Time-Varying Model Predictive Control for Train Regulation and Passenger Flow in Metro Lines with Sinusoidal Disturbance

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Abstract—This research intended to create a control strategy at each time step to optimize train regulation, and passenger flows with existing constraints to improve the regularity of headway and commercial speed on metro lines. Additionally, inevitable disturbances on the metro lines are considered as a periodic sine function, and the uncertainty of fluctuating passenger arrival flow was handled using time-varying MPC. The best solution was sought as a quadratic programming problem by using time-varying MPC to issues of train regulation and passenger flow control in which the systems were time-dependent. Moreover, time-varying MPC was utilized to predict future outputs and calculate optimal inputs for the objective function. Numerical examples were provided to illustrate the effectiveness of the proposed method.

Index Terms—Train regulation; Passenger Flow Control; Quadratic Programming; Time Varying MPC.

I. INTRODUCTION

Due to their inherent characteristics of speed, efficiency, and safety, metro systems have become an essential source of public transportation for passengers in urban centers ([1],[2]). However, metro systems frequently experience minor disruptions due to irregular occurrences of passenger demand fluctuations, equipment failure, and crises. They can significantly impact the service quality of service with a short headway. Variations in passenger demand result in an unanticipated crowded passenger arrival flow, which impacts dwell time ([3],[4]). Specifically, as the number of arriving passengers during peak hours increases, train delays caused by random disruptions will spread from one station to the next, making the system unstable. Periodic passenger arrival can be assumed to be a disruption and is considered a sine function [5]. Train regulation, which involves changing the operating duration and dwell time of each train, is necessary to recover from delays and prevent unstable operations ([6],[7]).

Various train regulation mechanisms have been proposed for metro lines. In order to guarantee system stability and the reduction of a particular performance index, a state feedback control strategy built on the linear quadratic controller was used in [6]. In [8], a genetic algorithm was successfully applied to the problem of optimal train regulation. However, train regulation, which controls each train’s operating duration and dwells time, can not handle the overloaded passenger flow during peak hours. The joint dynamic train regulation and passenger flow control design problem for metro lines was established in [9] to enhance commercial speed and headway regularity.

The amount of passengers boarding and departing each train has an impact on dwell time [9]. It is assumed that the number of passengers boarding the train is proportional to the time between trains ([3], [10]) and that the passenger arrival rate is uncertain ([11]), both of which are time-dependent. Therefore, there should be a proportionate relationship between the number of passengers entering and exiting trains [12], which depends on the time change. This study assumes that the uncertain passenger arrival rate is different at each station and train and that the sine-shaped passenger arrival rate causes disturbances. This is more relevant to factual problems. However, as the number of variables and constraints increases, the calculation time of conventional linear and nonlinear programming methods increases, making them unsuitable for calculating daily activity schedules.

One of the most promising subfields of modern control, model predictive control (MPC), is capable of effectively handling large-scale optimization problems with complex constraints. [13]. There are numerous MPC types, such as distributed MPC [18], time-varying MPC [15], SPF-MPC [19], and Nonlinear MPC [20]. MPC is an effective solver for real-time metro traffic regulation and passenger load due to its high predictive ability. A linear programming-based MPC approach was published in [14] to compute optimal train schedules on metro lines, which can successfully generate a daily timetable. In addition, [9] addressed a challenge in predictive design for metro lines, including dynamic train regulation and passenger flow control.

In addition, the unpredictability of passenger flow and periodic sinusoidal disturbances that cause train delays must be considered. Time-varying MPC can solve problems involving time-dependent parameters, such as uncertain passenger arrival rates, sinusoidal disturbances, and proportional factors, among others. Using time-varying algorithms, autonomous cars have been developed [15]. In order to deal with unpredictable changes in passenger flow, various proportional factors, and the sine disturbance, we examine the optimal train regulation and passenger load control within the framework of time-varying MPC.

Based on the time-varying MPC scheme, this research suggests a novel approach to train regulation and passenger flow management approaches in metro lines. A constrained state space model was utilized to take safety, passenger, and control constraints into account for the joint dynamic model of train regulation and passenger flow on the metro lines.
Furthermore, time-varying MPC regulates train operation and passenger flow. Using dynamic models of train traffic and passenger load on metro lines, time-varying MPC predicts future outcomes. The train departure time and the passenger load error are the system outputs. While passenger load error refers to the difference between the factual and nominal passenger load, train departure time error relates to the factual and nominal departure time.

The suggested approaches convert the optimization problem into an easily-solvable convex quadratic programming problem at each time step. In addition, the quadratic programming issue might have constraints for control, load, and safety headway. Therefore, the suggested technique offers a less computationally expensive solution and is easier to implement. This paper is organized as follows: the next section discusses a coupled relationship between the dynamics of train traffic and passenger load. The following sections describe the time-varying model predictive control method for train regulation and passenger flow. Numerical examples to illustrate the effectiveness of the proposed strategies are presented in the next section. Last section states the conclusion.

II. Problem Formulation

We considered a metro-type train line with one terminal station, N stations, and an ordered train set that stops between stations to allow passengers to enter and leave. Stations, trains, and passengers are components of the metro-type rail system. The metro line’s mission is to transport every passenger from their starting point to their final destination safely and efficiently.

Disruptions, such as equipment failure or non-compliant driver/passenger activity, are unavoidable in the real-time operation of metro lines. The optimal train schedule was no longer required when a disruption occurred, thus a train control plan was necessary to implement to decrease on delays. Some stations typically have a high number of passengers. Passenger demand is relatively high at several stations, especially during peak hours. At such a station, the number of passengers will exceed the train’s nominal passenger load. If passenger flow is not controlled, train delays will increase significantly. When a train deviates from its usual schedule due to a disruption, a train regulation and passenger flow control strategy must be implemented to improve the safety and efficiency of the metro line systems.

A train traffic dynamics model and a train passenger load dynamics model were created to solve this issue. This model integrated the two relationships between train traffic and passenger load dynamics to generate a train traffic and passenger flow dynamics model. This study applied a dynamic model of changes in passenger load between stations, as characterized by the number of people entering and exiting the train at each station. Previous research described passenger demand using a time-dependent origin-destination (OD) matrix [(16), (17)]. The number of passengers boarding the train was thought to be proportionate to the duration between trains [(3), (10)], and the passenger arrival rate was unpredictable [1], both of which were time-dependent. It was anticipated that the number of people boarding and departing the train would equal the number of passengers on board [12], which depends on the time change. Few trains were affected by the system’s disruption [9]. In this study, the disturbance was modeled as a periodic sine function occurring on every train. It was considered that the passenger arrival rate varied by train and station and was uncertain.

A. Train traffic dynamic model

Based on [6], the dynamics of train traffic for high-frequency metro lines were presented. Let \( t_{j+1}^i \) be the departure time of train \( i \) from station \( j \). The departure time of train \( i \) from station \( j + 1 \) was stated as

\[
 t_{j+1}^i = t_j^i + r_j^i + s_{j+1}^i \tag{1}
\]

where \( s_{j+1}^i \) was the dwell time for the train \( i \) at station \( j + 1 \). The running time for the train \( i \) from station \( j \) to station \( j + 1 \), \( r_j^i \), was

\[
 r_j^i = R_j^i + u_{1j}^i + w_{1j}^i \tag{2}
\]

where \( R_j^i \) was the nominal running time of train \( i \) from station \( j \) to station \( j + 1 \), \( u_{1j}^i \) was the control to adjust the running time of train \( i \) between station \( j \) to station \( j + 1 \), and \( w_{1j}^i \) were uncertain disturbances occurred when the \( i \) train ran from \( j \) station to \( j + 1 \) station. If \( u_{1j}^i > 0 \) it means that the running time was increased, while if \( u_{1j}^i < 0 \) it means that the running time was decreased.

Suppose that the dwell time of the trains at the station was affected by both the number of entering and leaving passengers [9]. According to this, the dwell time \( s_{j+1}^i \) was modeled as

\[
 s_{j+1}^i = \alpha (m_{j+1}^i + n_{j+1}^i) + D_{j+1} + u_{2j}^i + w_{2j}^i. \tag{3}
\]

where \( D_{j+1} \) was the minimum dwell time at the station \( j + 1 \) when there were no passengers, \( \alpha \) is the delay rate which represents the time it takes to get on or off the train when the train stops, \( \alpha \in [0.01, 0.06] \). The dwell time adjustment of train \( i \) at station \( j + 1 \) denoted as \( u_{2j}^i \). If \( u_{2j}^i > 0 \) it means that the dwell time was increased, while if \( u_{2j}^i < 0 \) it means that the dwell time was decreased. Furthermore, \( w_{2j}^i \) was a disturbance occurred when the \( i \) train stopped at station \( j + 1 \). From the Equation (1)-(3), the train traffic dynamic model is

\[
 t_{j+1}^i = t_j^i + R_j^i + \alpha (m_{j+1}^i + n_{j+1}^i) + D_{j+1} + u_j^i + w_j^i \tag{4}
\]

with \( u_j^i = u_{1j}^i + u_{2j}^i \) and \( w_j^i = w_{1j}^i + w_{2j}^i \).

B. The passenger load dynamic model

When a train arrives at a station, there are passengers enter the train, and there are others leaving it. According to [9], the dynamic change of the passenger load on the train at the station was

\[
 t_{j+1}^i = t_j^i + m_{j+1}^i - n_{j+1}^i + p_{j+1}^i \tag{5}
\]

where \( m_{j+1}^i \) and \( n_{j+1}^i \) were respectively the numbers of passengers entering and leaving the train \( i \) at station \( j + 1 \), and \( p_{j+1}^i \) was a control to increase the number of passengers entering the \( i \)th train at the \( j + 1 \) station. This control was implemented during peak hours or on holidays, especially for the sudden arrival of passengers in which the value was non-positive to reduce passenger load.
The number of passengers entering train $i$ at station $j + 1$ or $n^i_{j+1}$ was supposed to be proportional to the waiting time between consecutive trains and it satisfies

$$m^i_{j+1} = \gamma^i_{j+1} (t^i_{j+1} - t^i_{j+1} - t^j_{j+1})$$  \hspace{1cm} (6)

where $\gamma^i_{j+1}$ was the passenger arrival rate at station $j + 1$ for train $i$. According to [10], the passenger arrival rate would change with time and it was assumed that the value of $\gamma^i_{j+1}$ varies in a symmetrical range around $\gamma$ with half the length of $d$ was

$$\gamma^i_{j+1} = \gamma + \lambda^i_{j+1} d, \hspace{1cm} -1 \leq \lambda^i_{j+1} \leq 1$$  \hspace{1cm} (7)

with $\lambda^i_{j+1}$ different in each station. For simplicity, the parameter $\gamma$ and half the length of $d$ was assumed to be similar for each station. In this study, the parameter of $\lambda^i_{j+1}$ varied at each station $j + 1$ it was more general and realistic compared [10] that assumed that the average passenger arrival rate was the same for all stations. We might ensure a maximum allowable passengers arrival rate by modifying the parameter $\gamma$ and the half-length $d$, which would satisfy the trains limited capacity for transporting passengers. The $p^i_{j+1}$ control approach minimized the number of people entering the train in satisfying the train’s limited passenger capacity. The passenger flow control, in particular, induced a change in train dwell time from $s^i_{j+1}$ to $s^i_{j+1}$ was

$$s^i_{j+1} = \alpha (m^i_{j+1} + n^i_{j+1} + p^i_{j+1} + D_{j+1} + u^i_{j} + w^i_{j})$$  \hspace{1cm} (8)

The number of passengers leaving the train $i$ at station $j + 1$ was assumed to be proportional to the number of passengers on the train which satisfy

$$n^i_{j+1} = \beta^i_{j+1} l^i_j$$  \hspace{1cm} (9)

where $l^i_j$ was the passenger load of train $i$ between stations $j$ and $j + 1$, and $\beta^i_{j+1}$ was a proportional factor for passengers leaving the train. From the Equation (5)-(9), the dynamic model of passenger load on the train was

$$l^i_{j+1} = l^i_j + \gamma^i_{j+1} (t^i_{j+1} - t^j_{j+1}) - \beta^i_{j+1} l^i_j + p^i_{j+1}$$  \hspace{1cm} (10)

which indicated that the dynamic model of passenger loads on the train was influenced by the dynamic model of train traffic.

C. The joint dynamic model

From the Equation (4) and (10), we obtained a joint dynamic model of the factual departure time and passenger load on the train as

$$\begin{align*}
t^i_{j+1} &= t^i_j + R^i_j + \alpha (m^i_{j+1} + n^i_{j+1} + p^i_{j+1}) + D_{j+1} + u^i_{j} + w^i_{j} \\
\frac{t^i_{j+1}}{l^i_{j+1}} &= t^i_j + \gamma^i_{j+1} (t^i_{j+1} - t^j_{j+1}) - \beta^i_{j+1} l^i_j + p^i_{j+1}.
\end{align*}$$  \hspace{1cm} (11)

This demonstrated how the train’s factual departure time and passenger load interact. Based on Equation (11), it was able to determine that if one train is delayed, the train delay would increase from one station to the next, as would the aggregation of passengers potentially causing metro line instability.

By substituting Equation (6) and (9) to the first Equation (11), and $x^i_j = [t^i_j, l^i_j]^T$ and $u^i_j = [u^i_j, p^i_{j+1}]^T$ we acquired a joint dynamic model with departure time and load passengers on the train was

$$x^i_{j+1} = A^i_j x^i_j + B^i_j x^i_{j+1} + C^i_j u^i_j + G^i_j (D_{j+1} + R^i_j) + w^i_j \hspace{1cm} (12)$$

with

$$x^0 = [0, 0]^T, A^i_j = \begin{bmatrix} 1 - \frac{\alpha^i_{j+1}}{\lambda^i_{j+1}} & \frac{\alpha^i_{j+1}}{\lambda^i_{j+1}} \gamma^i_{j+1} + 1 \\ \gamma^i_{j+1} & 1 - \frac{\alpha^i_{j+1}}{\lambda^i_{j+1}} \end{bmatrix},$$

$$B^i_j = \begin{bmatrix} -\frac{\alpha^i_{j+1}}{\lambda^i_{j+1}} m^i_{j+1} + 1 \\ 0 \end{bmatrix},$$

$$G^i_j = \begin{bmatrix} \frac{1}{\lambda^i_{j+1}} \\ \frac{1}{\lambda^i_{j+1}} \end{bmatrix}. \hspace{1cm}$$

Equation (12) was a standard model for metro lines system operation management under disturbance and it described the change in train departure time and passenger load. Furthermore, the dynamic model for the nominal departure time and passenger load of the nominal train

$$T^i_{j+1} = T^i_j + R^i_j + \alpha (\gamma^i_{j+1} (T^i_{j+1} - T^j_{j+1}) + \beta^i_{j+1} L^i_j) + D_{j+1},$$

and

$$L^i_{j+1} = L^i_j + \gamma^i_{j+1} (T^i_{j+1} - T^j_{j+1}) - \beta^i_{j+1} L^i_j.$$  \hspace{1cm} (13)

The constant time difference between two successive trains determined the nominal departure time, denoted by $H = T^i_{j+1} - T^j_{j+1}$. In terms of service demands, train capacity, and passenger flow during operating hours, the $H$ headway schedule corresponded to operational hours. In particular, headway scheduling was reduced during peak hours.

The error vector is $e^i_j = [t^i_j - T^i_j, (L^i_j - L^i_j)]^T$, from Equation (11), we found the error dynamics for the joint dynamic model as

$$e^i_{j+1} = A^i_j e^i_j + B^i_j e^i_{j+1} + C^i_j u^i_j + G^i_j w^i_j \hspace{1cm} (15)$$

with $A^i_j, B^i_j, C^i_j$, and $G^i_j$ taken from Equation (12). For the dynamic model error Equation (15), the difference between the factual departure time and the nominal departure time was represented by $e^i_j$, as well as the difference between the factual passenger load on the train and its nominal passenger load. Minimizes $e^i_j$ referred to improving the metro lines operating efficiency in order to recover train delays caused by disturbances. Furthermore, if $e^i_j \rightarrow 0$ then $t^i_j \rightarrow T^i_j$ and $l^i_j \rightarrow L^i_j$ which prevent instability on metro lines.

According to Equation (12), information for $x^i_{j+1}$ was generated from $x^i_j$ and $x^i_{j+1}$ for each train $i$ and $j$ station. Let $X_k$ is the state vector of the joint dynamic model with $X_k = [x^i_{j-1}, x^i_{j-2}, \ldots, x^i_{jN}]^T, k > N$ which displayed the factual departure time of the train and the passenger load on the train at all stations. The dimension of the vector state is $2N$. It was assumed that every elements of the state $X_k$ vector lied in the same time interval. Furthermore, by using Equation (12) we obtained the form state space for the joint dynamic model as

$$X_{k+1} = A_k X_k + B_k U_k + G_k (w_k + R_k + D_k) \hspace{1cm} (16)$$
with $X(k)$ as the state vector, the input vector $U_k = \left[ u_{0k}, u_{1k}, \ldots, u_{N-1k} \right]^T$, the disturbance vector $w_k = \left[ w_{0k}, w_{1k}, \ldots, w_{N-1k} \right]^T$, $R_k = \left[ R_{0k}^1, R_{1k}^1, \ldots, R_{N-1k}^1 \right]^T$, $D = \left[ D_1, D_2, \ldots, D_N \right]^T$, and matrix $A_k, B_k$, and $G_k$ was

$$
\begin{align*}
\bar{A}_k &= \begin{bmatrix}
B_{0k} & 0 & 0 & \cdots & 0 \\
A_{1}^{k-1} & B_{1}^{k-1} & 0 & \cdots & 0 \\
0 & \cdots & A_{N-1}^{k-1} & B_{N-1}^{k-1} & 0
\end{bmatrix}, \\
\bar{B}_k &= \begin{bmatrix}
C_{0k} & 0 & 0 & \cdots & 0 \\
0 & C_{1}^{k-1} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & C_{N-1}^{k-1} & 0
\end{bmatrix}, \\
\bar{G}_k &= \begin{bmatrix}
G_{0k} & 0 & 0 & \cdots & 0 \\
0 & G_{1}^{k-1} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & G_{N-1}^{k-1} & 0
\end{bmatrix},
\end{align*}
$$

with the dimensions of matrix $A_k, B_k$, and $G_k$ were $2N \times 2N, 2N \times 2N$, and $2N \times N$, respectively.

According to Equation (16), $2N$ was the number of stations, not trains. The matrices $\bar{A}_k$ and $\bar{B}_k$ represented the dynamic relationship between train traffic and passenger load, and $\bar{G}_k$ represented the system disturbance parameter. Moreover, we obtained the state space model of the joint error dynamic

$$
E_{k+1} = \bar{A}_k E_k + \bar{B}_k U_k + \bar{G}_k w_k
$$

(17)

with $E_k = \left[ e_1^{k-1}, e_2^{k-2}, \ldots, e_N^{k-N} \right]^T$ which consisted of the departure time error and the passenger load error and with the matrix $\bar{A}_k, \bar{B}_k$ and $\bar{G}_k$ was same as in Equation (16).

**D. Objective Function and System Constraints**

To address this issue, metro lines that integrate train regulation and passenger flow control were designed to improve commercial speed and headway regularity. The cost function of the joint dynamic model of metro lines was defined to solve this problem

$$
J = \sum_{i,j} \left\{ e_i^T P_i e_j + (e_i - e_j) \right\} Q_j^T (e_j - e_i) + \left( \bar{u}_i^T \right) R_j \bar{u}_j
$$

(18)

with positive definite weighted matrix $P_i^j, Q_j^i, R_i^j$. The first term in (18) was used to reduce train delays by comparing factual and nominal timetables and passenger loads. The second term improved headway regularity and reduced average passenger waiting time by summing train headway deviations from the nominal value. The third term saved cost. The weight matrix $P_i^j, Q_j^i$, and $R_i^j$ were related to departure time deviations, headway deviations, and control action amplitude, respectively.

From the state space for the joint dynamic model (17), the objective function matrix (18) was formulated as

$$
J = \sum_{k=k_0}^{k_f} \left\{ E_k^T P E_k + (E_{k+1} - E_k)^T Q (E_{k+1} - E_k) + U_k^T R U_k \right\}
$$

(19)

where $k_0$ and $k_f$ were the initial and final stages, respectively. $P, Q$, and $R$ were positive definite weighted matrix.

In addition, to ensure safety on metro lines, we considered the following constraints.

1) State constraints for the departure time

To ensure that a safe distance exists between two adjacent trains is satisfied $t_j^i - t_{j-1}^i \geq \tau_{min}$, where $\tau_{min}$ represented the minimum allowed headway. Furthermore, the state constraint for each train’s departure time could be changed into an error state constraint for each train’s departure time that satisfy

$$
(t_j^i - T_j^i) - (t_{j-1}^i - T_{j-1}^i) \geq \tau_{min} - H
$$

(20)

with $\tau_{min}$ and $H$ were provided. Furthermore, from Equation (17) the constraint for departure time could be written as

$$
H_1 (E_{k-1} - E_k) \leq (H - \tau_{min}) I_{N \times 1}
$$

(21)

with $H_1$ as a matrix of dimension $N \times 2N$ where

$$
H_1 = \begin{bmatrix} h_{ij} \end{bmatrix}, \quad h_{ij} = \begin{cases} 1, & j = 2i - 1 \\ 0, & \text{otherwise}. \end{cases}
$$

2) State constraints for the passenger load

To meet the train’s capacity, the passenger load constraint was $t_j^l \leq l_{max}$, where $l_{max}$ was the maximum capacity of the train. To meet the capacity of the train, the passenger load constraint was $t_j^l \leq l_{max}$, where $l_{max}$ was the train’s maximum capacity for passengers. In addition, the state constraint for the train’s passenger load could be transformed into the state error constraint for the passenger load on each train that satisfy

$$
l_j^l - L_j^l \leq l_{max}s - L_j^l
$$

(22)

with $l_{max}s$ and $L_j^l$ were provided. Furthermore, from Equation (17) the constraint for passenger loads could be written as

$$
H_2 E_k \leq L_k
$$

(23)

with $H_2$ as a matrix dimension of $N \times 2N$ where

$$
H_2 = \begin{bmatrix} h_{ij} \end{bmatrix}, \quad h_{ij} = \begin{cases} 1, & j = 2i \\ 0, & \text{otherwise}. \end{cases}
$$

and $L_k = \left[ l_{max}s - L_1^{k-1}, l_{max}s - L_2^{k-2}, \ldots, l_{max}s - L_N^{k-N} \right]^T$.

3) The input constraint was

$$
[u_{min}, p_{min}]^T \leq \bar{u}_i^j \leq [u_{max}, p_{max}]^T
$$

(24)

where $[u_{min}, p_{min}]$ was the minimum allowed input vector and $[u_{max}, p_{max}]$ is the maximum allowed input vector. Furthermore, from Equation (17) the input constraint could be written as

$$
U_{min} \leq U_k \leq U_{max}
$$

(25)

where $U_{max}$ was a column vector of dimension $2N$ whose odd row elements were equal to $u_{max}$ and even rows were equal to $p_{max}$. Similarly, for $U_{min}$ was a column vector of dimension $2N$ whose odd row elements were equal to $u_{min}$ and even rows were equal to $p_{min}$.
III. TIME-VARYING MPC DESIGN

In this section, we developed a time-varying MPC algorithm for train regulation based on the model predictive control (MPC) method. In time-varying MPC methods, the optimal control input that minimized the specified cost function over a predetermined prediction horizon was calculated at each k step. In this case, the value of state $E_k$ calculated along the $H_p$ horizon prediction step $(k+1, \ldots, k+H_p)$, and the set of prediction sequences inputs was calculated as $U_k, U_{k+1}, \ldots, U_{k+H_p-1}$. The state $E_{k+j}$ prediction was calculated using the state change of the System (17). Only the first element in control $U_k$ was applied to the system in order to calculate for changes in disturbance and system parameters at each step of $k$. The process will be repeated until the horizon prediction is reached.

The optimization issue across a particular prediction horizon was solved directly at each step $k$ based on the dynamic model in the system by calculating the optimal control sequence. According to the most recent information, optimization was performed on the metro lines systems to determine control over the problem of train regulation and passenger flow on the train in order to increase headway regularity and commercial speed of high-frequency metro lines with constraints.

The objective function for each k step optimization problem to determine the control input was

$$\min_{U_{k+j}} \sum_{j=0}^{H_p-1} \left\{ E_{k+j+1}^T P E_{k+j+1} + (E_{k+j+1} - E_{k+j})^T Q \right\}$$

$$\text{s.t.} \quad e_{k+j+1} = A_{k+j} e_{k+j} + B_{k+j} U_{k+j} + G_{k+j} w_{k+j},$$

$$H_1 (e_{k+j} - e_{k+j+1}) \leq (H_t - t_{\min}) I_{N_{X_1}},$$

$$H_2 e_{k+j+1} \leq L_{k+j+1},$$

$$-U_{k+j} \leq -U_{\min},$$

$$U_{k+j} \leq U_{\max}, \quad j = 0, 1, \ldots, H_p - 1.$$  (26)

For each step of $k$, the optimization problem in Equation (26) could be converted into a quadratic programming problem. Furthermore, let $E = \begin{bmatrix} E_{k+1}^T, E_{k+2}^T, \ldots, E_{k+H_p}^T \end{bmatrix}^T$, $U = \begin{bmatrix} U_k^T, U_k^T, \ldots, U_{k+H_p-1}^T \end{bmatrix}^T$, and $W = \begin{bmatrix} w_k^T, w_k^T, \ldots, w_{k+H_p-1}^T \end{bmatrix}^T$ at each step $k$ for the prediction of state $E_k$ until the prediction horizon $H_p$ was calculated from Equation (26) as

$$E = F E_k + \Phi U + \Gamma W$$  (27)

with

$$F = \begin{bmatrix} A_k & \cdots & A_{k+H_p-2} \cdots & A_k \\ A_{k+1} & \cdots & A_{k+H_p-2} \cdots & A_{k+1} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ A_k & \cdots & A_{k+H_p-2} \cdots & A_k \end{bmatrix},$$

$$G = \begin{bmatrix} G_k & \cdots & G_{k+H_p-2} \cdots & G_k \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ G_k & \cdots & G_{k+H_p-2} \cdots & G_k \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} Z_0 G_k & \cdots & Z_0 G_{k+H_p-1} \cdots & Z_0 G_k \\ Z_1 G_k & \cdots & Z_1 G_{k+H_p-1} \cdots & Z_1 G_k \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ Z_{H_p} G_k & \cdots & Z_{H_p} G_{k+H_p-1} \cdots & Z_{H_p} G_k \end{bmatrix},$$

where $Z_j = \prod_{i=k+j}^{k+H_p-1} A_i$, $j$ was the column number in matrix $\Gamma$.

The following theorem provided the corresponding quadratic programming formulation at step $k$ for the optimization problem (26) associated with state prediction $E$.

**Theorem 3.1:** For $E = F E_k + \Phi U + \Gamma W$, the simplified quadratic programming formulation at step $k$ for the optimization problem (26) was provided as

$$\min_U \quad J = \frac{1}{2} U^T H U + U^T f + \Psi.$$  (28)

$$\text{s.t.} \quad \begin{bmatrix} H_3 H_4 \Phi \\ H_0 \Phi \\ I_{2H_{n+1}} \\ -I_{2H_{n+1}} \end{bmatrix} U \leq \begin{bmatrix} \tilde{Z} \\ \tilde{O} \\ -U_{\max} \end{bmatrix},$$

where the weight matrix $P, Q$, and $R$ could be directly found from the objective function (26),

$$\Psi = E_k^T F^T [P + \tilde{Q}] F E_k + W^T \Gamma^T [P + \tilde{Q}] \Gamma W + E_k^T [P + \tilde{Q}] \Gamma W + W^T \Gamma [P + \tilde{Q}] F E_k + U^T \Gamma^T [P + \tilde{Q}] \Gamma W$$

constant.

Matrix $H = 2 \begin{bmatrix} \Phi^T \tilde{P} \Phi + \Phi^T \tilde{Q} \Phi + \tilde{R} \\ \Phi^T \tilde{P} F E_k + \Phi^T \tilde{Q} F E_k \end{bmatrix}$, and $\tilde{Z} = (H - t_{\min}) I_{H_{n+1}} - H_3 H_4 F E_k - H_3 H_4 \Gamma W - H_3 H_5 E_k$.

Matrix $\tilde{O} = L - H_0 F E_k - H_4 \Gamma W,$

$L = [L_{k+1}, L_{k+2}, \ldots, L_{k+H_p}]^T,$

$\tilde{U}_{\max} = [U_{\max}^T, U_{\max}^T, \ldots, U_{\max}^T]^T,$

$\tilde{U}_{\min} = [U_{\min}^T, U_{\min}^T, \ldots, U_{\min}^T]^T,$ and matrix $H_3, H_4, H_5$ respectively were

$H_3 = [h_{ij}]_{H_{n+1} \times 2H_{n+1}}, \quad h_{ij} = \begin{cases} 1, & j = 2i - 1 \\ 0, & \text{otherwise} \end{cases}$,

$H_4 = \begin{bmatrix} -I_{2N} & 0_{2N} & 0_{2N} & \cdots \\ I_{2N} & -I_{2N} & 0_{2N} & \cdots \\ 0_{2N} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix},$

and

$H_5 = \begin{bmatrix} I_{2N} \\ 0_{2N} \\ \vdots \\ 0_{2N} \end{bmatrix}$.

**Proof:** Recalling $E = \begin{bmatrix} E_{k+1}^T, E_{k+2}^T, \ldots, E_{k+H_p}^T \end{bmatrix}^T$, $U = \begin{bmatrix} U_k^T, U_k^T, \ldots, U_{k+H_p-1}^T \end{bmatrix}^T$, and $W = \begin{bmatrix} w_k^T, w_k^T, \ldots, w_{k+H_p-1}^T \end{bmatrix}^T$, the objective function (26) could be writ-
The measured state $E_k$ for the error joint dynamic model (17) was calculated at each sample step $k$ using the updated system parameters $A_k$, $B_k$, and $G_k$, as well as disturbances.

Calculate $F$ and $\Phi$ for a selected prediction horizon $H_p$ and formulate the quadratic programming problem (28) based on Theorem 3.1.

The following $E_{k+1}$ would be computed by solving the quadratic programming problem efficiently (28), computing the optimal train regulation and passenger flow control $U$, and applying it to the joint dynamic model (17).

Steps 1-4 should be repeated based on the measured value $E_{k+1}$ until the step horizon $k_f$ is reached.

In the time-varying MPC algorithm, the metro line system's stability was a complex function parameters, namely $P, Q, R, A_k, B_k, L_k, U_{\text{max}}, U_{\text{min}}$. Based on [15], for the proposed time-varying MPC algorithm in this study, we constructed a Lyapunov function with state and control constraints (26). For stability analysis, the following theorem was utilized.

**Theorem 3.3**: Consider the joint error dynamic model (17), which is based on the following optimization problem and is subject to a time-varying MPC algorithm.

$$
\min_{U_{k+j}} \sum_{j=0}^{H_p-1} \left\{ E_{k+j+1}^T P E_{k+j+1} + (E_{k+j+1} - E_{k+j})^T Q (E_{k+j+1} - E_{k+j}) + U_{k+j}^T R U_{k+j} \right\}
$$

s.t. $E_{k+j+1} = A_k E_{k+j} + B_k U_{k+j} + G_k w_{k+j}$,

$$
H_1 (E_{k+j} - E_{k+j+1}) \leq (H - t_{\text{min}}) I_{N \times 1},
$$

$$
H_2 E_{k+j+1} \leq L_{k+j+1} + H_{k+j},
$$

$$
U_{k+j} \leq U_{\text{max}}, j = 0, 1, \ldots, H_p - 1.
$$

Assume that the optimization problem $k = k_0$ was feasible from the start, that the system parameters $A_k$ and $B_k$ were provided, and that $E_{k+H_p} = 0$. Then, for all $P > 0$, $Q > 0$, and $R > 0$, it held that $\lim_{k \to \infty} E_k = 0$, implying that the proposed time-varying MPC algorithm's joint error dynamic model (17) was stable at zero under constraints, and the factual timetable converged to the nominal timetable.

**Proof**: First, for the joint error dynamic model of Equation (17) under time-varying MPC, the function of the optimization problem (35) is chosen as the lyapunov function, i.e.

$$
V(k) = J(U^*(k), E_k),
$$

with $U^*(k) = \left\{ U_{k_0}^*, U_{k+1}^*, \ldots, U_{k+H_p-1}^* \right\}$ as the optimal control sequence for the problem (35). It is clear that $V(k)$ is non-negative.

The state vector for the optimal control solution $U^*(k)$ will be obtained in step $k$, $E(k) = \left\{ E_{k+1}^T, E_{k+2}^T, \ldots, E_{k+H_p}^T \right\}$.

The constraints are clearly satisfied by $U^*(k)$ and $E(k)$. As a result, a control sequence $U(k + 1) = \left\{ U_{k+1}^*, U_{k+2}^*, \ldots, U_{k+H_p-1}^*, 0 \right\}$ was formed for the next step $k + 1$. At step $k + 1$, it was obvious that $U(k + 1)$ is feasible for the problem (35). By substituting $U(k + 1)$ into the objective function, we get $J(U(k + 1), E_{k+1})$. Then, using the assumption that $E_{k+H_p} = 0$, we get

$$
V(k + 1) = J(U^*(k + 1), E_{k+1})
$$

$$
\leq J(U(k + 1), E_{k+1})
$$

$$
= V(k) - E_{k+1}^T P E_{k+1} + (E_{k+1} - E_k)^T Q (E_{k+1} - E_k) + U_{k+1}^T R U_{k+1}
$$

(37)

which means that $V(k + 1) - V(k) \leq 0$, and $V(k)$ is decreasing and lower-bounded by $0$. Then, using Lyapunov's theory of stability, it was claimed that $\lim_{k \to \infty} E_k = 0$, i.e. the joint error dynamic model (17) under the proposed time-varying MPC algorithm was stable at zero subject to the constraints, and the factual timetable converged to the nominal timetable.

**IV. NUMERICAL EXPERIMENT**

Consider the problem of train regulation and passenger flow on a metro lines consisting of 12 stations ($N = 12$) and 20 trains ($Z = 20$). The disturbance occurred on train $j$ at station $i$ was denoted by $w_{ij}$ and was assumed as a sine periodic function, $w_{ij} = \sin(\beta_i^j t)$ for $i = 1, 2, \ldots, Z, j = 1, 2, \ldots, Z$. The first constraint of (26) was equivalent to

$$
H_3 A_k E_k \leq (H - t_{\text{min}}) D_{\text{max}}.
$$

Similarly, the last two constraint (26) were respectively, identical with

$$
I_2 H_{\text{max}} \leq U_{\text{max}} - U_{\text{min}}, I_2 H_{\text{max}} \leq -U_{\text{min}}.
$$

The proof has been complete.

According to Theorem 3.1, the primary method for joint optimal train regulation and passenger flow control on metro lines with sinusoidal disturbances was given below. **Algorithm 3.2**:

- The measured state $E_k$ for the error joint dynamic model (17) was calculated at each sample step $k$ using the updated system parameters $A_k$, $B_k$, and $G_k$, as well as disturbances.

- Calculate $F$ and $\Phi$ for a selected prediction horizon $H_p$ and formulate the quadratic programming problem (28) based on Theorem 3.1.

- The following $E_{k+1}$ would be computed by solving the quadratic programming problem efficiently (28), computing the optimal train regulation and passenger flow control $U$, and applying it to the joint dynamic model (17).

- Steps 1-4 should be repeated based on the measured value $E_{k+1}$ until the step horizon $k_f$ is reached.

In the time-varying MPC algorithm, the metro line system's stability was a complex function parameters, namely $P, Q, R, A_k, B_k, L_k, U_{\text{max}}, U_{\text{min}}$. Based on [15], for the proposed time-varying MPC algorithm in this study, we constructed a Lyapunov function with state and control constraints (26). For stability analysis, the following theorem was utilized.
1, 2, ..., N, a > 0 constant. In this study, β = \frac{π}{4}, a = 5, and β_j^i was different at each station.

Given a delay rate (α) 0.03, the passenger arrival rate at station j for trains i or γ_j^i varied within a range of values symmetrically around γ = 0.35 with half the length of d = 0.2 at each station. The passenger arrival rates at station j for trains i or γ_j^i are presented in Table I.

Table I

<table>
<thead>
<tr>
<th>Index j</th>
<th>γ_1^i</th>
<th>γ_2^i</th>
<th>γ_3^i</th>
<th>γ_4^i</th>
<th>γ_5^i</th>
<th>γ_6^i</th>
</tr>
</thead>
<tbody>
<tr>
<td>i = 1 : 4</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
</tr>
<tr>
<td>i = 5 : 8</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.47</td>
</tr>
<tr>
<td>i = 9 : 12</td>
<td>0.47</td>
<td>0.47</td>
<td>0.47</td>
<td>0.47</td>
<td>0.47</td>
<td>0.49</td>
</tr>
<tr>
<td>i = 13 : 16</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.47</td>
</tr>
<tr>
<td>i = 17 : 20</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>0.45</td>
</tr>
<tr>
<td>Index j</td>
<td>γ_1^i</td>
<td>γ_2^i</td>
<td>γ_3^i</td>
<td>γ_4^i</td>
<td>γ_5^i</td>
<td>γ_6^i</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>i = 1 : 4</td>
<td>0.45</td>
<td>0.47</td>
<td>0.49</td>
<td>0.49</td>
<td>0.47</td>
<td>0.43</td>
</tr>
<tr>
<td>i = 5 : 8</td>
<td>0.49</td>
<td>0.45</td>
<td>0.51</td>
<td>0.49</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>i = 9 : 12</td>
<td>0.51</td>
<td>0.47</td>
<td>0.53</td>
<td>0.51</td>
<td>0.47</td>
<td>0.47</td>
</tr>
<tr>
<td>i = 13 : 16</td>
<td>0.49</td>
<td>0.45</td>
<td>0.51</td>
<td>0.49</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>i = 17 : 20</td>
<td>0.47</td>
<td>0.43</td>
<td>0.49</td>
<td>0.47</td>
<td>0.43</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Based on Table I, the passengers arrival rate increased from time k = 1 to k = 8 and maximum at k = 9 to k = 12, then decreased at k = 13 to k = 20. Furthermore, the passengers arrival rate (γ_j^i) was illustrated in Fig. 1.

Due to the state constraint for the departure time on the Inequality (21), the minimum allowable headway (t_{min}) was 125s and headway scheduling \( (H) \) of 150 seconds, therefore we obtained \( H - t_{min} = 25 \) seconds. Next, for the constraint state passenger load on Inequality (23) the maximum capacity of the train for passengers (t_{max}) was 2000 and it was assumed that \( t_{max} - L_j^i \leq 50 \). Table II contains the initial error for the departure time and the passenger load. Table II shows the initial error for departure time and passenger load. The maximum train delay and maximum number of overloaded passenger are 30s and 40, respectively, both of which are greater than the maximum timetable and passenger load capacity adjustments. Delays require multiple stations to keep trains on a nominal schedule.

From the initial conditions in Table II, it can be simulated the condition when the metro lines system was not regulated or in other words \( v_j^i = 0 \). The error of train departure time at stations 1-12 are illustrated in Fig. 2, which demonstrates that the disturbance causes large fluctuations in the nominal state and has a negative impact on passenger waiting times. The passenger load error on the train at stations 1-12 is illustrated in Fig. 3, which indicates that the passenger load fluctuated greatly from the nominal state. The fluctuations in departure time errors and passenger load errors have a negative impact on reducing train operational efficiency and passenger service levels.

In this study, time-varying MPC was performed with time step \( T = 20 \). The simulation was carried out with the aim that the departure time error and the passenger load become zero, which means that there is no delay in train departure time and passenger overload. The input control constraint on Inequality (25) are \( u_{min} = -20 \) and \( u_{max} = 20 \) which means it satisfies the constraint (24). For given \( p_{min} = -25 \) and \( p_{max} = 0 \) which satisfies the constraint (24).

The weights P, Q, and R, respectively, were \( P = \text{diag} \{0.5, 0.5, \ldots, 0.5\} \), \( Q = \text{diag} \{0.5, 0. \ldots, 0.5, 0\} \), and \( R = \text{diag} \{0.3, 0.3, \ldots, 0.3\} \). Let \( H_u = H_p = 5 \). With MATLAB, the simulation applied during the peak hour period at 07:00-09:00 with time interval is 6 minutes, therefore the rush hour period was equivalent to a 20 time step.

Using the initial conditions of the departure time error, parameter \( \gamma_j^i \), and parameter \( \beta_j^i \), we acquired the simulation results and input for the departure time error on each station can be seen in Fig. 4-7.

From Fig. 4-7, it can be concluded that the error in the departure time of the train at station 1 to station 12 converge to zero in several steps, it means that the control provided in the adjustment of waiting time and train travel time was successfully implemented efficiently. The input in Fig. 4-7 at each station in the k time step is less than zero or \( u_j^i < 0 \), which means the running time and dwell time are reduced to reduce train delays.

Furthermore, by using the initial conditions of the passenger load error on the train, the parameters \( \gamma_j^i \) in Table I and parameter \( \beta_j^i \). The simulation result and input for the passenger load error at each station are displayed in Fig. 8-11.
Fig. 2. The headway deviations of metro lines without train regulation.
Fig. 3. The passenger load errors of metro lines without train regulation.
Fig. 4. Train delay at different time $k$ in station 1, 2, and 3 under the time-varying MPC.

Fig. 5. Train delay at different time $k$ in station 4, 5, and 6 under the time-varying MPC.
Fig. 6. Train delay at different time \( k \) in station 7, 8, and 9 under the time-varying MPC.

Fig. 7. Train delay at different time \( k \) in station 10, 11, and 12 under the time-varying MPC.
Fig. 8. Passenger load error at different time $k$ in station 1, 2, and 3 under the time-varying MPC.

Fig. 9. Passenger load error at different time $k$ in station 4, 5, and 6 under the time-varying MPC.
Fig. 10. Passenger load error at different time $k$ in station 7, 8, and 9 under the time-varying MPC.

Fig. 11. Passenger load error at different time $k$ in station 10, 11, and 12 under the time-varying MPC.
From Fig. 8-11, it can be concluded that the error of passenger load on the train at station 1 to station 12 convergent to zero in several time steps. It means the control provided was successfully implemented efficiently. The input in Fig. 4-7 at each station in the k time step is less than zero or $p_j^k < 0$, which indicates that there was a decrease in the number of arriving passengers and that the train’s limited passenger capacity was met. Furthermore, the optimization problem in Equation (28), was solved using quadratic programming by quadprog in MATLAB. The result of objective function was 630.131.

V. CONCLUSION

The joint optimum train control and passenger flow strategy were investigated in this article to optimize headway regularity and commercial speed. The time-varying MPC approach was used to design an optimal control problem for the combined dynamic train regulation and passenger flow management strategy, and it was addressed by considering the headway regularity and commercial speed of the cost function. The numerical solution of a set of quadratic programming problems provided an optimal control strategy for the joint dynamic train regulation and passenger flow control method.

The suggested method offered a real-time train control and management technique for passenger flow that could be efficiently applied to real-time metro lines. The recommended joint optimum control strategy reduced train delays, passenger load errors, and train headway deviations, according to numerical experiments. Additionally, it improves passenger service standards and train operating efficiently.

REFERENCES


