (2.5), (2.10) is transformed into the following form

$$\widetilde{\omega}'(\xi) \leq -K\widetilde{\phi}(\xi)\widetilde{q}(\xi) + \widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi) - \frac{\widetilde{\phi}(\xi)\widetilde{\varphi}^2(\xi)}{\widetilde{r}(\xi)} + \frac{2\widetilde{\phi}(\xi)\widetilde{\varphi}(\xi) + \widetilde{\phi}'(\xi)\widetilde{r}(\xi)}{\widetilde{\phi}(\xi)\widetilde{r}(\xi)}\widetilde{\omega}(\xi) - \frac{1}{\widetilde{\phi}(\xi)\widetilde{r}(\xi)}\widetilde{\omega}^2(\xi), \xi \geq \xi_2. \tag{2.11}$$

Select $a,\ b,\ c$ arbitrarily in $[\xi_2,\infty)$ with $b > c > a$. Substituting ξ with s, multiplying both sides of (2.11) by $\widehat{H}(\xi,s)$ and integrating it with respect to s from c to ξ for $\xi \in [c,b)$, we get that

$$\int_c^\xi \widehat{H}(\xi,s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$\le -\int_c^\xi \widehat{H}(\xi,s)\widetilde{w}'(s)ds$$
$$+\int_c^\xi \widehat{H}(\xi,s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})\widetilde{w}(s)ds$$
$$-\frac{\widehat{H}(\xi,s)}{\widetilde{\phi}(s)\widetilde{r}(s)}\widetilde{\omega}^2(s)ds$$
$$=\widehat{H}(\xi,c)\widetilde{w}(c)-\int_c^\xi[\left(\frac{\widehat{H}(\xi,s)}{\widetilde{\phi}(s)\widetilde{r}(s)}\right)^{1/2}\widetilde{w}(s)$$
$$+\frac{1}{2}(\widetilde{\phi}(s)\widetilde{r}(s))^{1/2}Q_2(\xi,s)]^2ds+\int_c^\xi \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)ds$$
$$\le \widehat{H}(\xi,c)\widetilde{w}(c)+\int_c^\xi \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)ds. \quad (2.12)$$

Dividing both sides of the inequality (2.12) by $\widehat{H}(\xi,c)$ and let $\xi\to b^-$, we obtain

$$\frac{1}{\widehat{H}(b,c)}\int_c^b \widehat{H}(b,s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$\le \widetilde{w}(c)+\frac{1}{\widehat{H}(b,c)}\int_c^b \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(b,s)ds. \quad (2.13)$$

On the other hand, substituting ξ with s, multiplying both sides of (2.11) by $\widehat{H}(s,\xi)$ and integrating it with respect to s from ξ to c for $\xi\in(a,c]$, we get that

$$\int_\xi^c \widehat{H}(s,\xi)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$\le -\int_\xi^c \widehat{H}(s,\xi)\widetilde{w}'(s)ds$$
$$+\int_\xi^c \widehat{H}(s,\xi)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})\widetilde{w}(s)ds$$
$$-\int_\xi^c \widehat{H}(s,\xi)\frac{\widetilde{w}^2(s)}{\widetilde{\phi}(s)\widetilde{r}(s)}ds$$
$$=-\widehat{H}(c,\xi)\widetilde{w}(c)-\int_\xi^c[\left(\frac{\widehat{H}(s,\xi)}{\widetilde{\phi}(s)\widetilde{r}(s)}\right)^{1/2}\widetilde{w}(s)$$
$$+\frac{1}{2}(\widetilde{\phi}(s)\widetilde{r}(s))^{1/2}Q_1(s,\xi)]^2ds+\int_\xi^c \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi)ds$$
$$\le -\widehat{H}(c,\xi)\widetilde{w}(c)+\int_\xi^c \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,\xi)ds. \quad (2.14)$$

Dividing both sides of the inequality (2.14) by $\widehat{H}(c,\xi)$ and letting $\xi\to a^+$, we obtain

$$\frac{1}{\widehat{H}(c,a)}\int_a^c \widehat{H}(s,a)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$\le -\widetilde{w}(c)+\frac{1}{\widehat{H}(c,a)}\int_a^c \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,a)ds. \quad (2.15)$$

A combination of (2.13) and (2.15) yields

$$\frac{1}{\widehat{H}(c,a)}\int_a^c \widehat{H}(s,a)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)$$
$$+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$+\frac{1}{\widehat{H}(b,c)}\int_c^b \widehat{H}(b,s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)$$
$$+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$\le \frac{1}{4\widehat{H}(c,a)}\int_a^c \widetilde{\phi}(s)\widetilde{r}(s)Q_1^2(s,a)ds$$
$$+\frac{1}{4\widehat{H}(b,c)}\int_c^b \widetilde{\phi}(s)\widetilde{r}(s)Q_2^2(b,s)ds,$$

which contradicts to (2.9). So the proof is complete.

Theorem 7. Under the conditions of Theorem 6, if for any $l\ge\xi_0$,

$$\lim_{\xi\to\infty}\sup\int_l^\xi[\widehat{H}(s,l)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]$$
$$-\frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,l)]ds>0 \quad (2.16)$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi[\widehat{H}(\xi,s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]$$
$$-\frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(\xi,s)]ds>0, \quad (2.17)$$

then Eq. (1.1) is oscillatory.

Proof: For any $T\ge\xi_0$, let $a=T$. In (2.16) we choose $l=a$. Then there exists $c>a$ such that

$$\int_a^c[\widehat{H}(s,a)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]$$
$$-\frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_1^2(s,a)]ds>0. \quad (2.18)$$

In (2.17) we choose $l=c>a$. Then there exists $b>c$ such that

$$\int_c^b[\widehat{H}(b,s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]$$
$$-\frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}Q_2^2(b,s)]ds>0. \quad (2.19)$$

Combining (2.18) and (2.19) we obtain (2.9). The conclusion thus comes from Theorem 6, and the proof is complete.

In Theorems 6-7, if we choose $\widehat{H}(\xi,s)=(\xi-s)^\lambda$, $\xi\ge s\ge\xi_0$, where $\lambda>1$ is a constant, then we obtain the following two corollaries.

Corollary 8. Under the conditions of Theorem 6, if for any sufficiently large $T\ge\xi_0$, there exist $a,\ b,\ c$ with $T\le a<c<b$ satisfying

$$\frac{1}{(c-a)^\lambda}\int_a^c(s-a)^\lambda[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)$$
$$+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$+\frac{1}{(b-c)^\lambda}\int_c^b(b-s)^\lambda[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)$$
$$+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$

$$> \frac{1}{4(c-a)^\lambda}\int_a^c \widetilde{\phi}(s)\widetilde{r}(s)(s-a)^{\lambda-2}$$
$$(\lambda+(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})(s-a))^2ds$$
$$+\frac{1}{4(b-c)^\lambda}\int_c^b \widetilde{\phi}(s)\widetilde{r}(s)(b-s)^{\lambda-2}$$
$$(\lambda-(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})(b-s))^2ds, \quad (2.20)$$

then Eq. (1.1) is oscillatory.

Corollary 9. Under the conditions of Theorem 7, if for any $l \geq \xi_0$,

$$\lim_{\xi\to\infty}\sup\int_l^\xi\{(s-l)^\lambda[K\widetilde{\phi}(s)\widetilde{q}(s)$$
$$-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]-\frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}(s-l)^{\lambda-2}$$
$$[\lambda+(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})(s-l)]^2\}ds>0 \quad (2.21)$$

and

$$\lim_{\xi\to\infty}\sup\int_l^\xi\{(\xi-s)^\lambda[K\widetilde{\phi}(s)\widetilde{q}(s)$$
$$-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]-\frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}(\xi-s)^{\lambda-2}$$
$$[\lambda-(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})(\xi-s)]^2\}ds>0, \quad (2.22)$$

then Eq. (1.1) is oscillatory.

Theorem 10. Under the conditions of Theorem 6, furthermore, suppose (2.9) does not hold. If for any $T \geq \xi_0$, there exist $a,\ b$ with $b > a \geq T$ such that for any $u \in C[a,b]$, $u'(t) \in L^2[a,b]$, $u(a) = u(b) = 0$, the following inequality holds:

$$\int_a^b\{u^2(s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]-\widetilde{\phi}(s)\widetilde{r}(s)$$
$$(u'(s)+\tfrac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)}))^2\}ds>0, \quad (2.23)$$

where $\widetilde{\phi},\ \widetilde{\varphi},\ \widetilde{q},\ \widetilde{r}$ are defined as in Theorem 2, then Eq. (1.1) is oscillatory.

Proof: Assume (1.1) has a non-oscillatory solution x on $[t_0,\infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_2,\infty)$, where t_2 is sufficiently large. Let $\omega(t)$ and $\widetilde{\omega}(\xi)$ be defined as in Theorem 2. Similar to the proof of Theorem 6, we obtain (2.11). Select $a,\ b$ arbitrarily in $[\xi_2,\infty)$ with $b > a$ such that $u(a) = u(b) = 0$. Substituting ξ with s, multiplying both sides of (2.11) by $u^2(s)$, integrating it with respect to s from a to b, we get that

$$\int_a^b u^2(s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]ds$$
$$\leq -\int_a^b u^2(s)\widetilde{w}'(s)ds-\int_a^b u^2(s)\frac{\widetilde{w}^2(s)}{\widetilde{\phi}(s)\widetilde{r}(s)}ds$$
$$+\int_a^b u^2(s)\widetilde{w}(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})ds$$
$$=2\int_a^b u(s)u'(s)\widetilde{w}(s)ds-\int_a^b u^2(s)\frac{\widetilde{w}^2(s)}{\widetilde{\phi}(s)\widetilde{r}(s)}ds$$
$$+\int_a^b u^2(s)\widetilde{w}(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})ds$$
$$=-\int_a^b\{[\sqrt{\frac{1}{\widetilde{\phi}(s)\widetilde{r}(s)}}u(s)\widetilde{w}(s)-\sqrt{\widetilde{\phi}(s)\widetilde{r}(s)}$$
$$(u'(s)+\tfrac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)}))]^2+\widetilde{\phi}(s)\widetilde{r}(s)$$
$$\left(u'(s)+\tfrac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})\right)^2\}ds.$$

Moreover,

$$\int_a^b\{u^2(s)[K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]-\widetilde{\phi}(s)\widetilde{r}(s)$$
$$\left(u'(s)+\tfrac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})\right)^2\}ds\leq 0, \quad (2.24)$$

which contradicts to (2.23). So every solution of Eq. (1.1) is oscillatory, and the proof is complete.

Remark. In Theorems 2, 3, 6, 7, 10, if we assume $f \in C^1[R,R]$ satisfying $f'(x) \geq \mu > 0$ for $x \neq 0$, and modify the definition of $\omega(t)$ by $\omega(t) = \phi(t)\{\frac{r(t)D_t^\alpha x(t)}{f(x(t))} + \varphi(t)\}$, then following a similar process, we can obtain similar oscillatory criteria for Eq. (1.1), which are omitted here.

As of the term $f(x(t))$ in Eq. (1.1), the main results presented in the theorems are different from existing results in [10,12].

III. Applications

In this section, by presenting several examples, we apply the results established above to research oscillation of some fractional differential equations.

Example 1. Consider the following fractional differential equation:

$$D_t^\alpha(\sqrt{\frac{t^\alpha}{\alpha}}D_t^\alpha x(t))+(\frac{t^\alpha}{\alpha})^{-\frac{3}{2}}\frac{x(t)(2+x^2(t))}{1+x^2(t)}=0,$$
$$t\geq 2,\ 0<\alpha<1. \quad (3.1)$$

In fact, if we set in Eq. (1.1) $t_0 = 2$, $r(t)=\sqrt{\frac{t^\alpha}{\alpha}}$, $q(t)=(\frac{t^\alpha}{\alpha})^{-\frac{3}{2}}$, $f(x)=\frac{x(2+x^2)}{1+x^2}$, then we obtain (3.1). So $\widetilde{r}(\xi)=r(t)=\sqrt{\frac{t^\alpha}{\alpha}}=\sqrt{\xi}$, $\widetilde{q}(\xi)=q(t)=(\frac{t^\alpha}{\alpha})^{-\frac{3}{2}}=\xi^{-\frac{3}{2}}$, and $f(x)/x \geq 1$, which implies $K = 1$. Furthermore, in (2.1),

$$\int_{t_0}^\infty\frac{\alpha t^{\alpha-1}}{\alpha r(t)}dt=\int_{\xi_0}^\infty\frac{1}{\sqrt{\xi}}d\xi=\infty.$$

In (2.4), letting $\widetilde{\phi}(\xi)=\sqrt{\xi},\ \widetilde{\varphi}(\xi)=0$, we obtain

$$\int_{\xi_0}^\infty\{K\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}$$
$$-\frac{[2\widetilde{\phi}(s)\widetilde{\varphi}(s)+\widetilde{\phi}'(s)\widetilde{r}(s)]^2}{4\widetilde{\phi}(s)\widetilde{r}(s)}\}ds$$

$$= \int_{\xi_0}^{\infty} \frac{15}{16s} ds = \infty.$$

Therefore, Eq. (3.1) is oscillatory by Theorem 2.

Example 2. Consider the following fractional differential equation:

$$D_t^{2\alpha} x(t) + \frac{t^\alpha}{\alpha} x(t) e^{x^2(t)} = 0,$$

$$t \geq 5,\ 0 < \alpha < 1. \tag{3.2}$$

In fact, if we set in Eq. (1.1) $t_0 = 5,\ r(t) \equiv 1,\ q(t) = \frac{t^\alpha}{\alpha},\ f(x) = xe^{x^2}$, then we obtain (3.2). So $\widetilde{r}(\xi) \equiv 1$, $\widetilde{q}(\xi) = q(t) = \frac{t^\alpha}{\alpha} = \xi$, and $f(x)/x = e^{x^2} \geq 1$, which implies $K = 1$. Furthermore, in (2.21)-(2.22), after letting $\widetilde{\phi}(\xi) \equiv 1,\ \widetilde{\varphi}(\xi) = 0,\ \lambda = 2$, considering $\widetilde{q}(s) \equiv 1$, we obtain

$$\lim_{\xi\to\infty} \sup \int_l^\xi \{(s-l)^\lambda [K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s)$$
$$+ \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}] - \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}(s-l)^{\lambda-2}$$
$$[\lambda + (\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})(s-l)]^2\} ds$$
$$= \lim_{\xi\to\infty} \sup \int_l^\xi \left[s(s-l)^2 - 1\right] ds = \infty$$

and

$$\lim_{\xi\to\infty} \sup \int_l^\xi \{(\xi-s)^\lambda [K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s)$$
$$+ \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}] - \frac{\widetilde{\phi}(s)\widetilde{r}(s)}{4}(\xi-s)^{\lambda-2}$$
$$[\lambda - (\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})(\xi-s)]^2\} ds$$
$$= \lim_{\xi\to\infty} \sup \int_l^\xi \left[s(\xi-s)^2 - 1\right] ds = \infty.$$

So according to Corollary 9 we deduce that Eq. (3.2) is oscillatory.

Example 3. Consider the following fractional differential equation:

$$D_t^\alpha \left(\sin^2(\frac{t^\alpha}{\alpha}) D_t^\alpha x(t)\right) + x(t)(1 + x^2(t)) = 0,$$

$$t \geq 2,\ 0 < \alpha < 1. \tag{3.3}$$

If we set in Eq. (1.1) $t_0 = 2,\ r(t) = \sin^2(\frac{t^\alpha}{\alpha}),\ q(t) \equiv 1,\ f(x) = x(1+x^2)$, then we obtain (3.3). So $\widetilde{r}(\xi) = r(t) = \sin^2(\frac{t^\alpha}{\alpha}) = \sin^2\xi$, $\widetilde{q}(\xi) \equiv 1$, and $f(x)/x = 1 + x^2 \geq 1$, which implies $K = 1$. Furthermore, in (2.23), after letting $\widetilde{\phi}(\xi) \equiv 1,\ \widetilde{\varphi}(\xi) = 0,\ a = 2k\pi,\ b = 2k\pi + \pi,\ u(s) = \sin s$, then $u(a) = u(b) = 0$, and we obtain

$$\int_a^b \{u^2(s)[K\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}]$$
$$-\widetilde{\phi}(s)\widetilde{r}(s)\left(u'(s) + \tfrac{1}{2}u(s)(\frac{2\widetilde{\phi}(s)\widetilde{\varphi}(s) + \widetilde{\phi}'(s)\widetilde{r}(s)}{\widetilde{\phi}(s)\widetilde{r}(s)})\right)^2\} ds$$
$$= \int_{2k\pi}^{2k\pi+\pi} \left(\sin^2 s - \sin^2 s \cos^2 s\right) ds$$
$$= \int_{2k\pi}^{2k\pi+\pi} \sin^4 s ds > 0.$$

Therefore, Eq. (3.3) is oscillatory by Theorem 10.

Remark. We note that oscillation for the three examples above can not be obtained by existing results so far in the literature.

IV. Conclusions

We have established some new oscillatory criteria for a class of fractional differential equations with the fractional derivative defined in the sense of the conformable fractional derivative. Some applications for these established results are also presented. We note that the approach in establishing the main theorems above can be generalized to research oscillation of fractional differential equations with more complicated forms such as with damping term or with forced term, which are expected to further research.

For example, the following fractional differential equations with damping term and forced term respectively can be further investigated:

$$D_t^\alpha(r(t)D_t^\alpha x(t)) + r(t)D_t^\alpha x(t) + q(t)f(x(t)) = 0$$

and

$$D_t^\alpha(r(t)D_t^\alpha x(t)) + r(t)D_t^\alpha x(t) + q(t)f(x(t)) = p(t).$$

Also some higher order fractional differential equations can be further considered, such as

$$D_t^\alpha[D_t^\alpha(r(t)D_t^\alpha x(t))] + q(t)f(x(t)) = 0$$

and

$$D_t^\alpha[D_t^\alpha(r(t)D_t^\alpha x(t))] + D_t^\alpha(r(t)D_t^\alpha x(t)) + q(t)f(x(t)) = 0.$$

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