A New Approach for Seeking Exact Solutions of Fractional Partial Differential Equations in the Sense of Conformable Fractional Derivative

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Abstract—**In this paper, based on the combination of the Jacobi elliptic equation and the concept of the simple equation method, we introduce a new approach for solving fractional partial differential equations, where the fractional derivative is defined in the sense of the conformable fractional derivative. By use of a nonlinear transformation, the proved chain rule and the properties of fractional calculus, certain fractional partial differential equation can be converted into another ordinary differential equation of integer order. With general solutions of the Jacobi elliptic equation, a series of exact solutions for the ordinary differential equation can be obtained subsequently based on the homogeneous balance principle and with the aid of mathematical software. As for applications of this approach, we apply it to seek exact solutions for the space fractional (2+1)-dimensional breaking soliton equations and the space-time fractional BBM equation. As a result, abundant solitary wave solutions, periodic wave solutions, rational function solutions and Jacobi elliptic function solutions are successfully found.**

MSC 2010: 35C07; 35Q51; 35Q53

Index Terms—**Jacobi elliptic equation; fractional equation; exact solution; breaking soliton equations; space-time fractional BBM equation**

I. Introduction

The nonlinear phenomena exist in all the fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on [1-4]. It is well known that many nonlinear partial differential equations are widely used to describe these complex phenomena. Fractional differential equations are generalizations of classical differential equations of integer order. Recently, Fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics. In particular, fractional derivative is useful in describing the memory and hereditary properties of materials and processes. To illustrate better the physical phenomena denoted by fractional differential equations, it is necessary to obtain analytical or numerical solutions for fractional differential equations. Many efficient methods have been proposed so far to obtain numerical solutions and exact solutions of fractional differential equations. For example, these methods include the finite difference method [5,6], the $(\frac{G'}{G})$ method [7-11], the variational iterative method [12-15], the fractional Nikiforov-Uvarov Method [16], the modified Kudryashov method [17-21], the exp method [22,23], the first integral method [24,25], the sub-equation method [26-29], the coupled fractional reduced differential transform method [30], the Bernstein polynomials method [31], the residual power series method [32], the Jacobi elliptic function method [33] and so on.

In [34], Pandir and Duzgun developed a new version of F-expansion method based on the modified Riemann-Liouville derivative, which is defined as $D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi$ for $0<\alpha<1$. By the chain rule for fractional calculus, which is denoted by $\frac{\partial^\alpha u}{\partial t^\alpha} = \sigma' \frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha}$, where σ' denotes the sigma index, fractional differential equations can be converted into ordinary differential equations of integral order. Based on the Jacobi elliptic equation, a lot of exact solutions with Jacobi elliptic function forms were obtained.

However, we note that for different expressions of $u(t)$, the sigma indexes σ' are also different. For example, for $u(t)=t$, after one proposed transformation $s=\frac{t^\alpha}{\Gamma(1+\alpha)}$, where $0<\alpha<1$, we get that $u(t) = [\Gamma(1+\alpha)s]^{\frac{1}{\alpha}} = t$. By use of the definition of the modified Riemann-Liouville derivative, one can obtain that $\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$, while $\frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha} = \frac{t^{1-\alpha}\Gamma(1+\alpha)}{\alpha}$. So $\sigma' = \frac{\alpha}{\Gamma(2-\alpha)\Gamma(1+\alpha)}$. Otherwise, if we take $u(t)=t^2$, By use of the definition of the modified Riemann-Liouville derivative, one can obtain that $\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}$, while $\frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha} = \frac{2t^{2-\alpha}\Gamma(1+\alpha)}{\alpha}$. So $\sigma' = \frac{\alpha}{\Gamma(3-\alpha)\Gamma(1+\alpha)}$. Thus one can see that the sigma index is not always the same constant, which shows that the reduction from fractional differential equations to ordinary differential equations of integral order is of little flaw, and needs further improvement. In fact, the chain rule can not be effective any longer in the case of the modified Riemann-Liouville derivative.

Motivated by the analysis above, in this paper, we introduce a new approach to seek exact solutions for space-time fractional partial differential equations based on the combination of the simple equation method and the following Jacobi elliptic equation

$$(G')^2 = e_2G^4 + e_1G^2 + e_0, \tag{1}$$

where $e_0,\ e_1,\ e_2$ are arbitrary constants. The fractional partial differential equations are defined in the sense of the conformable fractional derivative, which is defined as follows

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$$D^\alpha f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},$$

and satisfies the following properties:

(i). $D_t^\alpha[af(t)+bg(t)] = aD^\alpha f(t)+bD^\alpha g(t)$.

(ii). $D_t^\alpha(t^\gamma) = \gamma t^{\gamma-\alpha}$.

(iii). $D_t^\alpha[f(t)g(t)] = f(t)D^\alpha g(t)+g(t)D^\alpha f(t)$.

(iv). $D_t^\alpha C = 0$, where C is a constant.

(v). $D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t)$.

(vi). $D_t^\alpha(\frac{f}{g})(t) = \frac{g(t)D^\alpha f(t)-f(t)D^\alpha g(t)}{g^2(t)}$.

(vii). $D_t^\alpha f(t) = t^{1-\alpha}f'(t)$.

The properties above can be easily proved due to the definition of the conformable fractional derivative. Under a given transformation $\xi = \frac{t^\alpha}{\alpha}$, by use of (ii) and (v) one can obtain that $\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial \xi}\frac{\partial^\alpha \xi}{\partial t^\alpha} = \frac{\partial u}{\partial \xi}$. So the chain rule holds, and then the fractional derivative can be converted into integer order case.

The main point of the present method lies in that by a nonlinear transformation for ξ, one certain fractional partial differential equation expressed in the variables $t, x_1, x_2, ..., x_n$ can be turned into another ordinary differential equation of integer order in ξ, the solution of which are supposed to have the form $U(\xi) = \sum_{i=0}^{m} a_i[\frac{G'(\xi)}{G(\xi)}]^i$, where the integer m can be determined by the homogeneous balancing principle, and $G = G(\xi)$ satisfies the Jacobi elliptic equation (1). By the general solutions of Eq. (1), we can deduce the expression for $(\frac{G'(\xi)}{G(\xi)})$, and then the exact solutions for the original fractional partial differential equation can be deduced subsequently.

The rest of this paper is organized as follows. In Section II, we give the description of the proposed method for solving fractional partial differential equations. In Section III, we apply this method to establish exact solutions for the space fractional (2+1)-dimensional breaking soliton equations and the space-time fractional BBM equation. In Section IV, we extend the present method in Section II in three aspects, and give the main points for these extensions. In Section V, some concluding comments are presented.

II. Summary of the Method

In this section we give the description of the present method for solving fractional partial differential equations.

Suppose that a fractional partial differential equation in the independent variables $t, x_1, x_2, ..., x_n$ is given by

$$P(u_1, ...u_k,\ D_t^\alpha u_1, ..., D_t^\alpha u_k, D_{x_1}^\beta u_1, ..., D_{x_1}^\beta u_k, ...$$
$$, D_{x_n}^\gamma u_1, ..., D_{x_n}^\gamma u_k, ...) = 0, \qquad (2)$$

where $u_i = u_i(t, x_1, x_2, ..., x_n),\ i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Execute a certain nonlinear fractional complex transformation for ξ

$$u_i(t, x_1, x_2, ..., x_n) = U_i(\xi), \quad \xi = \xi(t, x_1, x_2, ..., x_n), \ (3)$$

such that Eq. (2) can be turned into the following ordinary differential equation of integer order with respect to the variable ξ:

$$\widetilde{P}(U_1, ..., U_k,\ U'_1, ..., U'_k, U''_1, ..., U''_k,\ ...) = 0. \qquad (4)$$

In fact, take $D_t^\alpha u_1$ for example, one can suppose a nonlinear fractional complex transformation $\xi = c\frac{t^\alpha}{\alpha}$, and then using the properties (ii) and (v) one can obtain $D_t^\alpha u_1 = U'_1(\xi)D_t^\alpha \xi = cU'_1(\xi)$.

Step 2. Suppose that the solution of (4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i}(\frac{G'}{G})^i,\ j = 1, 2, ..., k, \qquad (5)$$

where $a_{j,i},\ i = 0, 1, ..., m_j,\ j = 1, 2, ..., k$ are constants to be determined later, $a_{j,m_j} \neq 0$, the positive integer m_j can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (4), $G = G(\xi)$ satisfies the Jacobi elliptic equation (1).

Step 3. Substituting (5) into (4) and using (1), we convert the left-hand side of (4) into another polynomial in $G^iG'^j$. Collecting all coefficients of the same power and Equating them to zero, one can obtain a set of algebraic equations for $a_{j,i},\ i = 0, 1, ..., m_j,\ j = 1, 2, ..., k$.

Step 4. Solving the equations system in Step 3, and using the general solutions of Eq. (1), we can construct a variety of exact solutions for Eq. (2).

Some general solutions of Eq. (1) are listed as follows.

$$G(\xi) = \begin{cases} sn(\xi), & e_2 = m^2,\ e_1 = -(1+m^2),\ e_0 = 0, \\ cn(\xi), & e_2 = -m^2,\ e_1 = 2m^2-1,\ e_0 = 1-m^2, \\ dn(\xi), & e_2 = -1,\ e_1 = 2-m^2,\ e_0 = m^2-1, \\ cs(\xi), & e_2 = 1,\ e_1 = 2-m^2,\ e_0 = 1-m^2, \\ sd(\xi), & e_2 = m^2(m^2-1),\ e_1 = 2m^2-1,\ e_0 = 1, \\ dc(\xi), & e_2 = 1,\ e_1 = -(m^2+1),\ e_0 = m^2, \\ -\sqrt{e_1}sech(\sqrt{e_1}\xi), & e_2 = -1,\ e_1 > 0,\ e_0 = 0, \\ -\sqrt{e_1}csch(\sqrt{e_1}\xi), & e_2 = 1,\ e_1 > 0,\ e_0 = 0, \\ \sqrt{-e_1}sec(\sqrt{-e_1}\xi), & e_2 = 1,\ e_1 < 0,\ e_0 = 0, \\ \frac{1}{\xi+C_0}, & e_2 = 1,\ e_1 = 0,\ e_0 = 0, \end{cases}$$

where $sn(\xi),\ cn(\xi),\ dn(\xi)$ denote the Jacobi elliptic sine function, Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind respectively, m is the modulus of Jacobi elliptic functions, and

$$cs(\xi) = \frac{cn(\xi)}{sn(\xi)},\ sd(\xi) = \frac{sn(\xi)}{dn(\xi)},\ dc(\xi) = \frac{dn(\xi)}{cn(\xi)},$$
$$sc(\xi) = \frac{1}{cs(\xi)},\ ds(\xi) = \frac{1}{sd(\xi)},\ cd(\xi) = \frac{1}{dc(\xi)},$$
$$nd(\xi) = \frac{1}{dn(\xi)},\ ns(\xi) = \frac{1}{sn(\xi)},\ nc(\xi) = \frac{1}{cn(\xi)}.$$

Furthermore, one has

$$
\left(\frac{G'(\xi)}{G(\xi)}\right)=\begin{cases}
cn(\xi)ds(\xi), & e_2=m^2,\\
 & e_1=-(1+m^2),\ e_0=0,\\
-sn(\xi)dc(\xi), & e_2=-m^2,\ e_1=2m^2-1,\\
 & e_0=1-m^2,\\
-m^2sn(\xi)cd(\xi), & e_2=-1,\ e_1=2-m^2,\\
 & e_0=m^2-1,\\
-\dfrac{dc(\xi)}{sn(\xi)}, & e_2=1,\ e_1=2-m^2,\\
 & e_0=1-m^2,\\
\dfrac{cs(\xi)}{dn(\xi)}, & e_2=m^2(m^2-1),\\
 & e_1=2m^2-1,\ e_0=1,\\
(1-m^2)\dfrac{sd(\xi)}{cn(\xi)}, & e_2=1,\\
 & e_1=-(m^2+1),\ e_0=m^2,\\
-\sqrt{e_1}tanh(\sqrt{e_1}\xi), & e_2=-1,\ e_1>0,\ e_0=0,\\
-\sqrt{e_1}coth(\sqrt{e_1}\xi), & e_2=1,\ e_1>0,\ e_0=0,\\
\sqrt{-e_1}tan(\sqrt{-e_1}\xi), & e_2=1,\ e_1<0,\ e_0=0,\\
-\dfrac{1}{\xi+C_0}, & e_2=1,\ e_1=0,\ e_0=0.
\end{cases} \tag{6}
$$

III. Application of the present method to some fractional partial differential equations

In this section, we present some applications for the method described in Section II to seek exact solutions for some fractional partial differential equations.

A. Space fractional (2+1)-dimensional breaking soliton equations

Consider the space fractional (2+1)-dimensional breaking soliton equations [1]

$$
\begin{cases}
u_t+a\dfrac{\partial^{2\alpha+\beta}u}{\partial x^\alpha\partial x^\alpha y^\beta}+4au\dfrac{\partial^\alpha v}{\partial x^\alpha}+4a\dfrac{\partial^\alpha u}{\partial x^\alpha}v=0,\\
\dfrac{\partial^\beta u}{\partial y^\beta}=\dfrac{\partial^\alpha v}{\partial x^\alpha},
\end{cases},
$$

$$
0<\alpha,\ \beta\le 1, \tag{7}
$$

where the contained fractional derivative is defined as the conformable fractional derivative.

The corresponding integer order equation to Eqs. (7) can be found in [35-38]. In [1], the authors solved Eqs. (7) by a fractional sub-equation method based on the known (G'/G) method, and obtained some exact solutions including hyperbolic function solutions, trigonometric function solutions, rational function solutions and so on. In the following, we will apply the described method in Section II to solve Eqs. (7). To begin with, we suppose $u(x,y,t)=U(\xi),\ v(x,y,t)=V(\xi)$, where $\xi=ct+\frac{k_1}{\alpha}x^\alpha+\frac{k_2}{\beta}y^\beta+\xi_0,\ k_1,\ k_2,\ c,\ \xi_0$ are all constants with $k_1,\ k_2,\ c\neq 0$. Then by use of the properties (ii) and (v), we obtain that

$$
\begin{cases}
D_x^\alpha u=D_x^\alpha U(\xi)=U'(\xi)D_x^\alpha\xi=k_1U'(\xi),\\
D_y^\beta u=D_y^\beta U(\xi)=U'(\xi)D_y^\beta\xi=k_2U'(\xi),\\
u_t=cU'(\xi),
\end{cases} \tag{8}
$$

and then Eqs. (7) can be turned into the following form:

$$
\begin{cases}
cU'+ak_1^2k_2U'''+4ak_1UV'+4ak_1VU'=0,\\
k_2U'=k_1V'.
\end{cases} \tag{9}
$$

Suppose that the solution of Eqs. (9) can be expressed by

$$
\begin{cases}
U(\xi)=\sum\limits_{i=0}^{m_1}a_i(\frac{G'}{G})^i,\\
V(\xi)=\sum\limits_{i=0}^{m_2}b_i(\frac{G'}{G})^i,
\end{cases} \tag{10}
$$

where $G=G(\xi)$ satisfies the Jacobo elliptic equation (1).

Balancing the order of U''' and UV' in the first equation in (9), U' and V' in the second equation in (9) we obtain $m_1=m_2=2$. So

$$
\begin{cases}
U(\xi)=a_0+a_1(\frac{G'}{G})+a_2(\frac{G'}{G})^2,\\
V(\xi)=b_0+b_1(\frac{G'}{G})+b_2(\frac{G'}{G})^2.
\end{cases} \tag{11}
$$

Substituting (11) into (9), using (1) and collecting all the terms with the same power of $G^iG'^j$ together, equating each coefficient to zero, we obtain a set of algebraic equations. Solving these equations, we get that

$$
a_0=a_0,\ a_1=0,\ a_2=-\frac{3}{2}k_1^2,
$$

$$
b_0=-\frac{-8ak_1^2k_2e_1+c+4aa_0k_2}{4ak_1},\ b_1=0,\ b_2=-\frac{3}{2}k_1k_2,
$$

where a_0 is an arbitrary constant.

Substituting the result above into Eqs. (11), and combining with (6), we can obtain the following exact solutions in the forms of the Jacobi elliptic functions for Eqs. (7), where $\xi=ct+\frac{k_1}{\alpha}x^\alpha+\frac{k_2}{\beta}y^\beta+\xi_0$.

Family 1: when $e_2=m^2,\ e_1=-(1+m^2),\ e_0=0$, the following Jacobi elliptic function solutions can be obtained:

$$
\begin{cases}
u_1(x,y,t)=a_0-\frac{3}{2}k_1^2[cn(\xi)ds(\xi)]^2,\\
v_1(x,y,t)=-\dfrac{8ak_1^2k_2(1+m^2)+c+4aa_0k_2}{4ak_1}\\
\qquad-\frac{3}{2}k_1k_2[cn(\xi)ds(\xi)]^2.
\end{cases} \tag{12}
$$

Family 2: when $e_2=-m^2,\ e_1=2m^2-1,\ e_0=1-m^2$,

$$
\begin{cases}
u_2(x,y,t)=a_0-\frac{3}{2}k_1^2[sn(\xi)dc(\xi)]^2,\\
v_2(x,y,t)=-\dfrac{-8ak_1^2k_2(2m^2-1)+c+4aa_0k_2}{4ak_1}\\
\qquad-\frac{3}{2}k_1k_2[sn(\xi)dc(\xi)]^2.
\end{cases} \tag{13}
$$

Family 3: when $e_2=-1,\ e_1=2-m^2,\ e_0=m^2-1$,

$$
\begin{cases}
u_3(x,y,t)=a_0-\frac{3}{2}k_1^2m^4[sn(\xi)cd(\xi)]^2,\\
v_3(x,y,t)=-\dfrac{-8ak_1^2k_2(2-m^2)+c+4aa_0k_2}{4ak_1}\\
\qquad-\frac{3}{2}k_1k_2m^4[sn(\xi)cd(\xi)]^2.
\end{cases} \tag{14}
$$

Family 4: when $e_2=1,\ e_1=2-m^2,\ e_0=1-m^2$,

$$
\begin{cases}
u_4(x,y,t)=a_0-\frac{3}{2}k_1^2[\frac{dc(\xi)}{sn(\xi)}]^2,\\
v_4(x,y,t)=-\dfrac{-8ak_1^2k_2(2-m^2)+c+4aa_0k_2}{4ak_1}\\
\qquad-\frac{3}{2}k_1k_2[\frac{dc(\xi)}{sn(\xi)}]^2.
\end{cases} \tag{15}
$$

Family 5: when $e_2 = m^2(m^2-1),\ e_1 = 2m^2-1,\ e_0 = 1,$

$$\begin{cases} u_5(x,y,t) = a_0 - \frac{3}{2}k_1^2[\frac{cs(\xi)}{dn(\xi)}]^2, \\ v_5(x,y,t) = -\frac{-8ak_1^2k_2(2m^2-1)+c+4aa_0k_2}{4ak_1} \\ \qquad -\frac{3}{2}k_1k_2[\frac{cs(\xi)}{dn(\xi)}]^2. \end{cases} \tag{16}$$

Family 6: when $e_2 = 1,\ e_1 = -(m^2+1),\ e_0 = m^2,$

$$\begin{cases} u_6(x,y,t) = a_0 - \frac{3}{2}k_1^2(1-m^2)^2[\frac{sd(\xi)}{cn(\xi)}]^2, \\ v_6(x,y,t) = -\frac{8ak_1^2k_2(m^2+1)+c+4aa_0k_2}{4ak_1} \\ \qquad -\frac{3}{2}k_1k_2(1-m^2)^2[\frac{sd(\xi)}{cn(\xi)}]^2. \end{cases} \tag{17}$$

One can also obtain corresponding solitary wave solutions, periodic wave solutions and rational function solutions as follows, where $\xi = ct + \frac{k_1}{\alpha}x^\alpha + \frac{k_2}{\beta}y^\beta + \xi_0$.

Family 7: when $e_2 = -1,\ e_1 > 0,\ e_0 = 0$, the following solitary wave solutions with hyperbolic function forms can be obtained:

$$\begin{cases} u_7(x,y,t) = a_0 - \frac{3}{2}k_1^2e_1\tanh^2(\sqrt{e_1}\xi), \\ v_7(x,y,t) = -\frac{-8ak_1^2k_2(2-m^2)+c+4aa_0k_2}{4ak_1} \\ \qquad -\frac{3}{2}k_1k_2e_1\tanh^2(\sqrt{e_1}\xi). \end{cases} \tag{18}$$

In Figs. 1-2, the solitary wave solutions $u_7(x,y,t),\ v_7(x,y,t)$ in (18) with some special parameters are demonstrated.

Family 8: when $e_2 = 1,\ e_1 > 0,\ e_0 = 0,$

$$\begin{cases} u_8(x,y,t) = a_0 - \frac{3}{2}k_1^2e_1\coth^2(\sqrt{e_1}\xi), \\ v_8(x,y,t) = -\frac{-8ak_1^2k_2(2-m^2)+c+4aa_0k_2}{4ak_1} \\ \qquad -\frac{3}{2}k_1k_2e_1\coth^2(\sqrt{e_1}\xi). \end{cases} \tag{19}$$

Family 9: when $e_2 = 1,\ e_1 < 0,\ e_0 = 0$, the following trigonometric function solutions can be obtained:

$$\begin{cases} u_9(x,y,t) = a_0 + \frac{3}{2}k_1^2e_1tan^2(\sqrt{-e_1}\xi), \\ v_9(x,y,t) = -\frac{-8ak_1^2k_2(2-m^2)+c+4aa_0k_2}{4ak_1} \\ \qquad +\frac{3}{2}k_1k_2e_1tan^2(\sqrt{-e_1}\xi). \end{cases} \tag{20}$$

In Figs. 3-4, the periodic wave solutions $u_9(x,y,t),\ v_9(x,y,t)$ in (20) with some special parameters are demonstrated.

Family 10: when $e_2 = 1,\ e_1 = 0,\ e_0 = 0$, the following rational function solutions can be obtained:

$$\begin{cases} u_{10}(x,y,t) = a_0 - \frac{3}{2}k_1^2(\frac{1}{\xi + C_0})^2, \\ v_{10}(x,y,t) = -\frac{-8ak_1^2k_2(2-m^2)+c+4aa_0k_2}{4ak_1} \\ \qquad -\frac{3}{2}k_1k_2(\frac{1}{\xi + C_0})^2. \end{cases} \tag{21}$$

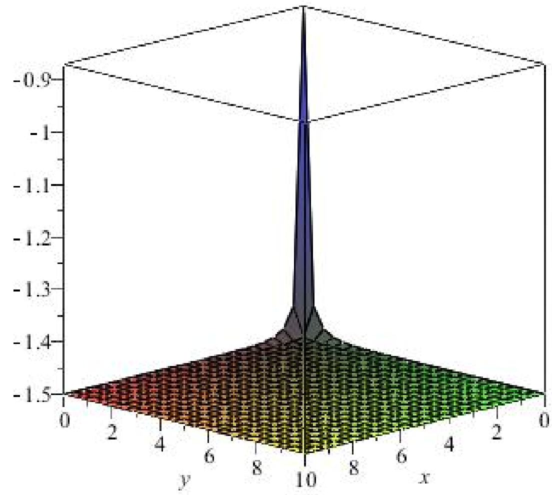

Fig. 1. The solitary wave solution u_7 with $c = k_1 = k_2 = e_1 = a = t = m = 1,\ \xi_0 = a_0 = 0,\ \alpha = \beta = 0.5$

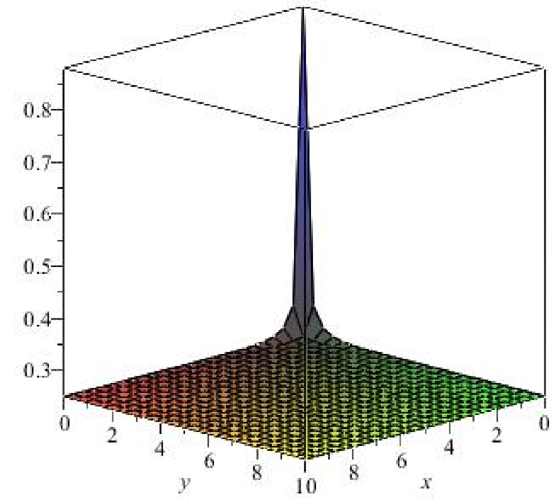

Fig. 2. The solitary wave solution v_7 with $c = k_1 = k_2 = e_1 = a = t = m = 1,\ \xi_0 = a_0 = 0,\ \alpha = \beta = 0.5$

B. Space-time fractional BBM equation

Consider the space-time fractional BBM equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial^\beta u}{\partial x^\beta} + \frac{\partial^\beta u}{\partial x^\beta} - \mu\frac{\partial^{2\beta+\alpha} u}{\partial x^\beta \partial x^\beta t^\alpha} = 0,\ 0 < \alpha,\ \beta \leq 1, \tag{22}$$

which is a variation of the following BBM equation of integer order:

$$u_t + uu_x + u_x - \mu u_{xxt} = 0. \tag{23}$$

In order to apply the present method described in Section II, suppose $u(x,t) = U(\xi)$, where $\xi = \frac{ct^\alpha}{\alpha} + \frac{kx^\beta}{\beta} + \xi_0$, $k,\ c,\ \xi_0$ are all constants with $k,\ c \neq 0$. Then similar to above, by use of the properties (ii) and (v), we obtain

$$\begin{cases} D_t^\alpha u = D_t^\alpha U(\xi) = U'(\xi)D_t^\alpha\xi = cU'(\xi), \\ D_x^\beta u = D_x^\beta U(\xi) = U'(\xi)D_x^\beta\xi = kU'(\xi), \end{cases} \tag{24}$$

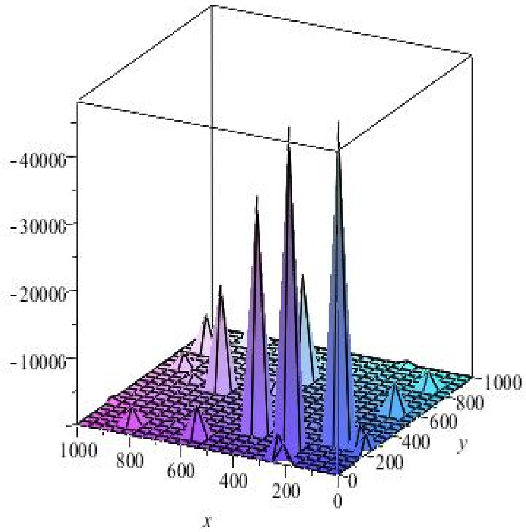

Fig. 3. The periodic wave solution u_9 with $e_1=-1$, $c=k_1=k_2=a=t=m=1$, $\xi_0=a_0=0$, $\alpha=\beta=0.5$

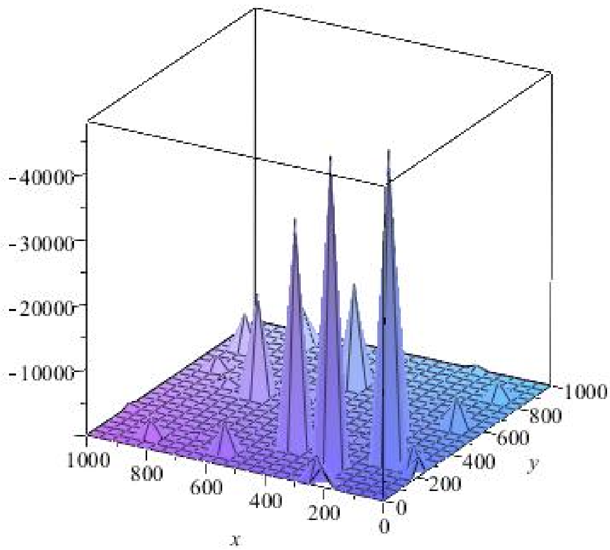

Fig. 4. The periodic wave solution v_9 with $e_1=-1$, $c=k_1=k_2=a=t=m=1$, $\xi_0=a_0=0$, $\alpha=\beta=0.5$

and then Eq. (22) can be turned into the following form:

$$cU' + kUU' + kU' - \mu ck^2U''' = 0. \tag{25}$$

Suppose that the solution of Eq. (25) can be expressed by

$$U(\xi) = \sum_{i=0}^{n} a_i(\frac{G'}{G})^i, \tag{26}$$

where $G = G(\xi)$ satisfies the Jacobo elliptic equation (1). By Balancing the order between the highest order derivative term and nonlinear term in Eq. (25), we can obtain $n = 2$. So we have

$$U(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2. \tag{27}$$

Substituting (27) into (25), using (1) and collecting all the terms with the same power of $G^iG'^j$ together, equating each coefficient to zero, we obtain a set of algebraic equations. Solving these equations, we get that

$$a_0 = -\frac{k + c + 8\mu ck^2e_1}{k},\ a_1 = 0,\ a_2 = 12\mu ck.$$

Substituting the result above into Eq. (27), and combining with (11), we can obtain the following exact solutions in the forms of the Jacobi elliptic functions for Eq. (22), where $\xi = \frac{ct^\alpha}{\alpha} + \frac{kx^\beta}{\beta} + \xi_0$.

Family 1: when $e_2 = m^2,\ e_1 = -(1 + m^2),\ e_0 = 0$,

$$u_1(x, t) = -\frac{k + c + 8\mu ck^2e_1}{k} + 12\mu ck[cn(\xi)ds(\xi)]^2, \tag{28}$$

Family 2: when $e_2 = -m^2,\ e_1 = 2m^2 - 1,\ e_0 = 1 - m^2$,

$$u_2(x, t) = -\frac{k + c + 8\mu ck^2e_1}{k} + 12\mu ck[sn(\xi)dc(\xi)]^2, \tag{29}$$

Family 3: when $e_2 = -1,\ e_1 = 2 - m^2,\ e_0 = m^2 - 1$,

$$u_3(x, t) = -\frac{k + c + 8\mu ck^2e_1}{k} + 12\mu ckm^4[sn(\xi)cd(\xi)]^2. \tag{30}$$

Family 4: when $e_2 = 1,\ e_1 = 2 - m^2,\ e_0 = 1 - m^2$,

$$u_4(x, t) = -\frac{k + c + 8\mu ck^2e_1}{k} + 12\mu ck[\frac{dc(\xi)}{sn(\xi)}]^2. \tag{31}$$

Family 5: when $e_2 = m^2(m^2 - 1),\ e_1 = 2m^2 - 1,\ e_0 = 1$,

$$u_5(x, t) = -\frac{k + c + 8\mu ck^2e_1}{k} + 12\mu ck[\frac{cs(\xi)}{dn(\xi)}]^2. \tag{32}$$

Family 6: when $e_2 = 1,\ e_1 = -(m^2 + 1),\ e_0 = m^2$,

$$u_6(x, t) = -\frac{k + c + 8\mu ck^2e_1}{k} + 12\mu ck(1 - m^2)^2[\frac{sd(\xi)}{cn(\xi)}]^2. \tag{33}$$

Remark. Combining with other general solutions of the Jacobi elliptic equation (1) where $e_2,\ e_1,\ e_0$ taken different values, one can obtain corresponding hyperbolic function solutions, trigonometric function solutions and rational function solutions for space-time fractional BBM equation, which are omitted here for the sake of simplicity.

IV. Further extensions of the present method

In this section, we extend the present method in Section II in three aspects.

First, if we change the form of the polynomial in $G(\xi)$ in (5), such as

$$U_j(\xi) = \sum_{i=0}^{m_j}[a_{j,i}(\frac{G'}{G})^i + b_{j,i}(\frac{G'}{G})^{-i}],\ j = 1, 2, ..., k, \tag{34}$$

where $a_{j,i},\ b_{j,i},\ i = 0, 1, ..., m_j,\ j = 1, 2, ..., k$ are constants to be determined later, $a_{j,m_j} \neq 0$, then following a similar process as Sections II and III, one can obtain some more new exact solutions for the space fractional (2+1)-dimensional breaking soliton equations and the space-time fractional BBM equation.

Second, if we change the form of (5) to

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i}(\frac{D^\alpha G}{G})^i, \ j = 1, 2, ..., k, \qquad (35)$$

or

$$U_j(\xi) = \sum_{i=0}^{m_j} [a_{j,i}(\frac{D^\alpha G}{G})^i + b_{j,i}(\frac{D^\alpha G}{G})^{-i}], \ j = 1, 2, ..., k, \qquad (36)$$

where G satisfies the following fractional Jacobi elliptic equation

$$(D^\alpha G)^2 = e_2 G^4 + e_1 G^2 + e_0, \qquad (37)$$

then combing the properties of the conformable fractional derivative and (1), one can obtain the solutions of (37), and furthermore following a similar process as Section III, we can obtain other new exact solutions for the space fractional (2+1)-dimensional breaking soliton equations and the space-time fractional BBM equation.

At last, if we select other different sub-equations from (1), we can obtain other exact solutions of new forms for the two fractional differential equations.

V. Conclusions

In this paper, we have introduced a new approach for solving fractional partial differential equations in the sense of the conformable fractional derivative. The most important point here lies in that certain fractional partial differential equation can be converted into another ordinary differential equation of integer order by use of a nonlinear transformation for ξ, and the exact solutions of the converted ordinary differential equation can be determined by use of a combination of the simple equation method and the Jacobi elliptic equation. For illustrating the validity of this method, we apply it to seek exact solutions for the space fractional (2+1)-dimensional breaking soliton equations and the space-time fractional BBM equation, and as a result, a series of exact solutions in various forms including the Jacobi elliptic function solutions, hyperbolic function solutions, trigonometric function solutions and rational function solutions for the two fractional partial differential equations have been successfully found. We also extend the present method in three aspects to get more new exact solutions.

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