Fuzzy Higher Derivative Block Method with Generalised Steplength for Direct Solution of Second-Order Fuzzy Ordinary Differential Equations

Kashif Hussain, Oluwaseun Adeyeye, and Nazihah Ahmad

Abstract—**This study intends to improve the solution accuracy of the second-order fuzzy ordinary differential equation (FODEs). As a result, it introduces two fuzzy higher derivative terms into the block scheme with generalised steplength. The generalised steplength scheme is then adopted to develop a four-step block method for solving directly second-order fuzzy initial and boundary value problems. Its properties to ensure convergence and show the region of absolute stability are investigated in fuzzy form. The numerical results, compared to the exact solution of the numerical problems under consideration (applications in engineering), showed the new block method performs better than existing numerical methods in terms of solution accuracy. Therefore, the proposed method is suitable for directly solving models defined as second-order fuzzy ordinary differential equations.**

Index Terms—**fuzzy differential equations, second-order, four-step, block method.**

I. Introduction

SECOND-ORDER differential equations have many applications, especially in mechanical and electrical engineering, chemistry, biology, physics, electronics, etc. However, unpredictable circumstances may arise, introducing the concept of uncertainty and using fuzzy derivatives and fuzzy differential equations (FDEs) to deal with these situations [1]. There are three differentiations used to define the differential of fuzzy functions, the Hukuhara derivative (H-derivative) [2], the Seikkala derivative [3], and the generalized derivative (g-derivative) [4]. This research focuses on the H-derivative to define the differential equations covered in this article, which follows the definition used by authors' whose findings were compared in the numerical examples with the newly developed block method in this article.

Most second-order problems modelled as FODEs may be difficult to solve directly, and it is not always possible to obtain their exact solution. As a result, experts were interested in employing various numerical methods to obtain an approximate solution. Numerous scholars have developed a variety of numerical approaches for solving second-order FODEs [5]-[13]. Reducing the second-order FODE system to first-order FODEs is the most significant disadvantage of these techniques since it increases computing effort and compromises solution accuracy. Therefore, block methods were used as a direct numerical solution to the second-order FODEs to avoid the rigour of reduction [14]-[16]. However, because of the order of the developed techniques, the accuracy of their obtained findings in terms of absolute error is low and might be improved. As a result, this article's motivation is to develop a block method with two fuzzy higher derivative terms to improve accuracy.

This article is organized as follows: Section 2 presents the basic definitions for fuzzy set theory. Section 3 offers the development of the k-step (four-step) block method with the presence of third and fourth derivatives using linear block approach. The basic properties of the block method are highlighted in Section 4, while Section 5 shows the results obtained for the linear and non-linear numerical examples. The article is concluded in Section 6.

II. Basic Definitions

This section recalls some basic definitions which will be adopted in this article.

Definition 2.1 [17]

The link between the crisp and fuzzy domain is represented by the α-level set (cuts) of a fuzzy set A with the crisp set X, denoted by

$$\left[A\right]=\left\{x\in X \mid \mu_A(x)\geq\alpha,\alpha\in[0,1]\right\}.$$

Definition 2.2 [17]

Fuzzy numbers are a subset of real numbers and represent uncertain values. A fuzzy number is connected to the degree of membership which states how genuine it is to say in case something has a place or not in a decided set.

A fuzzy number M is called a triangular fuzzy number $(a^1,a^2,a^3)\in\mathbb{R}^3, a^1\leq a^2\leq a^3$ with membership degree $M(x)$ given as

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$$M(x)=\begin{cases}0, x<a^1\\ \frac{x-a^1}{a^2-a^1}, a^1\le x\le a^2\\ \frac{a^3-x}{a^3-a^2}, a^2<x\le a^3\\ 0, \mathrm{x}>a^3\end{cases}, \quad (1)$$

and corresponding α-level set

$$M_\alpha=\left\{a^1-\alpha(a^2-a^1), a^3-\alpha(a^3-a^2)\right\}, \alpha\in[0,1]. \quad (2)$$

A fuzzy number M is called a trapezoidal fuzzy number $(a^1,a^2,a^3,a^4)\in\mathbb{R}^4, a^1\le a^2\le a^3\le a^4$ with membership degree $M(x)$ given as

$$M(x)=\begin{cases}0, x<a^1\\ \frac{x-a^1}{a^2-a^1}, a^1\le x<a^2\\ 1, a^2\le x\le a^3\\ \frac{a^3-x}{a^3-a^2}, a^3<x\le a^4\\ 0, \mathrm{x}>a^4\end{cases}, \quad (3)$$

and corresponding α-level set

$$M_\alpha=\left\{a^1-\alpha(a^2-a^1), a^4-\alpha(a^4-a^3)\right\}, \alpha\in[0,1]. \quad (4)$$

Definition 2.3 [18]

A fuzzy function $f(x)$ is called Hukuhara differentiable if h>0 is sufficiently small, then H-difference exists $f(x+h)-f(x), f(x)-f(x-h),$ such that $f'(x)=\lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to0}\frac{f(x)-f(x-h)}{h}$.

III. Methodology

Consider the second-order FODE of the form

$$y^{''}(x)=f(x,y(x),y^{'}(x)) \quad (5)$$

where $y^{''}(x)$ is an H-derivative of y, which is a fuzzy function of crisp variable x. Two lower and upper solutions exist since the given function is fuzzy and the parametric form in the α-level set is defined as

$$\overline{\underline{y}}^{''}(x,\alpha)=\overline{\underline{f}}(x,y(x,\alpha),y^{'}(x,\alpha))$$

$$\underline{f}=\min(x,\underline{y}(x,\alpha),\underline{y}^{'}(x,\alpha)), \overline{f}=\max(x,\overline{y}(x,\alpha),\overline{y}^{'}(x,\alpha)).$$

The generalised k-step block method with the presence of third and fourth fuzzy derivatives for the direct solution of Equation (5) form is stated as,

$$y_{n+\eta}=\left(\sum_{\zeta=0}^{1}\frac{(\eta h)^\zeta}{\zeta!}y_n^\zeta+\sum_{d-0}^{2}\left[\sum_{\zeta=0}^{k}\psi_{d\zeta\eta}f_{n+\zeta}^{d}\right]\right)_{\underline{\alpha}}^{\overline{\alpha}}, \eta=1,2,3,\ldots,k \quad (6)$$

with derivative expression

$$y'_{n+\eta}=\left(y_n^\zeta+\sum_{d-0}^{2}\left[\sum_{\zeta=0}^{k}\varphi_{d\zeta\eta}f_{n+\zeta}^{d}\right]\right)_{\underline{\alpha}}^{\overline{\alpha}}, \eta=1,2,3,\ldots,k. \quad (7)$$

Expanding Equations (6) and (7) produces the expressions in Equations (8) and (9) respectively

$$y_{n+\eta}=\begin{pmatrix}y_n+hy_n^{'}+\\ \begin{bmatrix}\psi_{00\eta}f_n+\psi_{01\eta}f_{n+1}+\psi_{02\eta}f_{n+2}+,\ldots,+\psi_{0k\eta}f_{n+k}+\\ \psi_{10\eta}f_n^{'}+\psi_{11\eta}f_{n+1}^{'}+\psi_{12\eta}f_{n+2}^{'}+,\ldots,+\psi_{1k\eta}f_{n+k}^{'}+\\ \psi_{20\eta}f_n^{''}+\psi_{21\eta}f_{n+1}^{''}+\psi_{22\eta}f_{n+2}^{''}+,\ldots,+\psi_{2k\eta}f_{n+k}^{''}+\end{bmatrix}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}} \quad (8)$$

$$y_{n+\eta}^{'}=\begin{pmatrix}y_n^{'}+\\ \begin{bmatrix}\varphi_{00\eta}f_n+\varphi_{01\eta}f_{n+1}+\varphi_{02\eta}f_{n+2}+,\ldots,+\varphi_{0k\eta}f_{n+k}+\\ \varphi_{10\eta}f_n^{'}+\varphi_{11\eta}f_{n+1}^{'}+\varphi_{12\eta}f_{n+2}^{'}+,\ldots,+\varphi_{1k\eta}f_{n+k}^{'}+\\ \varphi_{20\eta}f_n^{''}+\varphi_{21\eta}f_{n+1}^{''}+\varphi_{22\eta}f_{n+2}^{''}+,\ldots,+\varphi_{2k\eta}f_{n+k}^{''}+\end{bmatrix}\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}. \quad (9)$$

By applying Taylor series expansions by [19],

$$y(x+h;\alpha)=\left(\sum_{i=0}^{n}\frac{h^i}{i!}f^{(i)}(x,\alpha)\right)_{\underline{\alpha}}^{\overline{\alpha}} \quad (10)$$

to expand each term in Equations (8) and (9) yields

$$y_{n+\eta}=y(x+\eta h;\alpha)=\left(\sum_{i=0}^{n}\frac{(\eta h)^i}{i!}f^{(i)}(x_n,\alpha)\right)_{\underline{\alpha}}^{\overline{\alpha}}, \eta=0,1,\ldots,k$$

$$y_{n+\eta}=\begin{pmatrix}y(x_n;\alpha)+\eta hy'(x_n;\alpha)+\frac{(\eta h)^2}{2!}y''(x_n;\alpha)\\ +\frac{(\eta h)^3}{3!}y'''(x_n;\alpha)+\ldots++\frac{(\eta h)^n}{\mathrm{n}!}y^n(x_n;\alpha)\end{pmatrix}_{\underline{\alpha}}^{\overline{\alpha}}.$$

The unknown coefficients $\psi_{d\zeta\eta}$ and $\phi_{d\zeta\eta}$ are obtained using the matrix inverse method, where $\psi_{d\zeta\eta}=A^{-1}B$, $\phi_{d\zeta\eta}=A^{-1}D$, with

$$A=\begin{pmatrix}1&1&\ldots&1&0&0&\ldots&0&0&0&\ldots&0\\ 0&h&\ldots&kh&1&1&\ldots&1&0&0&\ldots&0\\ 0&\frac{h^2}{2!}&\ldots&\frac{(kh)^2}{2!}&0&h&\ldots&kh&1&1&\ldots&1\\ 0&\frac{h^3}{3!}&\ldots&\frac{(kh)^3}{3!}&0&\frac{h^2}{2!}&\ldots&\frac{(kh)^2}{2!}&0&h&\ldots&kh\\ .&.&\ldots&.&.&.&\ldots&.&.&.&\ldots&.\\ .&.&\ldots&.&.&.&\ldots&.&.&.&\ldots&.\\ .&.&\ldots&.&.&.&\ldots&.&.&.&\ldots&.\\ 0&\frac{h^{3k+2}}{(3k+2)!}&\ldots&\frac{(kh)^{3k+2}}{(3k+2)!}&0&\frac{h^{3k+1}}{(3\mathrm{k}+1)!}&\ldots&\frac{(kh)^{3k+1}}{(3\mathrm{k}+1)!}&0&\frac{(h)^{3k}}{3\mathrm{k}!}&\ldots&\frac{(kh)^{3k}}{5\mathrm{k}!}\end{pmatrix}$$

$$B=\begin{pmatrix}\frac{(\eta h)^2}{2!}\\ \frac{(\eta h)^3}{3!}\\ \frac{(\eta h)^4}{4!}\\ \frac{(\eta h)^5}{5!}\\ .\\ .\\ .\\ \frac{(\eta h)^{(3k+4)}}{(3\mathrm{k}+4!}\end{pmatrix}, \text{ and } D=\begin{pmatrix}\eta h\\ \frac{(\eta h)^2}{2!}\\ \frac{(\eta h)^3}{3!}\\ \frac{(\eta h)^4}{4!}\\ .\\ .\\ .\\ \frac{(\eta h)^{(3k+3)}}{(3\mathrm{k}+3)!}\end{pmatrix}.$$

The values obtained are substituted in Equations (8) and (9) to get the desired generalized k-step block method with the presence of third and fourth derivatives for solving second-order FODEs. A more detailed explanation is given in the following subsection, where the generalized steplength (k-step) block method scheme with the presence of third and fourth derivatives is adopted to develop a four-step (k=4) block method for second-order FODEs.

Development of Four-Step Block Method

Consider developing a four-step block method (FSBM) with third and fourth fuzzy derivatives for second-order FODEs by substituting k=4 in Equations (8) and (9) which results in Equations (11) and (12) below

$$y_{n+\eta} = \left(\begin{array}{l} y_n + h y_n' + \\ \left[\begin{array}{l} \psi_{00\eta} f_n + \psi_{01\eta} f_{n+1} + \psi_{02\eta} f_{n+2} + \psi_{03\eta} f_{n+3} + \psi_{04\eta} f_{n+4} + \\ \psi_{10\eta} f_n' + \psi_{11\eta} f_{n+1}' + \psi_{12\eta} f_{n+2}' + \psi_{13\eta} f_{n+3}' + \psi_{14\eta} f_{n+4}' + \\ \psi_{20\eta} f_n'' + \psi_{21\eta} f_{n+1}'' + \psi_{22\eta} f_{n+2}'' + \psi_{23\eta} f_{n+3}'' + \psi_{24\eta} f_{n+4}'' \end{array} \right] \end{array} \right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$\eta = 1,2,3,4, \tag{11}$$

$$y_{n+\eta}' = \left(\begin{array}{l} y_n' + \\ \left[\begin{array}{l} \varphi_{00\eta} f_n + \varphi_{01\eta} f_{n+1} + \varphi_{02\eta} f_{n+2} + \varphi_{03\eta} f_{n+3} + \varphi_{04\eta} f_{n+4} + \\ \varphi_{10\eta} f_n' + \varphi_{11\eta} f_{n+1}' + \varphi_{12\eta} f_{n+2}' +, \varphi_{13\eta} f_{n+3}' + \varphi_{14\eta} f_{n+4}' + \\ \varphi_{20\eta} f_n'' + \varphi_{21\eta} f_{n+1}'' + \varphi_{22\eta} f_{n+2}'' + \varphi_{23\eta} f_{n+3}'' + \varphi_{24\eta} f_{n+4}'' \end{array} \right] \end{array} \right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$\eta = 1,2,3,4. \tag{12}$$

The unknown coefficients in Equations (11) and (12) are obtained using matrix A for k=4 and matrices B and D for $\eta = 1,2,3,4$ as

$$\begin{pmatrix} \psi_{001} \\ \psi_{011} \\ \psi_{021} \\ \psi_{031} \\ \psi_{041} \\ \psi_{101} \\ \psi_{111} \\ \psi_{121} \\ \psi_{131} \\ \psi_{141} \\ \psi_{201} \\ \psi_{211} \\ \psi_{221} \\ \psi_{231} \\ \psi_{241} \end{pmatrix} = \begin{pmatrix} \frac{1278669671h^2}{4598415360} \\ \frac{20793569h^2}{38918880} \\ \frac{442685h^2}{946176} \\ \frac{18801383h^2}{116756640} \\ \frac{-15685603h^2}{2846638090} \\ \frac{531347h^3}{16220160} \\ \frac{19491781h^3}{311351040} \\ \frac{230389h^3}{3294720} \\ \frac{-18603301h^3}{311351040} \\ \frac{61072477h^3}{39852933120} \\ \frac{29662123h^4}{19926466560} \\ \frac{81h^4}{1792} \\ \frac{-5218561h^4}{92252160} \\ \frac{3229249h^4}{311351040} \\ \frac{-779951h^4}{6642155520} \end{pmatrix}, \quad \begin{pmatrix} \psi_{002} \\ \psi_{012} \\ \psi_{022} \\ \psi_{032} \\ \psi_{042} \\ \psi_{102} \\ \psi_{112} \\ \psi_{122} \\ \psi_{132} \\ \psi_{142} \\ \psi_{202} \\ \psi_{212} \\ \psi_{222} \\ \psi_{232} \\ \psi_{242} \end{pmatrix} = \begin{pmatrix} \frac{2305208h^2}{3648645} \\ \frac{7490816h^2}{3648645} \\ \frac{-38h^2}{35} \\ \frac{1514752h^2}{3648645} \\ \frac{-10420}{729729} \\ \frac{763657h^3}{9729720} \\ \frac{24928h^3}{135135} \\ \frac{6784h^3}{45045} \\ \frac{-992h^3}{6435} \\ \frac{5521h^3}{1389960} \\ \frac{17783h^4}{4864860} \\ \frac{159584h^4}{1216215} \\ \frac{-1h^4}{7} \\ \frac{32608h^4}{1216215} \\ \frac{-1481h^4}{4864860} \end{pmatrix}$$

$$\begin{pmatrix} \psi_{003} \\ \psi_{013} \\ \psi_{023} \\ \psi_{033} \\ \psi_{043} \\ \psi_{103} \\ \psi_{113} \\ \psi_{123} \\ \psi_{133} \\ \psi_{143} \\ \psi_{203} \\ \psi_{213} \\ \psi_{223} \\ \psi_{233} \\ \psi_{243} \end{pmatrix} = \begin{pmatrix} \frac{80789193h^2}{82001920} \\ \frac{16821h^2}{4576} \\ \frac{-1436859h^2}{1576960} \\ \frac{123867h^2}{160160} \\ \frac{-147393h^2}{6307840} \\ \frac{20358801h^3}{164003840} \\ \frac{421767h^3}{1281280} \\ \frac{592677h^3}{2562560} \\ \frac{-347751h^3}{1281280} \\ \frac{81891h^3}{12615680} \\ \frac{477441h^4}{82001920} \\ \frac{280827h^4}{1281280} \\ \frac{-2123577h^4}{10250240} \\ \frac{81h^4}{1792} \\ \frac{-40743h^4}{82001920} \end{pmatrix}, \quad \begin{pmatrix} \psi_{004} \\ \psi_{014} \\ \psi_{024} \\ \psi_{034} \\ \psi_{044} \\ \psi_{104} \\ \psi_{114} \\ \psi_{124} \\ \psi_{134} \\ \psi_{144} \\ \psi_{204} \\ \psi_{214} \\ \psi_{224} \\ \psi_{234} \\ \psi_{244} \end{pmatrix} = \begin{pmatrix} \frac{4872272h^2}{3648645} \\ \frac{6557696h^2}{1216215} \\ \frac{-1024h^2}{1155} \\ \frac{118784h^2}{56133} \\ \frac{52552h^2}{1216215} \\ \frac{6224h^3}{36855} \\ \frac{618496h^3}{1216215} \\ \frac{13568h^3}{45045} \\ \frac{-77824h^3}{173745} \\ \frac{-688h^3}{173745} \\ \frac{1376h^4}{173745} \\ \frac{126976h^4}{405405} \\ \frac{-13568h^4}{45045} \\ \frac{126976h^4}{1216215} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \varphi_{001} \\ \varphi_{011} \\ \varphi_{021} \\ \varphi_{031} \\ \varphi_{041} \\ \varphi_{101} \\ \varphi_{111} \\ \varphi_{121} \\ \varphi_{131} \\ \varphi_{141} \\ \varphi_{201} \\ \varphi_{211} \\ \varphi_{221} \\ \varphi_{231} \\ \varphi_{241} \end{pmatrix} = \begin{pmatrix} \frac{3028574741h}{8539914240} \\ \frac{6272003h}{5307120} \\ \frac{-14753h}{18480} \\ \frac{1438733h}{5307120} \\ \frac{-551113363h}{59779399680} \\ \frac{457963769h^2}{9963233280} \\ \frac{1331003h^2}{19459440} \\ \frac{2187h^2}{17920} \\ \frac{-391297h^2}{3891888} \\ \frac{25533817h^2}{9963233280} \\ \frac{10906367h^3}{4981616640} \\ \frac{794921h^3}{9729720} \\ \frac{-34589h^3}{360360} \\ \frac{169439h^3}{9729720} \\ \frac{-977791h^3}{4981616640} \end{pmatrix}, \quad \begin{pmatrix} \varphi_{002} \\ \varphi_{012} \\ \varphi_{022} \\ \varphi_{032} \\ \varphi_{042} \\ \varphi_{102} \\ \varphi_{112} \\ \varphi_{122} \\ \varphi_{132} \\ \varphi_{142} \\ \varphi_{202} \\ \varphi_{212} \\ \varphi_{222} \\ \varphi_{232} \\ \varphi_{242} \end{pmatrix} = \begin{pmatrix} \frac{82429429h}{233513280} \\ \frac{543736h}{331695} \\ \frac{-256h}{1155} \\ \frac{78856h}{331695} \\ \frac{-1950581h}{233513280} \\ \frac{1772191h^2}{38918880} \\ \frac{36632h^2}{243243} \\ \frac{h^2}{70} \\ \frac{-107656h^2}{1216215} \\ \frac{90527h^2}{38918880} \\ \frac{42001h^3}{19459440} \\ \frac{108104h^3}{1216215} \\ \frac{-3392h^3}{45045} \\ \frac{2696h^3}{173745} \\ \frac{-3473h^3}{19459440} \end{pmatrix}$$

$$\begin{pmatrix} \varphi_{003} \\ \varphi_{013} \\ \varphi_{023} \\ \varphi_{033} \\ \varphi_{043} \\ \varphi_{103} \\ \varphi_{113} \\ \varphi_{123} \\ \varphi_{133} \\ \varphi_{143} \\ \varphi_{203} \\ \varphi_{213} \\ \varphi_{223} \\ \varphi_{233} \\ \varphi_{243} \end{pmatrix} = \begin{pmatrix} \frac{29017419h}{82001920} \\ \frac{11691h}{7280} \\ \frac{2187h}{6160} \\ \frac{723h}{1040} \\ \frac{-819531h}{82001920} \\ \frac{268101h^2}{5857280} \\ \frac{11097h^2}{80080} \\ \frac{\in 87h^2}{17920} \\ \frac{-1953h^2}{11440} \\ \frac{22599h^2}{82001920} \\ \frac{44613h^3}{20500480} \\ \frac{3483h^3}{40040} \\ \frac{-2187h^3}{40040} \\ \frac{909h^3}{40040} \\ \frac{-4293h^3}{20500480} \end{pmatrix}, \quad \begin{pmatrix} \varphi_{004} \\ \varphi_{014} \\ \varphi_{024} \\ \varphi_{034} \\ \varphi_{044} \\ \varphi_{104} \\ \varphi_{114} \\ \varphi_{124} \\ \varphi_{134} \\ \varphi_{144} \\ \varphi_{204} \\ \varphi_{214} \\ \varphi_{224} \\ \varphi_{234} \\ \varphi_{244} \end{pmatrix} = \begin{pmatrix} \frac{1257482h}{3648645} \\ \frac{622592h}{331695} \\ \frac{-512h}{1155} \\ \frac{622592h}{331695} \\ \frac{1257482h}{3648645} \\ \frac{52552h^2}{1216215} \\ \frac{290816h^2}{1216215} \\ 0 \\ \frac{-290816h^2}{1216215} \\ \frac{-52552h^2}{1216215} \\ \frac{344h^3}{173745} \\ \frac{126976h^3}{1216215} \\ \frac{-6784h^3}{45045} \\ \frac{126976h^3}{1216215} \\ \frac{344h^3}{173745} \end{pmatrix}.$$

Substituting the obtained coefficient values in Equations (11) and (12) gives the four-step block method with the presence of third and fourth fuzzy derivatives in the form of Equations (13) and (14).

$$y_{n+1}=\left(\begin{array}{l} y_n+hy_n'+ \\ \left[\begin{array}{l} h^2(\frac{1278669671}{4598415360}f_n+\frac{20793569}{38918880}f_{n+1}+\frac{442685}{946176}f_{n+2}+ \\ \frac{18801383}{116756640}f_{n+3}-\frac{15685603}{2846638090}f_{n+4})+h^3(\frac{531347}{16220160}g_n \\ +\frac{19491781}{311351040}g_{n+1}+\frac{230389}{3294720}g_{n+2}-\frac{18603301}{311351040}g_{n+3}+ \\ \frac{61072477}{39852933120}g_{n+4})+h^4(\frac{29662123}{19926466560}m_n+\frac{81}{1792}m_{n+1} \\ -\frac{5218561}{92252160}m_{n+2}+\frac{3229249}{311351040}m_{n+3}-\frac{779951}{6642155520}m_{n+4}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+2}=\left(\begin{array}{l} y_n+hy_n'+ \\ \left[\begin{array}{l} h^2(\frac{2305208}{3648645}f_n+\frac{7490816}{3648645}f_{n+1}-\frac{38}{35}f_{n+2}+\frac{1514752}{3648645}f_{n+3} \\ -\frac{10420}{729729}f_{n+4})+h^3(\frac{763657}{9729720}g_n+\frac{24928}{135135}g_{n+1}+ \\ \frac{6784}{45045}g_{n+2}-\frac{992}{6435}g_{n+3}+\frac{5521}{1389960}g_{n+4})+h^4(\frac{17783}{4864860}m_n \\ +\frac{159584}{1216215}m_{n+1}-\frac{1}{7}m_{n+1}+\frac{32608}{1216215}m_{n+3}-\frac{1481}{4864860}m_{n+4}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+3}=\left(\begin{array}{l} y_n+hy_n'+ \\ \left[\begin{array}{l} h^2(\frac{80789193}{82001920}f_n+\frac{16821}{4576}f_{n+1}-\frac{1436859}{1576960}f_{n+2}+\frac{123867}{160160}f_{n+3} \\ -\frac{147393}{6307840}f_{n+4})+h^3(\frac{20358801}{164003840}g_n+\frac{421767}{1281280}g_{n+1}+ \\ \frac{592677}{2562560}g_{n+2}-\frac{347751}{1281280}g_{n+3}+\frac{81891}{12615680}g_{n+4})+ \\ h^4(\frac{477441}{82001920}m_n+\frac{280827}{1281280}m_{n+1}-\frac{2123577}{10250240}m_{n+2}+ \\ \frac{81}{1792}m_{n+3}-\frac{40743}{82001920}m_{n+4}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+4}=\left(\begin{array}{l} y_n+hy_n'+ \\ \left[\begin{array}{l} h^2(\frac{4872272}{3648645}f_n+\frac{6557696}{1216215}f_{n+1}-\frac{1024}{1155}f_{n+2}+\frac{118784}{56133}f_{n+3} \\ +\frac{52552}{1216215}f_{n+4})+h^3(\frac{6224}{36855}g_n+\frac{618496}{1216215}g_{n+1}+\frac{13568}{45045}g_{n+2} \\ -\frac{77824}{173745}g_{n+3}-\frac{688}{173745}g_{n+4})+h^4(\frac{1376}{173745}m_n+\frac{126976}{405405}m_{n+1} \\ -\frac{-3568}{45045}m_{n+2}+\frac{126976}{1216215}m_{n+3}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}} \tag{13}$$

$$y_{n+1}'=\left(\begin{array}{l} y_n'+ \\ \left[\begin{array}{l} h(\frac{3028574741}{8539914240}f_n+\frac{6272003}{5307120}f_{n+1}-\frac{14753}{18480}f_{n+2}+\frac{1438733}{5307120}f_{n+3} \\ -\frac{551113363}{59779399680}f_{n+4})+h^2(\frac{457963769}{9963233280}g_n+\frac{1331003}{19459440}g_{n+1}+ \\ \frac{2187}{17920}g_{n+2}-\frac{391297}{3891888}g_{n+3}+\frac{25533817}{9963233280}g_{n+4})+ \\ h^3(\frac{10906367}{4981616640}m_n+\frac{794921}{9729720}m_{n+1}-\frac{34589}{360360}m_{n+2}+ \\ \frac{169439}{9729720}m_{n+3}-\frac{977791}{4981616640}m_{n+4}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+2}'=\left(\begin{array}{l} y_n'+ \\ \left[\begin{array}{l} h(\frac{82429429}{233513280}f_n+\frac{543736}{331695}f_{n+1}-\frac{256}{1155}f_{n+2}+\frac{78856}{331695}f_{n+3}- \\ \frac{1950581}{233513280}f_{n+4})+h^2(\frac{1772191}{38918880}g_n+\frac{36632}{243243}g_{n+1}+\frac{1}{70}g_{n+2} \\ -\frac{107656}{1216215}g_{n+3}+\frac{90527}{38918880}g_{n+4})+h^3(\frac{42001}{19459440}m_n+ \\ \frac{108104}{1216215}m_{n+1}-\frac{3392}{45045}m_{n+2}+\frac{2696}{173745}m_{n+3}-\frac{3473}{19459440}m_{n+4}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+3}'=\left(\begin{array}{l} y_n'+ \\ \left[\begin{array}{l} h(\frac{29017419}{82001920}f_n+\frac{11691}{7280}f_{n+1}+\frac{2187}{6160}f_{n+2}+\frac{723}{1040}f_{n+3}- \\ \frac{819531}{82001920}f_{n+4})+h^2(\frac{268101}{5857280}g_n+\frac{11097}{80080}g_{n+1}+ \\ \frac{\in 87}{17920}g_{n+2}-\frac{1953}{11440}g_{n+3}+\frac{22599}{82001920}g_{n+4})+h^3(\frac{44613}{20500480}m_n \\ +\frac{3483}{40040}m_{n+1}-\frac{2187}{40040}m_{n+2}+\frac{909}{40040}m_{n+3}-\frac{-4293}{20500480}m_{n+4}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}}$$

$$y_{n+4}'=\left(\begin{array}{l} y_n'+ \\ \left[\begin{array}{l} h(\frac{1257482}{3648645}f_n+\frac{622592}{331695}f_{n+1}-\frac{512}{1155}f_{n+2}+\frac{622592}{331695}f_{n+3} \\ +\frac{1257482}{3648645}f_{n+4})+h^2(\frac{52552}{1216215}g_n+\frac{290816}{1216215}g_{n+1}- \\ \frac{290816}{1216215}g_{n+3}-\frac{-52552}{1216215}g_{n+4})+h^3(\frac{344}{173745}m_n+ \\ \frac{126976}{1216215}m_{n+1}-\frac{-6784}{45045}m_{n+2}+\frac{126976}{1216215}m_{n+3}+\frac{344}{173745}m_{n+4}) \end{array}\right] \end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}} \tag{14}$$

IV. CONVERGENCE AND STABILITY PROPERTIES

This section will detail the convergence and stability properties of the developed four-step third-fourth derivative scheme. The following definitions are used: consistency, zero-stability, and region of absolute stability from [20]-[21]. These definitions for block methods in the crisp form are adopted to the proposed method for fuzzy initial and boundary value problems to prove the convergence properties of the proposed method.

Order and Error Constant

The linear operator which is associated with Equation (6) for the four-step block method is defined as:

$$L(y(x),h)=y_{n+\eta}-\left(\sum_{\zeta=0}^{1}\frac{(\eta h)^{\zeta}}{\zeta!}y_n^{\zeta}+\sum_{d-0}^{2}\left[\sum_{\zeta=0}^{4}\psi_{d\zeta\eta}f_{n+\zeta}^{d}\right]\right)_{\underline{\alpha}}^{\overline{\alpha}} \tag{15}$$

$$L(y(x),h)=\left(\begin{array}{l}\delta_0\,y(x_n)+\delta_1 hy'(x_n)+\delta_2 h^2\,y''(x_n)+,\ldots,+ \\ +\delta_{z+1}h^{z+1}y^{z+1}(x_n)+\delta_{z+2}h^{z+2}y^{z+2}(x_n)\end{array}\right)_{\underline{\alpha}}^{\overline{\alpha}}.$$

The order of this method is z if $\delta_0=\delta_1=\delta_1=,\ldots,=\delta_{z+1}=0$ and $\delta_{z+2}\neq 0$ is the error constant.

Using the definition of order and error constant, the developed block method has order fifteen with error constant

$$\left[\frac{48248741}{5340646732861440000},\ \frac{3824}{162983603908125},\ \frac{92529}{2441996677120000},\ \frac{7648}{162983603908125}\right]$$

So, the developed block method is consistent.

Zero-Stability

Applying the definition of zero stability in fuzzy form for the proposed method gives

$$P(\phi)=\left(\left|\phi A^0+A^1\right|\right)_{\underline{\alpha}}^{\overline{\alpha}} \tag{16}$$

$$P(\phi)=\left(\left|\begin{pmatrix}\phi&0&0&0\\0&\phi&0&0\\0&0&\phi&0\\0&0&0&\phi\end{pmatrix}+\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right|\right)_{\underline{\alpha}}^{\overline{\alpha}}=\phi^{2}(\phi-1)=0\,.$$

Hence the roots $P(\phi)=0$ satisfy the condition $\left|\phi_\eta\right|\leq 1,\eta=1,2,3,4$.

Since the obtained roots satisfy the above condition, the proposed four-step block method is zero-stable. The developed method is thus convergent having satisfied the properties of consistency and zero-stability.

Region of Absolute Stability

Determining the absolute stability region uses the characteristic polynomial of the developed block method obtained as

$$\left(\det\begin{bmatrix} -(w)^k + A^1 + q\left[\sum_{j=0}^{k} B^j w^{k-j}\right] + q^2\left[\sum_{j=0}^{k} C^j w^{k-j}\right] + \\ q^3\left[\sum_{j=0}^{k} D^j w^{k-j}\right] + q^4\left[\sum_{j=0}^{k} E^j w^{k-j}\right] \end{bmatrix}\right)_{\underline{\alpha}}^{\bar{\alpha}},$$

$$q = \lambda h.$$

The region of absolute stability is determined by plotting the roots of the polynomial using the boundary locus approach, as shown in Figure 1.

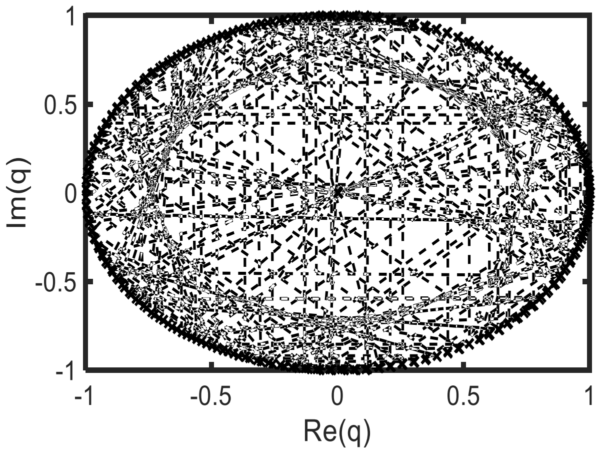

Fig. 1. Region of absolute stability of the proposed method

V. Results and Discussion

This section details the application of the developed four-step block method for the numerical solution of second-order FODEs. The results are compared with the exact solution and existing methods. The comparisons between exact and approximate solutions are shown in tables and graphs where the x-axis shows the value of the approximate solution and the y-axis shows α-level values.

Applications of FIVPS in Mechanical and Electrical Engineering

Here we consider the application of fuzzy differential equations in mechanical and electrical engineering concepts. The first one is the vibrating mass system without a damping effect, the second one is the displacement of the pendulum due to damping, and the third one is the electric circuit of charging.

Example 1 [22]

Consider the vibrating mass system. The mass $m = 1$, the spring constant $k = 4\ lbs/ft$ and there is no, or negligible, damping. The forcing function is $2\cos(x)$ and the differential equation of motion is

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2\cos(x) = 0$$

with fuzzy initial conditions

$y(0,\alpha) = (2r, 4-2\alpha), y'(0,\alpha) = (2\alpha-2, 2-2\alpha)$.

The lower and upper solution of the vibrating mass system is given as

$$\underline{Y}(x,\alpha) = 2\alpha\cos(2x) + \alpha\sin(2x) - \sin(2x) + \tfrac{2}{3}\cos(x) - \tfrac{2}{3}\cos(2x),$$

$$\bar{Y}(x,\alpha) = (4-2\alpha)\cos(2x) + \sin(2x) - \alpha\sin(2x) + \tfrac{2}{3}\cos(x) - \tfrac{2}{3}\cos(2x).$$

The solution of this vibrating mass system as an FIVP is compared with [22], where Laplace Transformation Method (LTM) was presented with h=0.1. The solution's accuracy in terms of absolute error with lower and upper bounds in presented in Table 1. The drawback of the study by [22] is that the second-order FIVP was reduced to first-order FIVPs and then LTM was applied for the approximate solution. Whereas, the developed method in this article directly solves the second-order FIVP with improved accuracy in terms of absolute error when comparing the approximate solution with the exact solution as seen in Table 1. Figures 2-6 with approximate solutions show the uncertain behaviour of the vibrating mass system with different values of α=0, 0,25, 0.5, 0.75, 1 for Example 1. It is observed that the approximate solution completely overlaps the exact solution.

TABLE 1
APPROXIMATE SOLUTION (LOWER/UPPER SOLUTION) OF EXAMPLE 1

α	Lower Solution	Error
0	-0.271664665215493650	0.000000e+00
0.25	-0.252413726782644400	0.000000e+00
0.5	-0.233162788349795170	0.000000e+00
0.75	-0.213911849916945920	1.387779e-18
1	-0.194660911484096670	0.000000e+00
α	**Upper Solution**	**Error**
0	-0.117657157752699650	0.000000e+00
0.25	-0.136908096185548920	0.000000e+00
0.5	-0.156159034618398170	0.000000e+00
0.75	-0.175409973051247420	0.000000e+00
1	-0.194660911484096670	0.000000e+00

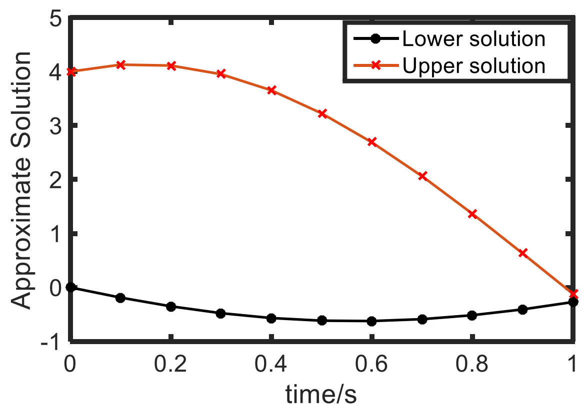

Fig 2. $\underline{y}(1,0)$ and $\overrightarrow{y}(1,0)$

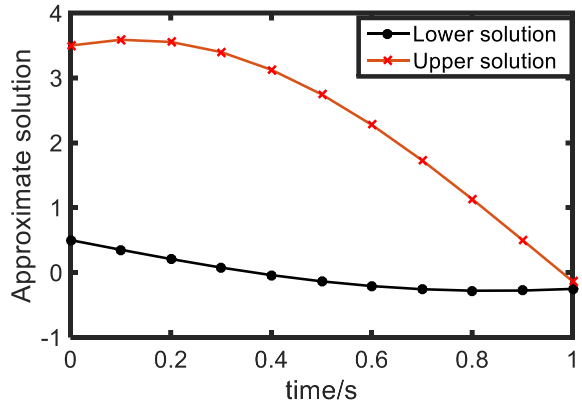

Fig 3. $\underline{y}(1,0.25)$ and $\vec{y}(1,0.25)$

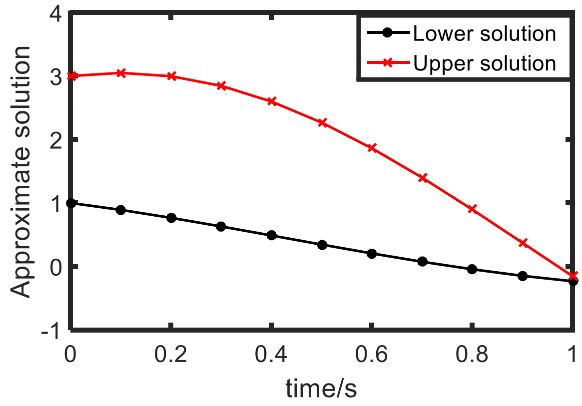

Fig 4. $\underline{y}(1,0.5)$ and $\vec{y}(1,0.5)$

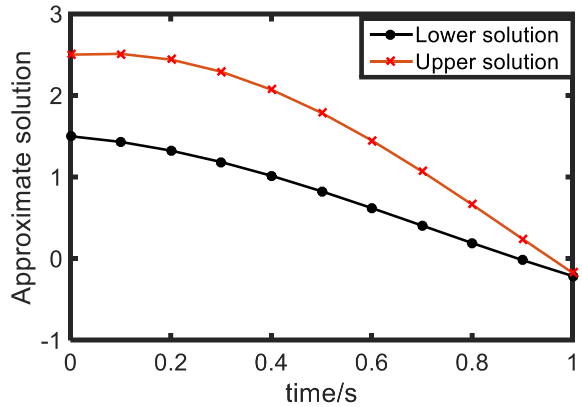

Fig 5. $\underline{y}(1,0.75)$ and $\vec{y}(1,0.75)$

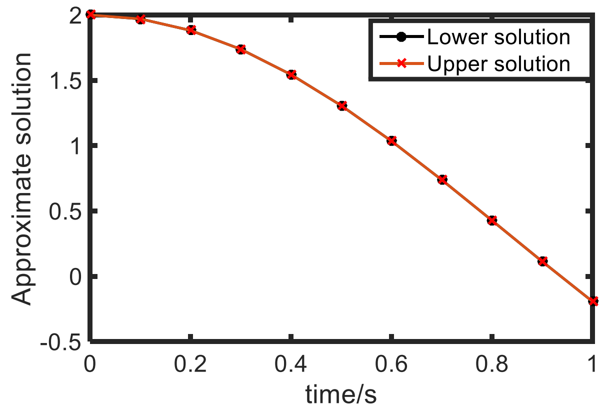

Fig 6. $\underline{y}(1,1)$ and $\vec{y}(1,1)$

Example 2 [22]

A pendulum of length L=8/5 ft is subject to resistive force $FR = 32/5\, d\theta/dt$ due to damping. To determine the displacement function if θ (0) = 1, and θ' (0) = 2, the resulting differential equation is

$$\frac{8}{5}.\frac{d^2\theta}{dx^2}+\frac{32}{5}.\frac{d\theta}{dx}+32\theta=0\,.$$

This crisp differential equation is converted to a fuzzy differential equation as FIVPs using definitions in Section 2 such that

$$\frac{d^2\theta}{dx^2}+4\frac{d\theta}{dx}+20\theta=0$$

with given fuzzy initial conditions

$$\theta(0,\alpha)=(\alpha,2-\alpha),\theta'(0,\alpha)=(1+\alpha,3-\alpha)$$

The lower and upper solutions of the damping pendulum problem is given as

$$\underline{\theta}(x,\alpha)=\alpha e^{-2x}\cos(4x)+(\tfrac{3\alpha+1}{4})e^{-2x}\sin(4x)$$

$$\overline{\theta}(x,\alpha)=(2-\alpha)e^{-2x}\cos(4x)+(\tfrac{7-3\alpha}{4})e^{-2x}\sin(4x)$$

The solution of this FIVP damping pendulum problem is also compared with [22] with h=0.1. The accuracy of the solution in terms of absolute error with lower and upper bounds is presented in Table 2. The drawback of the approach by [22] is still the reduction process as mentioned in Example 1. Comparison of the approximate solution with the exact solution is seen in Table 2 and Figures 7-11 with approximate solution show the uncertain behaviour of the damping pendulum problem with different values of α=0, 0,25, 0.5, 0.75, 1 for Example 2. Likewise, the approximate solution overlaps the exact solution.

TABLE 2
APPROXIMATE SOLUTION (LOWER/UPPER SOLUTION) OF EXAMPLE 2

α	Lower Solution	Error
0	-0.025605520014168430	0.000000e+00
0.25	-0.066924921166140253	0.000000e+00
0.5	-0.108244322318112080	0.000000e+00
0.75	-0.149563723470083930	2.775558e-18
1	-0.190883124622055760	2.775558e-17
α	**Upper Solution**	**Error**
0	-0.356160729229943020	0.000000e+00
0.25	-0.314841328077971220	0.000000e+00
0.5	-0.273521926925999410	0.000000e+00
0.75	-0.232202525774027560	0.000000e+00
1	-0.190883124622055760	2.775558e-17

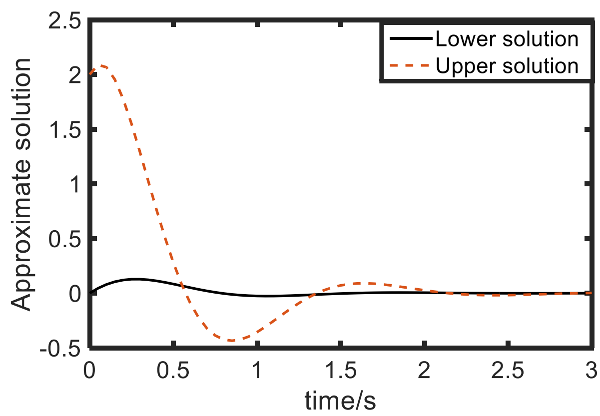

Fig 7. $\underline{\theta}(3,0)$ and $\vec{\theta}(3,0)$

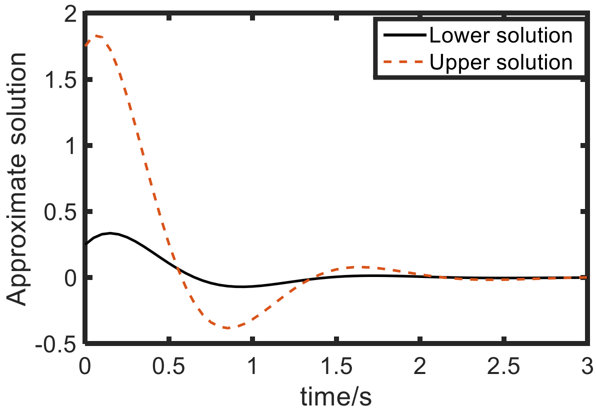

Fig 8. $\underline{\theta}(3,0.25)$ and $\overline{\theta}(3,0.25)$

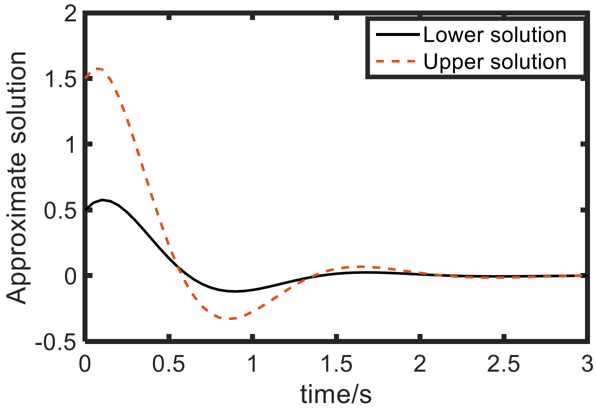

Fig 9. $\underline{\theta}(3,0.5)$ and $\overline{\theta}(3,0.5)$

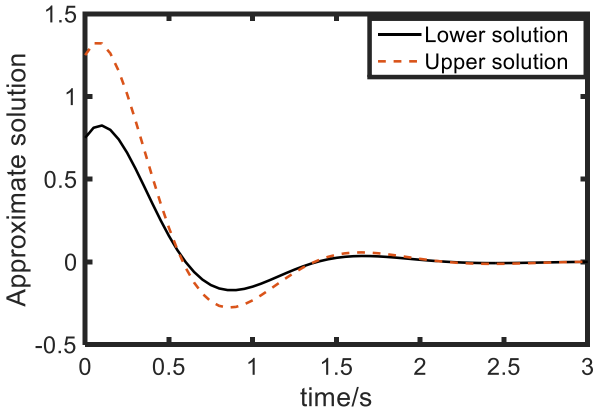

Fig 10. $\underline{\theta}(3,0.75)$ and $\overline{\theta}(3,0.75)$

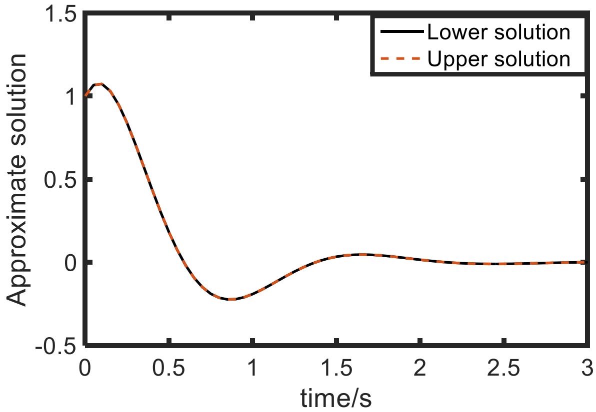

Fig 11. $\underline{\theta}(3,1)$ and $\overline{\theta}(3,1)$

Example 3 [23]

The differential equation describing an electrical circuit if Q is the charge of the capacitor of time x>0 is given as

$$\frac{d^2Q}{dx^2}+2\frac{dQ}{dx}+4Q-50\cos(x)=0$$

With fuzzy initial conditions

$Q(0,r)=(4+r,6-r),Q(0,r)=(r,2-r)$.

The lower and upper solution of the capacitor charge problem is given as

$$\underline{Q}(x,r)=\frac{50}{3}\cos(x)+(\frac{-38}{3}+r-\frac{3}{2}tr+\frac{38}{3}t)\cos(2x)$$
$$+\frac{100}{9}\sin(x)+(\frac{-107}{9}+\frac{5}{4}r-\frac{1}{2}tr)\sin(rx)$$
$$\overline{Q}(x,r)=\frac{50}{3}\cos(x)+(\frac{-32}{3}-r+\frac{3}{2}xr+\frac{29}{3}t)\cos(2x)$$
$$+\frac{100}{9}\sin(x)+(\frac{-169}{9}-\frac{5}{4}r+\frac{1}{2}xr-x)\sin(rx)$$

This solution of this problem is compared with [23], where the variational iteration method (VIM) was presented with h=0.1. The drawback of the study by [23] is also reduction of the second-order FIVP to first-order FIVPs. The direct solution using the developed method in this article is compared with [23] as shown in Table 3. Figures 12-16 also display the approximate solution showing the electrical circuit's uncertain behaviour with different values of α=0, 0,25, 0.5, 0.75, 1.

TABLE 3
APPROXIMATE SOLUTION (LOWER/UPPER SOLUTION) OF EXAMPLE 3

α	Lower Solution	Error
0	7.869505223941254	0.000000e+00
0.25	7.858577006672877	0.000000e+00
0.5	7.847648789404501	0.000000e+00
0.75	7.836720572136122	0.000000e+00
1	7.825792354867745	0.000000e+00
α	**Upper Solution**	**Error**
0	7.782079485794236	0.000000e+00
0.25	7.793007703062613	0.000000e+00
0.5	7.803935920330990	0.000000e+00
0.75	7.814864137599368	0.000000e+00
1	7.825792354867745	0.000000e+00

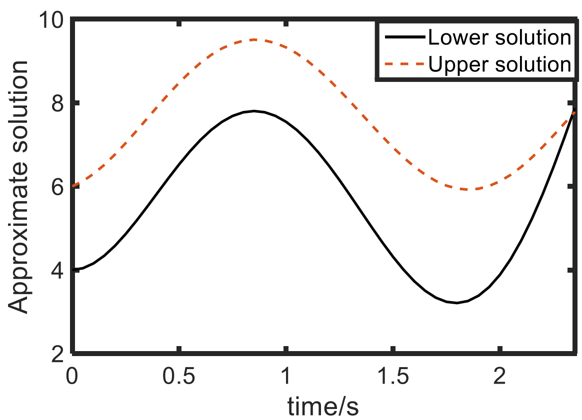

Fig 12. $\underline{Q}(2,0)$ and $\overline{Q}(2,0)$

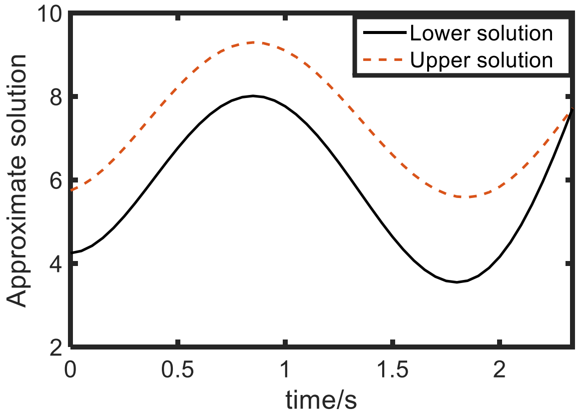

Fig 13. $Q(2,0.25)$ and $Q(2,0.25)$

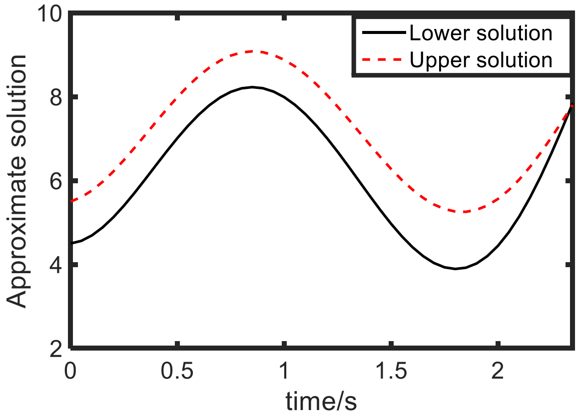

Fig 14. $Q(2,0.5)$ and $Q(2,0.5)$

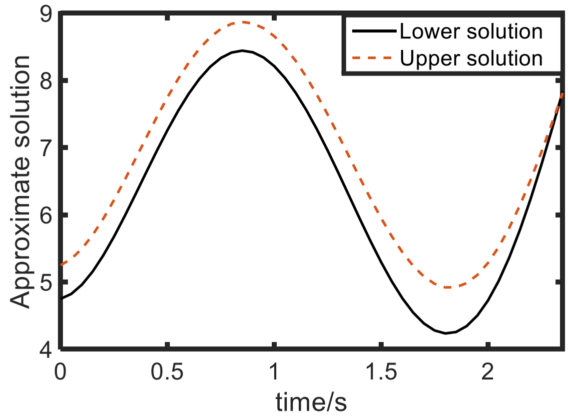

Fig 15. $Q(2,0.75)$ and $Q(2,0.75)$

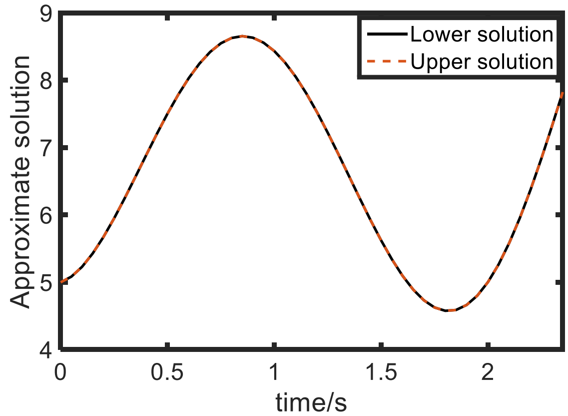

Fig 16. $Q(2,1)$ and $Q(2,1)$

Example 4. [24]

Consider the second-order FIVP

$$\begin{cases} y''(x) = \frac{1}{2}\left(y(x)+y(x)\right) \\ y(0) = (2+\alpha, 4-\alpha) \\ y'(0) = (2+\alpha, 4-\alpha) \end{cases}$$

with exact solution

$$\begin{cases} \underline{Y}(x,r) = (2+\alpha)e^{x} \\ \overline{Y}(x,r) = (4-\alpha)e^{x} \end{cases}.$$

This solution of Example 4 is compared with [24] where VIM was presented with h=0.1. VIM was adopted after reducing the second-order FIVP to first-order FIVPs, whereas the block method developed in this article solved the problem directly. Table 4 shows the comparison of results and Figure 17 shows the comparison between the exact and approximate solution for Example 4 with the approximate solution overlapping the exact solution.

TABLE 4
APPROXIMATE SOLUTION (LOWER/UPPER SOLUTION) OF EXAMPLE 4

α	Lower solution	FSBM Error	VIM Error [24]
0	5.436563656918091100	8.881784e-18	2.6645e-15
0.2	5.980220022609900700	8.881784e-18	2.6645e-15
0.4	6.523876388301709500	8.881784e-18	4.3476e-12
0.6	7.067532753993518200	8.881784e-18	3.8261e-10
0.8	7.611189119685327000	0.000000e+00	9.2191e-09
1	8.154845485377137500	1.776357e-18	1.0925e-07
α	Upper solution	FSBM Error	VIM Error [24]
0	10.873127313836182000	1.776357e-18	2.6645e-15
0.2	10.057642765298469000	1.776357e-18	2.6645e-15
0.4	9.513986399606659800	1.776357e-18	4.3476e-12
0.6	8.970330033914850200	1.776357e-17	3.8261e-10
0.8	8.426673668223040500	0.000000e+00	9.2191e-09
1	8.154845485377137500	1.776357e-17	1.0925e-07

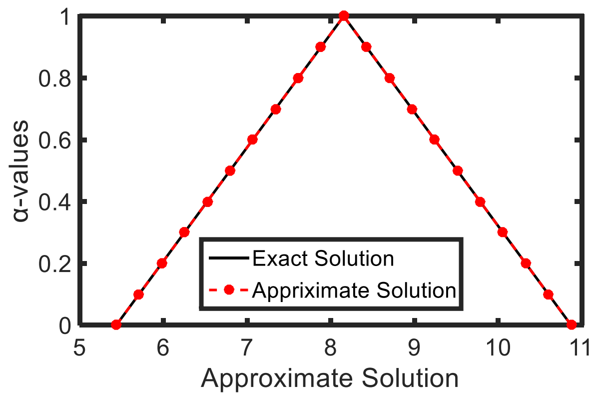

Fig 17. $y(1,1)$ and $y(1,1)$

Example 5. [25]

Consider the second-order FBVP

$$\begin{cases} y''(x) + y(x) = -x, x \in [0, \frac{\pi}{2}] \\ y(0,\alpha) = (0.1\alpha - 0.1, 0.1 - 0,1\alpha) \\ y'(\frac{\pi}{2},\alpha) = (-\frac{\pi}{2} + 0.1\alpha, 1 + \frac{\pi}{2} - 0.1\alpha) \end{cases}$$

with the exact solution as follows

$$\begin{cases}\underline{Y}(x,\alpha)=(0.1\alpha-0.1)\cos(x)+(0.1\alpha)\sin(x)-x\\ \overline{Y}(x,\alpha)=(0.1-0.1\alpha)\cos(x)+(1+\pi-0.1\alpha)\sin(x)-x\end{cases}$$

The solution of this FBVP is compared with [25], where the undetermined fuzzy coefficients method (UFCM) was presented with h=0.1. The accuracy of the solution in terms of absolute error with lower and upper bounds in given in Table 5. The drawback of the study by [25] is also the reduction approach and the improved accuracy of the developed block method as a direct approach is shown in Table 5. Figure 18 also displays the exact and approximate solution of Example 5 where the approximate solution completely overlaps the exact solution.

TABLE 5
APPROXIMATE SOLUTION (LOWER/UPPER SOLUTION) OF EXAMPLE 5

α	Lower solution	FSBM Error	VIM Error [25]
0	-1.054030230586813900	0.000000e+00	0.116492e-4
0.2	-1.026394764773293200	0.000000e+00	0.111074e-4
0.4	-0.998759298959772560	0.000000e+00	0.105656e-4
0.6	-0.971123833146251840	0.000000e+00	0.100238e-4
0.8	-0.943488367332731000	0.000000e+00	0.094820e-4
1	-0.915852901519210280	1.1102237e-16	0.089402e-4
α	**Upper solution**	**FSBM Error**	**VIM Error [25]**
0	2.539060279476166700	0.000000e+00	0.6231166e-5
0.2	2.511424813662646000	0.000000e+00	0.6772968e-5
0.4	2.483789347849124900	4.440892e-16	0.7314771e-5
0.6	2.456153882035604600	4.440892e-16	0.7856573e-5
0.8	2.428518416222083900	4.440892e-16	0.8398376e-5
1	2.400882950408563100	4.440892e-16	0.8940178e-5

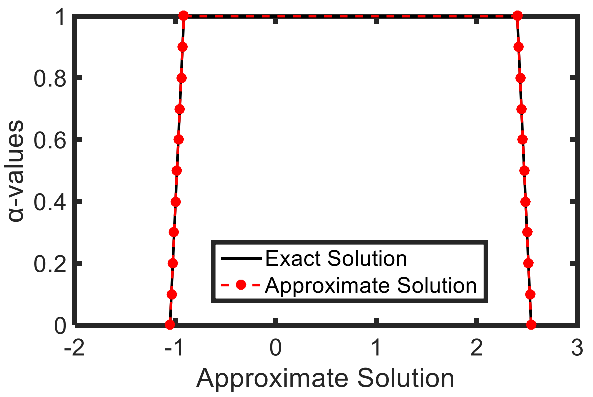

Fig 18. $\underline{y}(1,1)$ and $\overline{y}(1,1)$

VI. CONCLUSION

The main goal of this study is to develop a numerical approach for solving second-order FODEs (FIVPs and FBVPs) that will improve the accuracy of the solution in terms of absolute error. As a result, this paper developed a block method with generalised steplength with third and fourth derivatives for solving second-order FODEs. As indicated in the tables and graphs of the numerical results obtained, the developed four-step block method surpasses previous methods identified in the literature. Furthermore, the traditional approach of reduction to a system of first-order differential equations was bypassed, implying that the method does not require complicated subroutines. The developed block scheme is a feasible strategy for solving linear and nonlinear FIVPs and FBVPs with higher accuracy. The scheme was created employing a linear block approach with minimal computing complexity and fulfilled all convergence conditions. As a result, the approach proposed in this article is more suitable for solving second-order FIVPs and FBVPs directly.

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