

Critical Metrics of the Schouten Functional on a 4-manifold

L.F. Zhang

Abstract—The existence of critical metrics of the Schouten functional on $S^1 \times S^3$ is studied in this paper. We define a family of Riemannian metrics $(g_\lambda)$ on $S^1 \times S^3$, where $\lambda$ is the interval $(0, 1)$ or $(0, +\infty)$. By investigating these Riemannian metrics’ properties about the Schouten functional on $S^1 \times S^3$, we obtain critical metrics’ results of the Schouten functional on $S^1 \times S^3$.

Index Terms—Schouten tensor, Riemannian functional, critical metric, flat metric, 4-manifold

I. INTRODUCTION

Let $M^*(n \geq 3)$ be an n-dimensional compact, orientable, connected smooth manifold. We denote by $\mathcal{M}(M)$ and $\mathcal{D}(M)$ the space of smooth metrics on $M^*$ and the group of diffeomorphisms of $M^*$, respectively. A functional $\mathcal{F} : \mathcal{M}(M) \rightarrow R$ is called Riemannian if $\mathcal{F}$ is invariant under the action of $\mathcal{D}(M)$, that is, $\mathcal{F}(\varphi^*g) = \mathcal{F}(g)$ for each $\varphi \in \mathcal{D}(M)$ and $g \in \mathcal{M}(M)$.

A very typical example of Riemannian functionals is the Einstein-Hilbert functional defined by

$$\mathcal{H}[g] = \left(\int_{M^*} \text{dvol}_g\right)^{(2-n)/n} \int_{M^*} R_g \text{dvol}_g,$$

(1.1)

where $R_g$ is the scalar curvature of $g$. Hilbert pointed out that the critical points of this functional are Einstein metrics [1]. The critical points of this functional restricted to the space of metrics with constant scalar curvature of unitary volume was studied, the CPE metric was simplified. A necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein was obtained [2]. Baltazar et al. studied weakly Einstein critical metrics of the volume functional on a compact manifold with smooth boundary, and gave a complete classification for an n-dimensional weakly Einstein critical metrics of the volume functional with nonnegative scalar curvature [3].

In addition to the above functionals, we can get other examples of Riemannian functionals by integrating the quadratic polynomial for curvature tensor [1]. Among them, the most interesting Riemannian functional is

$$\mathcal{S}[g] = \frac{1}{n} \int_{M^*} \text{dvol}_g \left(\frac{1}{n} \int_{M^*} R_g \text{dvol}_g\right),$$

(1.2)

where $\sigma_j(A)$ denotes the k-th elementary symmetric function of the eigenvalues (with respect to $g$ ) of the Schouten tensor defined by

$$A_g = \text{Ric}_g - \frac{R_g}{2(n-1)} g.$$  

(1.3)

The study of this functional was first initiated by I. Viallovsky [4], and many in-depth results have been obtained [5-8]. It is the main result of [9] that, for compact, boundless, smooth 3-dimensional manifolds, the space forms are the only critical points of $\mathcal{S}_3$ if the critical value is nonnegative. In [10], Hu et al. extended this result to higher dimensions, and proved that, for n-dimensional ($n \geq 5$), compact, boundless, smooth manifolds, the space forms are the only locally conformally flat critical points of $\mathcal{S}_n$ if the critical value of $\mathcal{S}_n$ is nonnegative. Gursky et al. developed a gluing procedure designed to obtain canonical metrics on connected sums of Einstein four-manifolds. By using certain quotients of $S^1 \times S^3$ as one of the gluing factors, critical metrics on several non-simply-connected manifolds were obtained [11]. By using the curvature identity on 4-dimensional Riemannian manifolds, critical metrics for the squared L2-norm functionals of the curvature tensor, the Ricci tensor and the scalar curvature were studied [12].

In addition to the above functionals, we can get other examples of Riemannian functionals by integrating the quadratic polynomial for curvature tensor [1]. Among them, the most interesting Riemannian functional is$\mathcal{S}(g) = \frac{1}{n} \int_{M^*} \text{dvol}_g \left(\frac{1}{n} \int_{M^*} R_g \text{dvol}_g\right)$.

(1.4)

In this paper, we also call $\mathcal{S}$ the Schouten functional. We know

$$|A_g|^2 = |\text{Ric}_g|^2 - \frac{3n-4}{4(n-1)^2} R_g^2 = |E_g|^2 + \frac{(n-2)^2}{4n(n-1)^2} R_g^2,$$

where $E_g = \text{Ric}_g - (R_g / n) g$ is the trace-free Ricci tensor.

Obviously, $\mathcal{S}(g) = 0$ if and only if $(M^*, g)$ is Ricci-flat.

In [13], the authors proved that the infimum of the Schouten functional on $S^1 \times S^3$ is zero. However, only the Ricci-flat metric is the metric with zero Schouten functional on the manifold, but there is no Ricci-flat metric on $S^1 \times S^3$. A question naturally arises: “Are there critical metrics of $\mathcal{S}$ on $S^1 \times S^3$?" This problem is the creative motivation of this paper.

Remark. 1. In general, if $M^*(n \geq 3)$ is an orientable, noncompact or compact n-dimensional smooth manifold with boundary, we can consider the critical point of the functional with respect to the variation with compact support. Then the Euler-Lagrange equation of the Schouten functional in [13] is still valid for noncompact or compact manifold with boundary (see proposition 3.5). Thus, all the conclusions derived from

Manuscript received September 1, 2022; revised January 4, 2023.

This work was supported by the Education Department of Henan Province Key Foundation under Grant 23B110001.

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Volume 50, Issue 1: March 2023
Euler-Lagrange equation in [13] are also valid for noncompact or compact manifold with boundary. We firstly consider the manifold \( M = I \times S^3 \), where \( I \) is the interval \((0,1)\) or \((0,\infty)\). We write the coordinate function on \( I \) as \( r \), the tangent vector as \( v_i = \frac{\partial}{\partial r} \), and the cotangent vector as \( \theta^i = dr \). Take the tangent frame field on \( S^3 \) according to the method in section 5 of [13], that is, let \( S^3 = \{(x^i, x^j, x^k, x^l) \in R^4 | (x^i)^2 + (x^j)^2 + (x^k)^2 + (x^l)^2 = 1\} \) be the unit 3-sphere in \( R^4 \).

Define vector fields \( v_i, v_j \) and \( v_k \) on \( S^3 \) by
\[
v_i(x^i, x^j, x^k, x^l) = (x^i, -x^j, x^k, x^l),
\]
\[
v_j(x^i, x^j, x^k, x^l) = (x^i, x^j, x^k, x^l),
\]
\[
v_k(x^i, x^j, x^k, x^l) = (x^i, -x^j, -x^k, x^l),
\]
which constitute a basis for tangent vector fields on \( S^3 \). We denote the dual coframe field of \( \{v_i, v_j, v_k\} \) as \( \{\theta^i, \theta^j, \theta^k\} \).

Define a family of Riemannian metrics
\[
g_{ij} = dr^2 + f^i(r)(\theta^j)^2 + (\theta^j)^2 + r^i(\theta^j)^2
\]
on \( M = I \times S^3 \), where \( f(r) > 0 \) is the parameter and satisfies \( f(0) = 0, f'(0) = 1 \). Then \( \{v_i, v_j, v_k\} \) constitutes the standard, orthogonal frame field on \( S^3 \), and its dual coframe field is \( \{\theta^i, \theta^j, \theta^k\} \).

Remark 1.2. Based on the standard metric, the Riemannian metric \( g \) in the form of \( (1.5) \) is constructed by introducing parameters in order to find the critical metrics of the Schouten functional on \( I \times S^3 \).

Now let's describe the main results of this paper.

Theorem 1. For \( t \neq 0 \), \( g \) is not the nontrivial, critical metric of the Schouten functional on \( I \times S^3 \).

Theorem 2. There is no metric which likes \( g \), where \( f(r) \) is a periodic function on \( S^3 \), making it a nontrivial, critical metric of the Schouten functional on \( I \times S^3 \).

Here, the critical metric is called nontrivial, which means that it is neither locally conformally flat nor Einstein.

Remark 1.3. According to the inference 3.1 in [13], every Einstein metric is the critical point of the Schouten functional on a smooth n-dimensional manifold, so every Einstein metric is the critical metric of \( g \) on \( S^3 \times S^3 \).

The structure of this paper is as follows. In section 2, we firstly review the knowledge of Riemannian manifolds by using the moving frame method, and then give a fact. In section 3, we give some lemmas and propositions, as well as the proofs of theorem 1 and 2. In section 4, we summarize the research contents of this paper.

II. PRELIMINARIES

Let \( M^n \) denote an n-dimensional smooth manifold and \((M^n, g)\) be an n-dimensional Riemannian manifold with metric \( g \). Let \( \{e_i, \cdots, e_n\} \) be a local frame field of the tangent bundle of \( M^n \), and \( \{\omega^i, \cdots, \omega^n\} \) its dual coframe field. Throughout the paper we use the standard local notation and adopt the Einstein summation convention. For example, we write \( g = g^i \omega_i \otimes \omega^j \) for the local expression of \( g \) and set \( (g^i) = (g^j) \). We use \( g^i \) and \( g^j \) to raise and lower indices as usual.

Recall that the connection forms \( \{\omega_{ij}^l\} \) of the Levi-Civita connection of \((M^n, g)\) satisfy the structure equations:
\[
d\omega^i = \omega^i \wedge \omega^j + \omega^j \wedge \omega^i = 0, \quad (1.1)
\]
\[
d\omega^i = \omega^i \wedge \omega^j + R^j_{\mu\nu} \omega^\mu \wedge \omega^j, \quad (1.2)
\]
where \( R^i_{\mu\nu} \) are the components of the curvature tensor of \((M^n, g)\). It is well-known that the Riemannian curvature tensor \( R_{\mu\nu} = g_{\mu\lambda} R_{\lambda\nu} \) has an irreducible decomposition given by [1]
\[
R_{\mu\nu} = R_{\mu\nu} + \frac{1}{n-1} (R g_{\mu\nu} - R_{\mu\lambda} g_{\nu} - R_{\nu\lambda} g_{\mu} - R_{\mu\nu} g_{\lambda})
\]
\[
= R_{\mu\nu} + \frac{1}{n-2} (A_{\mu\lambda} g_{\nu} - A_{\nu\lambda} g_{\mu} - A_{\mu\nu} g_{\lambda}),
\]
where \( R_{\mu\nu} \), \( R_{\lambda\nu} \) and \( R_{\mu\nu} \) are the components of the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of \( g \), respectively, while \( A_{\mu\nu} \) are the components of the Schouten tensor \( A \). It is well-known that the Weyl conformal curvature tensor in the decomposition (2.3) yields the conformally invariant part of the Riemannian curvature tensor, while the Schouten tensor describes the curvature property that is not conformally invariant.

The Ricci tensor \( R_{ij} \) and the scalar curvature \( R \) of \((M^n, g)\) are defined by
\[
R_{ij} = \sum_{k,l} g^{kl} R_{ijkl}, \quad R = \sum_{ijkl} g^{ij} R_{ijkl}.
\]

For an arbitrary tensor, such as \( T_{ijkl} \), we define its norm as
\[
|T| = T_{ijkl} T_{ijkl} g^{im} g^{jn} g^{kl} g^{km}.
\]

Denote by \( V \) and \( A \) the covariant differentiation operator of the Levi-Civita connection and the Laplacian on \((M^n, g)\), respectively. For local expressions of the covariant derivatives we use the notation
\[
R_{\mu\nu} = V_k R_{\mu\nu}, \quad R_{\mu\lambda} = V_k V_{\lambda} R_{\mu\nu}, \quad A_{\mu\nu} = V_k R_{\nu\lambda} = g_{\mu\lambda} R_{\nu\lambda} = A_{\mu\nu} g_{\lambda}, \quad A_{\mu\nu} = A_{\mu\lambda} g_{\nu} - A_{\nu\lambda} g_{\mu} - A_{\mu\nu}.
\]

The first and the second order covariant derivatives of \( A_{\mu\nu} \) are given by
\[
A_{\mu\nu} = dA_{\mu\nu} - A_{\mu\lambda} g^{\lambda}_{\nu} - A_{\nu\lambda} g^{\lambda}_{\mu} - A_{\mu\nu} g_{\lambda} = \sum_{i,j} T_{ij} A_{\mu\nu} g^{ij},
\]
\[
A_{\mu\nu} = dA_{\mu\nu} - A_{\mu\lambda} g^{\lambda}_{\nu} - A_{\nu\lambda} g^{\lambda}_{\mu} - A_{\mu\nu} g_{\lambda} = \sum_{i,j} T_{ij} A_{\mu\nu} g^{ij}.
\]

An n-dimensional Riemannian manifold \((M^n, g)\) is called locally conformally flat if \( \bar{g} \) is locally conformally equivalent to a flat Riemannian metric, that is, for any \( p \in M \), there is a neighborhood \( U \) of point \( p \) and a smooth function \( \rho \) on \( U \), so that \( \bar{g} = e^{2\rho} g \) is a flat Riemannian metric. The following proposition is well-known.

Proposition 2.1. For a Riemannian manifold \((M^n, g)\), if \( n \geq 4 \), then \((M^n, g)\) is locally conformally flat if and only if \( W_{ijkl} = 0 \).
III. PROOF OF MAIN RESULTS

For manifold \( I \times S^3 \), we select
\[
e_0 = \frac{\partial}{\partial r}, e_i = f^{-1} v_i, e_2 = f^{-1} v_2, e_3 = f^{-1} \frac{1}{l} v_1.
\]

Then \( \{e_i\}_{i=0}^3 \) is the integral, standard, orthogonal frame field on \( (I \times S^3, g_r) \), whose dual coframe field \( \{\omega^i, \omega^0, \omega^\xi, \omega^\zeta\} \) is given by
\[
\omega^0 = dr, \\
\omega^i = f \xi^i, \\
\omega^\xi = f \xi^\xi, \\
\omega^\zeta = f \xi^\zeta.
\]

Lemma 3.1. If we define a family of Riemannian metrics 
\( g_i = dr^2 + f^2(r)(\theta^2) + (\phi^2) + \gamma^2(\phi^2), \quad t > 0 \)
on \( I \times S^3 \), then the connection form of the Riemannian connection of \((I \times S^3, g_t)\) under the above standard, orthogonal frame field \( \{e_i\}_{i=0}^3 \) can be given by
\[
\omega^0 = f^{-1} \omega^\vartheta, \\
\omega^i = f^{-1} \omega^\phi, \\
\omega^\xi = f^{-1} \omega^\phi, \\
\omega^\zeta = f^{-1} \omega^\phi. 
\]

Proof. Under the standard, orthogonal frame \( \{e_i\}_{i=0}^3 \), the connection form \( \{\omega^i\} \) of the Riemannian connection of \((I \times S^3, g_t)\) satisfies the following structural equation
\[
d\omega = \omega^0 \wedge \omega^i, \\
\omega^0 + \omega^i = 0.
\]
and is uniquely determined by it. Next, we just need to verify that (3.5) satisfies (3.6).

Thus, from (3.1)-(3.4) and (3.7), we obtain
\[
d\omega^0 = 0, \\
d\omega^i = f^1 dr \wedge \xi^i + f d \xi^i = f^1 dr \wedge \xi^i + 2 f^{-1} \xi^2 \wedge \xi^3, \\
d\omega^\xi = f^1 dr \wedge \xi^\xi + f d \xi^\xi = f^1 dr \wedge \xi^\xi + 2 f^{-1} \xi^2 \wedge \xi^3, \\
d\omega^\zeta = f^1 dr \wedge \xi^\zeta + f d \xi^\zeta = f^1 dr \wedge \xi^\zeta + 2 f^{-1} \xi^2 \wedge \xi^3, \\
\omega^0 = \omega^0 \wedge \omega^i + \omega^i \wedge \omega^0 + \omega^\phi \wedge \omega^0 + \omega^0 \wedge \omega^\phi = 0, \\
\omega^i = \omega^i \wedge \omega^0 + \omega^0 \wedge \omega^i + \omega^\phi \wedge \omega^i + \omega^i \wedge \omega^\phi = 0, \\
\omega^\xi = \omega^\xi \wedge \omega^0 + \omega^0 \wedge \omega^\xi + \omega^\phi \wedge \omega^\xi + \omega^\xi \wedge \omega^\phi = 0, \\
\omega^\zeta = \omega^\zeta \wedge \omega^0 + \omega^0 \wedge \omega^\zeta + \omega^\phi \wedge \omega^\zeta + \omega^\zeta \wedge \omega^\phi = 0, \\
\omega^0 = \omega^0 \wedge \omega^i + \omega^i \wedge \omega^0 + \omega^\phi \wedge \omega^i + \omega^i \wedge \omega^\phi = 0, \\
\omega^\xi = \omega^\xi \wedge \omega^i + \omega^i \wedge \omega^\xi + \omega^\phi \wedge \omega^\xi + \omega^\xi \wedge \omega^\phi = 0, \\
\omega^\zeta = \omega^\zeta \wedge \omega^i + \omega^i \wedge \omega^\zeta + \omega^\phi \wedge \omega^\zeta + \omega^\zeta \wedge \omega^\phi = 0.
\]

Thus, from (3.8) and (3.9), it is known that (3.5) satisfies (3.6).

Proposition 3.1. If we define a family of Riemannian metrics 
\( g_i = dr^2 + f^2(r)(\theta^2) + (\phi^2) + \gamma^2(\phi^2), \quad t > 0 \)
on \( I \times S^3 \), then the Ricci curvature tensor \( R_{ijkl} \) of \((I \times S^3, g_t)\) under the above standard, orthogonal frame field \( \{e_i\}_{i=0}^3 \) can be given by
\[
R_{0111} = -f^{-1} f^1 (\delta_{\theta\phi} \delta_{\phi\phi} - \delta_{\phi\phi} \delta_{\theta\phi}), \\
R_{0221} = -f^{-1} f^1 (\delta_{\theta\phi} \delta_{\phi\phi} - \delta_{\phi\phi} \delta_{\theta\phi}), \\
R_{2121} = -f^{-1} f^1 (\delta_{\theta\phi} \delta_{\phi\phi} - \delta_{\phi\phi} \delta_{\theta\phi}), \\
R_{2221} = [f^{-1}(4-3\gamma^2) - f^{-2} \gamma^2] (\delta_{\theta\phi} \delta_{\phi\phi} - \delta_{\phi\phi} \delta_{\theta\phi}).
\]

Proof. From (3.5), we know \( \omega^0 = f^{-1} f^i \omega^i \), therefore
\[
d\omega^0 = d(f^{-1} f^i \wedge \omega^i) + f^{-1} f d \omega^i = (f^{-1} f^i \wedge f^j \wedge \omega^i \wedge \omega^j - 2 f^{-1} f^{-1} (f^i \wedge f^j \wedge \omega^i \wedge \omega^j)) \]
\[
= -2 f^{-1} f^{-1} (f^i \wedge f^j \wedge \omega^i \wedge \omega^j).
\]

On the other hand, it is obtained from the structural equations (2.2) and (3.5)
\[
d\omega^0 = \omega^0 \wedge \omega^i, \\
\omega^i + \omega^0 = 0,
\]
and
\[
\omega^0 = -f^{-1} f^i \omega^i \wedge \omega^j.
\]
Therefore
\[
R_{ijkl} = -f^{-1} f^i \delta_{\theta\phi} \delta_{\phi\phi} - \delta_{\phi\phi} \delta_{\theta\phi}, \\
0 \leq k, l \leq 3.
\]

Similarly, we can obtain other Riemannian curvature tensor \( R_{ijkl} \) under the standard, orthogonal frame field \( \{e_i\}_{i=0}^3 \).

Proposition 3.2. If we define a family of Riemannian metrics 
\( g_i = dr^2 + f^2(r)(\theta^2) + (\phi^2) + \gamma^2(\phi^2), \quad t > 0 \)
on \( I \times S^3 \), then the Ricci curvature tensor \( R_{ijkl} \) of \((I \times S^3, g_t)\) under the above standard, orthogonal frame field \( \{e_i\}_{i=0}^3 \) can be given by
\[
R_{0001} = -f^{-1} f^i \omega^i, \\
R_{0011} = -f^{-1} f^i \omega^i, \\
R_{1111} = -f^{-1} f^i \omega^i, \\
R_{2222} = -f^{-1} f^i \omega^i, \\
R_{3333} = -f^{-1} f^i \omega^i, \\
R_{ijkj} = 0, \\
0 \leq k, l \leq 3,
\]

where \( 0 \leq i, j \leq 3 \).

From proposition 3.2, we can easily get the scalar curvature \( R \) of \((I \times S^3, g_t)\) as follows
\[
R = 6 f^{-1} f^i + \gamma^2 f^{-2} f^{-2} f^{-2} + 2 f^{-2} (4 - \gamma^2).
\]

Let's consider the Weyl curvature tensor of \((I \times S^3, g_t)\).

From (3.3), \((I \times S^3, g_t)\), we have
\[
W_{ijkl} = R_{ijkl} - \frac{1}{2} (R_{ij} \delta_{kl} - R_{il} \delta_{jk} + R_{kl} \delta_{ij} - R_{jk} \delta_{il}),
\]

By simple calculation, we can get
\[
W_{ijkj} = \frac{1}{2} (R_{ij} \delta_{kl} - R_{il} \delta_{jk} + R_{kl} \delta_{ij} - R_{jk} \delta_{il}),
\]

\[
W_{ijkl} = -\frac{1}{2} f^{-1} f^{-1} (f^i \wedge f^j \wedge \omega^i \wedge \omega^j).
\]
The first and the second order covariant derivatives of $E_6$ of $(I \times S^3, g_6)$ are given by

$$
E_{00} = E_{02} = -2f^2 f'(f'' f'' f' - f'' f' f) - 2f^2 f'(2 - t'),
E_{01} = E_{03} = -2f^2 f'(f' f'' f' - f' f'' f) - 2f^2 f',
E_{04}, E_{05}, E_{06}, E_{07} = E_{08} = E_{09} = 0,
$$

where $0 \leq k, l \leq 3$.

From proposition 2.1 and (3.16), we immediately obtain the following proposition.

**Proposition 3.1.** If $(I \times S^3, g_6)$ is locally conformally flat if and only if $t' = 1$.

**Remark 3.1.** If $(I \times S^3, g_6)$ is locally conformally flat and $f(r)$ satisfies $f'' + (f')^2 - 1 = 0$, then the scalar curvature $R_{(f)} = 0$.

According to theorem 4.1 in [13], at this time, the locally conformally flat metric $g_6$ is the critical point of the Schouten functional $S^f$.

Under the standard, orthogonally frame field, $E_6 = R_6 - \frac{1}{2} R_6 g_6$, so it's easy to get from propositions 3.2 and (3.14)

$$
E_{00} = -\frac{1}{2} \left( f^2 f' - f^2 f' ight) - \frac{1}{2} f^2 (4 - t'),
E_{11} = \frac{1}{2} \left( f^2 f' - f^2 f' ight) + \frac{1}{2} f^2 (4 - 3t'),
E_{22} = \frac{1}{2} \left( f^2 f' - f^2 f' ight) + \frac{1}{2} f^2 (4 - 3t'),
E_{33} = \frac{1}{2} \left( f^2 f' - f^2 f' ight) - \frac{1}{2} f^2 (4 - 5t'),
E_{ij} = 0 \quad \text{if} \quad i \neq j, \quad \text{and for} \quad 0 \leq i, j \leq 3.
$$

where $E_6$ denotes the components of the trace-free Ricci tensor $E_6$ of $(I \times S^3, g_6)$.

Therefore we can obtain the following Proposition.

**Proposition 3.2.** $(I \times S^3, g_6)$ is Einstein manifold if and only if $t' = 1$, and $f(r)$ satisfies the equation $f'' + (f')^2 + 1 = 0$.

**Proof.** $(I \times S^3, g_6)$ is Einstein manifold if and only if the trace-free Ricci tensor $E_6$ satisfies $E_{06} = 0$, i.e. $E_{06} = 0$ ($0 \leq i, j \leq 3$).

Hence, we obtain

$$
E_{00} = -\frac{1}{2} \left( f^2 f' - f^2 f' ight) - \frac{1}{2} f^2 (4 - t') = 0,
E_{11} = \frac{1}{2} \left( f^2 f' - f^2 f' ight) + \frac{1}{2} f^2 (4 - 3t') = 0,
E_{22} = \frac{1}{2} \left( f^2 f' - f^2 f' ight) + \frac{1}{2} f^2 (4 - 3t') = 0,
E_{33} = \frac{1}{2} \left( f^2 f' - f^2 f' ight) - \frac{1}{2} f^2 (4 - 5t') = 0.
$$

By solving the above equations, we can get $t' = 1$, and $f(r)$ satisfies the equation

$$
f'' = (f')^2 + 1 = 0.
$$

The first and the second order covariant derivatives of $E_6$ are defined by

$$
E_{ij} \omega^k = \epsilon_{ijk} E_{kl},
E_{ij} \theta^k = \epsilon_{jkl} E_{ik},
$$

With(3.17)and(3.18), we obtain

$$
E_{ij} \omega^k = (E_{ij} - E_{ij}) \theta^k \quad \text{if} \quad i \neq j,
E_{ij} \theta^k = E_{ij} \theta^k \quad \text{if} \quad i = j.
$$

By simple calculation, we get

**Lemma 3.2.** The first covariant derivatives of $E_6$ of $(I \times S^3, g_6)$ are given by
$E_{3,30} = E_{3,31} = E_{13,32} = E_{13,30} = E_{13,32} = E_{13,33} = 0,$

$E_{3,31} = E_{3,32} = -12 f^{-1}(f')^2(1-r^2),$

$E_{23,33} = -2 f^{-3}(f')^2(f^{-3} - f^{-2} f') - 2 f^{-4} f'(2-r^2),$

$E_{23,32} = -2 f^{-3}(f')^2(f^{-3} - f^{-2} f') - 4 f^{-4} r^2(1-r^2),$

$E_{23,00} = E_{23,01} = E_{23,02} = E_{23,03} = E_{23,12} = E_{23,13} = 0,$

$E_{23,20} = E_{23,21} = E_{23,22} = E_{23,30} = E_{23,31} = 0,$

$E_{00,30} = -\frac{1}{12} f^{-4}(f')^4 f^{*} - 3 f^{-3}(f')^2 - 4 f^{-2} f'' + f^{-1} f'(4 - 4 r^2),$

$E_{00,31} = E_{00,32} = \frac{1}{2} f^{-1} f'(17 f^{-2} f'' - 14 f^{-3}(f')^3 - 3 f^{-1} f'' f') + f^{-1} f'' f'(2 - r^2),$

$E_{00,33} = \frac{1}{2} f^{-1} f'(17 f^{-2} f'' - 14 f^{-3}(f')^3 - 3 f^{-1} f'' f') + f^{-1} f'' f'(4 - 4 r^2),$

$E_{00,01} = E_{00,02} = E_{00,03} = E_{00,30} = E_{00,31} = 0,$

$E_{00,10} = E_{00,12} = E_{00,13} = E_{00,21} = E_{00,22} = 0,$

$E_{11,00} = \frac{1}{12} f^{-3}(f')^2 f^{*} - 3 f^{-2}(f')^2 - 4 f^{-2} f'' + f^{-1} f'(4 - 4 r^2),$

$E_{11,11} = \frac{1}{2} f^{-1} f'(4 - 12 r^2),$

$E_{11,22} = \frac{1}{12} f^{-3}(f')^2 f^{*} - 3 f^{-2}(f')^2 - 2 f^{-1} f'' f' + f^{-1} f'' f'(4 - 4 r^2),$

$E_{11,33} = \frac{1}{2} f^{-1} f'(4 - 4 r^2),$

$E_{11,10} = E_{11,12} = E_{11,20} = E_{11,22} = E_{11,30} = E_{11,32} = 0,$

$E_{11,20} = E_{11,22} = E_{11,30} = E_{11,32} = 0,$

$E_{22,00} = \frac{1}{12} f^{-3}(f')^2 f^{*} - 3 f^{-2}(f')^2 - 4 f^{-2} f'' + f^{-1} f'(4 - 4 r^2),$

$E_{22,11} = \frac{1}{2} f^{-1} f'(4 - 12 r^2),$

$E_{22,22} = \frac{1}{2} f^{-1} f'(4 - 4 r^2),$

$E_{22,33} = \frac{1}{2} f^{-1} f'(4 - 4 r^2),$

$E_{22,01} = E_{22,02} = E_{22,03} = E_{22,10} = E_{22,12} = E_{22,13} = 0,$

$E_{22,20} = E_{22,21} = E_{22,22} = E_{22,30} = E_{22,31} = E_{22,32} = 0,$

$E_{33,00} = \frac{1}{12} f^{-3}(f')^2 f^{*} - 3 f^{-2}(f')^2 - 4 f^{-2} f'' + f^{-1} f'(4 - 4 r^2),$

$E_{33,11} = \frac{1}{2} f^{-1} f'(4 - 4 r^2),$

$E_{33,22} = \frac{1}{2} f^{-1} f'(4 - 4 r^2),$

$E_{33,01} = E_{33,02} = E_{33,03} = E_{33,30} = E_{33,31} = 0,$

$E_{33,20} = E_{33,21} = E_{33,30} = E_{33,31} = E_{33,32} = 0.$

Lemma 3.3 immediately implies

Lemma 3.4. For $(I \times S^3, g)$, we have

$\Delta_g E_{00} = \frac{1}{2} [5 f^{-3}(f')^2 f''^* - 8 f^{-4}(f')^4 + f^{-2} f''(f')^2 + 3 f^{-3}(f')^4 - f^{-1} f''(f')^2] + [4 f^{-4}(f')^2 + f^{-3} f''(4 - r^2),$

$\Delta_g E_{11} = \Delta_g E_{22} = -\frac{1}{2} [5 f^{-3}(f')^2 f''^* - 8 f^{-4}(f')^4 + f^{-2} f''(f')^2 + 3 f^{-3}(f')^4 - f^{-1} f''(f')^2] + 8 f^{-4} r^2(1 - r^2)$

$- 4 f^{-4}(f')^2(2 - r^2) - f^{-3} f''(4 - 3 r^2),$

$\Delta_g E_{33} = -\frac{1}{2} [5 f^{-3}(f')^2 f''^* - 8 f^{-4}(f')^4 + f^{-2} f''(f')^2 + 3 f^{-3}(f')^4 - f^{-1} f''(f')^2] + 16 f^{-4} r^2(1 - r^2)$

$- 4 f^{-4}(f')^2 r^2 + f^{-3} f''(4 - 5 r^2),$

$\Delta_g E_{01} = \Delta_g E_{02} = \Delta_g E_{03} = \Delta_g E_{12} = \Delta_g E_{13} = \Delta_g E_{23} = 0.$

(3.20)

Proposition 3.5. Let $M^n (n \geq 3)$ be a smooth $n$-manifold. Then a metric $g \in \mathcal{M}(M)$ is a critical point of the Schouten functional $\mathcal{S}^*$ if and only if it satisfies the following equations $(1 \leq i, j, k, l \leq n)$

$\Delta_g E_{ij} = \frac{(n - 2)(2n - 3)}{2(n - 1)^2} R_{ij} - \frac{(n - 2)(2n - 3)}{2(n - 1)^2} \Delta_g R_{ij}$

$+ 2 \sum_{k \neq i, j} \mathcal{E}_{ik} W_{kj} - 4 \sum_{i = 1}^{n} \mathcal{E}_{ik} E_{ij} + \frac{n^2 - 8n + 8}{2(n - 1)^2} \mathcal{R}_{ij} + \frac{4}{n(n - 2)} \mathcal{E}_{ij}^2 \nabla g_{ij} = 0,$

where $1 \leq i, j, k, l \leq n$, $n \leq 4,$

$\frac{(n - 2)^2}{n - 1} R_{ij} - \frac{(n - 2)^2(n - 4)}{4(n - 1)^2} R_{ij}$

$+ (n - 4) \mathcal{V}(M^n, g) \mathcal{S}^* [g] = 0.$

Proof. In [13], the authors give a detailed proof when the manifold is compact, boundless. If the manifold is noncompact or compact with boundary, the proof is similar to that of compact without boundary. At this time, we need to consider the critical point of the functional with respect to the variation with compact support set. Following the notation in [13], when the manifold is noncompact, we need to consider the case that the variation function $h_\delta$ has a compact support set. When the manifold is compact with boundary, we must require the variation function $h_\delta|_{\partial M} = 0$.

Here we will not give a proof. Please refer to proposition 3.1 and theorem 3.1 in [13].

Thus, under the standard, orthogonal frame field, we get

Corollary 3.1. Let $M^n$ be a smooth $n$-manifold. Then a metric $g \in \mathcal{M}(M)$ is a critical point of $\mathcal{S}^*$ if and only if it satisfies the following equations

$\Delta_g E_{ij} = \frac{1}{2} R_{ij} + \frac{1}{2} \Delta_g R_{ij} + 2 \sum_{k \neq i, j} \mathcal{E}_{ik} W_{kj} - 2 \sum_{i = 1}^{n} \mathcal{E}_{ik} E_{ij}$

$- \frac{1}{2} \Delta_g R_{ij},$

(3.21)

where $0 \leq i, j, k, l \leq 3,$

$\Delta_g R = 0.$

(3.22)

The following corollary is immediately obtained from (3.22).

Corollary 3.2. Let $M^4$ be a smooth $4$-manifold. If $g \in \mathcal{M}(M)$ is a critical point of $\mathcal{S}^*$, then the scalar curvature $R$ of $\mathcal{S}^*$ is constant.

Substituting (3.22) into (3.21), yields
\[ \Delta_{ij}E_{ij} + 2 \sum_{k} E_{ik}W_{kij} - 2 \sum_{k} E_{kj}W_{ik} - \frac{1}{2} RE_{ij} + \frac{1}{2} |E|^2 \delta_{ij} = 0. \]  
(3.23)

### 3.1 Proof of Theorem 1

**Proof.** For \((I \times S^3, g)\), under the above standard, orthogonal frame field \(\{e_i\}_{i=0}^3\), when \(i \neq j\), (3.23) can be written as

\[ \Delta_{ij}E_{ij} + 2 \sum_{k} E_{ik}W_{kij} - 2 \sum_{k} E_{kj}W_{ik} - \frac{1}{2} RE_{ij} + \frac{1}{2} |E|^2 \delta_{ij} = 0. \]  
(3.24)

When \(i \neq j\), (3.23) can be written as

\[ \sum_{k} E_{ik}W_{kij} = 0. \]  
(3.25)

From (3.16) and (3.17), when \(i \neq j\), (3.25) holds.

Now we only need to solve (3.24). From (3.16) and (3.17), we have

\[ \sum_{k} E_{ik}W_{kij} = -\frac{1}{2} f^4 (1-r^2)^2, \]
\[ \sum_{k} E_{ik}W_{kij} = \frac{1}{2} f^2 (1-r^2)(f^{-1} f^* - f^{-2} f^2)^2 (4 - 4r^2), \]
\[ \sum_{k} E_{ik}W_{kij} = \frac{1}{2} f^4 (4 - 4r^2)^2, \]
\[ \sum_{k} E_{ik}W_{kij} = \frac{1}{2} f^4 (4 - 4r^2)^2, \]
\[ \sum_{k} E_{ik}W_{kij} = \frac{1}{2} f^4 (4 - 4r^2)^2, \]
\[ \sum_{k} E_{ik}W_{kij} = \frac{1}{2} f^4 (4 - 4r^2)^2. \]  
(3.26)

and

\[ E_{ii} = \frac{1}{2} (f^{-1} f^* - f^{-2} f^2)^2 + \frac{1}{2} f^2 (f^{-1} f^* - f^{-2} f^2)^2 (4 - 4r^2), \]
\[ E_{ii} = \frac{1}{2} f^4 (4 - 4r^2)^2, \]
\[ E_{ii} = \frac{1}{2} f^4 (4 - 4r^2)^2, \]
\[ E_{ii} = \frac{1}{2} f^4 (4 - 4r^2)^2, \]
\[ E_{ii} = \frac{1}{2} f^4 (4 - 4r^2)^2. \]  
(3.27)

Therefore we obtain

\[ |E|^2 = 3 (f^{-1} f^* - f^{-2} f^2)^2 + 2 f^2 (f^{-1} f^* - f^{-2} f^2)^2 (4 - 4r^2)^2 (16 - 24r^2 + 11r^4). \]  
(3.28)

Substituting (3.20), (3.26), (3.27), (3.28), (3.14) and (3.17) into (3.24), yields the following four equations

\[ 9 f^4 f^{-4} - 9 f^{-2} f^* - 81 f^{-3} (f')^2 f^* + 84 f^4 (f')^4 = -3 f^{-2} (f')^2 + 6 f^3 f^* (4 - r^2) - 32 f^{-4} (f')^2 (4 - r^2), \]
\[ + 4 f^{-4} (40 - 56r^2 + 25r^4) = 0. \]  
(3.29)

\[ 9 f^4 f^{-4} - 9 f^{-2} f^* - 81 f^{-3} (f')^2 f^* + 84 f^4 (f')^4 = -3 f^{-2} (f')^2 + 2 f^3 f^* (4 - 13r^2) - 32 f^{-4} (f')^2 (5 - 2r^2) \]
\[ + 4 f^{-4} (40 - 112r^2 + 75r^4) = 0. \]  
(3.30)

\[ 9 f^4 f^{-4} - 9 f^{-2} f^* - 81 f^{-3} (f')^2 f^* + 84 f^4 (f')^4 = -3 f^{-2} (f')^2 - 2 f^3 f^* (4 - 13r^2) - 32 f^{-4} (f')^2 (5 - 2r^2) \]
\[ + 4 f^{-4} (40 - 112r^2 + 75r^4) = 0. \]  
(3.31)

\[ 9 f^4 f^{-4} - 9 f^{-2} f^* - 81 f^{-3} (f')^2 f^* + 84 f^4 (f')^4 = -3 f^{-2} (f')^2 + 2 f^3 f^* (44 - 35r^2) - 32 f^{-4} (f')^2 (2 + r^2) \]
\[ - 4 f^{-4} (40 - 168r^2 + 125r^4) = 0. \]  
(3.32)

We first prove this conclusion: for \(\forall t > 0\), there is no metric \(g\), with \(f(t)\) as a constant, so that it is a nontrivial, critical metric of \(\mathcal{S}[g] \) on \(I \times S^3\). We prove it by means of reduction to absurdity. Assuming that \(f(t)\) is a constant, the equations are obtained from (3.33)

\[ 40 - 56r^2 + 25r^4 = 0, \]
\[ 40 - 112r^2 + 75r^4 = 0, \]
\[ 40 - 168r^2 + 125r^4 = 0. \]  
(3.33)

By solving the above equations, we know that for \(\forall t > 0\), the equations have no solution. The proof is completed.

Next, we solve the equations (3.33) with assuming that \(f(t)\) is not a constant.

In the equations (3.33), the first equation minus the second equation, and the third equation minus the second equation, we get

\[ 32 f^{-3} f^* (1 - r^2) + 32 f^{-4} (f')^2 (1 - r^2) \]
\[ - \frac{32}{3} f^{-3} f^* (1 - r^2) = 0, \]  
(3.34)

\[ 96 f^{-3} f^* (1 - r^2) + 96 f^{-4} (f')^2 (1 - r^2) \]
\[ - 160 f^{-4} (2 - 5r^2) (1 - r^2) = 0, \]

Obviously, the two equations in (3.34) are actually the same equation. After simplification, we get

\[ 3 f^{-3} f^* (1 - r^2) + 3 f^{-4} (f')^2 (1 - r^2) - 5 f^{-4} (2 - 5r^2) (1 - r^2) = 0. \]

Now we only consider the case of \(t \neq 1\). So the above equation can be written as

\[ f f^* + (f')^2 \frac{f}{3} (2 - 5r^2) = 0. \]  
(3.35)

Next, we use the reduction method to solve this equation. Let

\[ y = \frac{df}{dr} = f', \]

then

\[ f' = \frac{dy}{dr} = \frac{dy}{df} \frac{df}{dr} = \frac{y}{f}. \]  
(3.36)

Substituting the above equation into (3.35), we obtain a variable separable equation

\[ y dy = \left[ \frac{1}{3} (2 - 5r^2) - y^2 \right] df. \]  
(3.37)

Due to \(f(r) > 0\), when \(\frac{1}{3} (2 - 5r^2) - y^2 \neq 0\), the equation (3.37) can be written as the following equivalent equation

\[ y dy = \frac{df}{f}. \]  
(3.38)
Integrating the above equation to get
\[ \ln f + \frac{1}{4} \ln |\frac{1}{2} (2 - 5t^2) - y^2| = C_1, \]
where \( C_1 \) is an arbitrary constant.
Therefore
\[ y^2 = \frac{1}{16} (2 - 5t^2) + C_2 f^{-2}, \]
where \( C_2 = \pm e^{2C_1} \neq 0 \). In addition, the special solution
\[ y^2 = \frac{1}{16} (2 - 5t^2) \]
is obtained from \( y^2 = \frac{1}{16} (2 - 5t^2) - y^2 = 0 \).
Therefore, the general solution of equation (3.36) is
\[ y^2 = \frac{1}{16} (2 - 5t^2) + C f^{-2}, \]
(3.37)
where \( C \) is an arbitrary constant.

The equation (3.37) is actually a first-order differential equation for a fixed \( C \), i.e.
\[ \frac{df}{dr} = \pm \frac{1}{\sqrt{16} (2 - 5t^2) + C f^{-2}}. \]
(3.38)
The equation (3.38) is also a variable separable equation.
\[ \frac{df}{dr} = \pm \sqrt{C} dr. \]
By solving this equation, we obtain
\[ f = \sqrt{\pm 2 \sqrt{C} r + 2C_1}, \]
(3.39)
where \( C_1 \) is an arbitrary constant.
When \( t = \sqrt{\frac{5}{2}} \), substituting (3.39) into equations (3.33), yields
\[ 4 f^2 - 19 C = 0, \]
i.e.
\[ 4 (\pm 2 \sqrt{C} r + 2C_1) - 19 C = 0. \]
Obviously, this equation does not hold. Thus, we can get that when \( t = \sqrt{\frac{5}{2}} \), \( g \), is not a nontrivial, critical metric of \( \mathcal{S}' \).
\[ \frac{1}{4 \sqrt{2} (2 - 5t^2) \neq 0}, \]
the general solution of the equation (3.38) is
\[ f = \sqrt{\frac{1}{16} (2 - 5t^2) (r + C_4)^2 - \frac{C}{\frac{1}{4} (2 - 5t^2)}}, \]
(3.40)
where \( C_4 \) is an arbitrary constant.
Substituting (3.40) into (3.33), yields
\[ 4 f^{-2} \left( \frac{1}{2} (2 - 5t^2) (r + C_4)^2 (35 - 223 r^2 + 350 r^4) \right) + \frac{C}{\frac{1}{4} (2 - 5t^2)} \left( 50 - 343 r^2 + 500 r^4 \right) = 0. \]
Owing to the arbitrariness of \( C \) and \( C_4 \), we can get
\[ \begin{align*}
35 - 223 r^2 + 350 r^4 & = 0, \\
50 - 343 r^2 + 500 r^4 & = 0.
\end{align*} \]
Obviously, this equation also does not hold. Thus, we can get that for \( \forall t > 0 \) and \( t \neq \frac{5}{\sqrt{2}} \), \( g \), is not a nontrivial, critical metric of \( \mathcal{S}' \) on \( I \times S^3 \).

To sum up, for \( \forall t > 0 \), \( g \), is not a nontrivial, critical metric of \( \mathcal{S}' \) on \( I \times S^3 \).
\[ \blacksquare \]

**Remark 3.2.** In theorem 1, the existence problem of nontrivial, critical metric \( g_0 \) of the Schouten functional on \( I \times S^3 \) is transformed into the existence problem of solutions of differential equations (3.33). Now we will apply the conclusion of theorem 1 to \( S^1 \times S^3 \).

### 3.2 Proof of Theorem 2

**Proof.** In the proof of theorem 1, we can easily see that there is no periodic function \( f(r) \) and parameter \( t \), so that \( g = g_0 \) is a nontrivial, critical metric of \( \mathcal{S} \) on \( I \times S^3 \). So we have the same conclusion on \( S^1 \times S^3 \).

**Remark 3.3.** In theorem 2, unfortunately, the Riemannian metric we defined on the manifold, although there is a critical metric of the Schouten functional, there is no nontrivial critical metric. In the subsequent research process, we can further explore the nontrivial critical metric of the Schouten functional by modifying the parameters of \( g \).

### IV. Conclusion

In this paper, we define the Riemannian metrics of the form \( g_0 \) on \( I \times S^3 \), and then transform the existence problem of the critical metric of the Schouten functional into the existence problem of solutions of the differential equation. This paper provides an idea and method to define and find the critical metric of the Schouten functional on a 4-manifold. Moreover, the method for finding critical metric given in this paper can be extended to other 4-manifolds, which is also our future research work.

### REFERENCES