Critical Metrics of the Schouten Functional on a 4-manifold

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Abstract—**The existence of critical metrics of the Schouten functional on $S^1\times S^3$ is studied in this paper. We define a family of Riemannian metrics $\{g_t\}_{t\in R^+}$ on $I\times S^3$, where I is the interval $(0,\ l)$ or $(0,\ +\infty)$. By investigating these Riemannian metrics' properties about the Schouten functional on $I\times S^3$, we obtain critical metrics' results of the Schouten functional on $S^1\times S^3$.**

Index Terms—**Schouten tensor, Riemannian functional, critical metric, flat metric, 4-manifold**

I. Introduction

Let $M^n(n\geq 3)$ be an n-dimensional compact, orientable, connected smooth manifold. We denote by $\mathcal{M}(M)$ and $\mathcal{D}(M)$ the space of smooth Riemannian metrics on M^n and the group of diffeomorphisms of M^n, respectively. A functional $\mathcal{F}:\mathcal{M}(M)\to R$ is called Riemannian if $\mathcal{F}$ is invariant under the action of $\mathcal{D}(M)$, that is, $\mathcal{F}(\varphi^*g)=\mathcal{F}(g)$ for each $\varphi\in\mathcal{D}(M)$ and $g\in\mathcal{M}(M)$.

A very typical example of Riemannian functionals is the Einstein-Hilbert functional defined by

$$\mathcal{H}[g]:=\left(\int_{M^n}dvol_g\right)^{(2-n)/n}\int_{M^n}R_g dvol_g, \tag{1.1}$$

where R_g is the scalar curvature of g. Hilbert pointed out that the critical points of this functional are Einstein metrics [1]. The critical points of this functional restricted to the space of metrics with constant scalar curvature of unitary volume was studied, the CPE metric was simplified. A necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein was obtained [2]. Baltazar et al. studied weakly Einstein critical metrics of the volume functional on a compact manifold with smooth boundary, and gave a complete classification for an n-dimensional weakly Einstein critical metrics of the volume functional with nonnegative scalar curvature [3].

In addition to Einstein-Hilbert functional, in connection with the generalized σ_k-Yamabe problem, the Riemannian functional defined by

$$\mathcal{F}_k[g]:=\left(\int_{M^n}dvol_g\right)^{(2k-n)/n}\int_{M^n}\sigma_k(A_g)dvol_g, \tag{1.2}$$

where $\sigma_k(A_g)$ denotes the k-th elementary symmetric function of the eigenvalues (with respect to g) of the Schouten tensor defined by

$$A_g:=Ric_g-\frac{R_g}{2(n-1)}g. \tag{1.3}$$

The study of this functional was first initiated by J. viaclovsky[4], and many in-depth results have been obtained [5-8]. It is the main result of [9] that, for compact, boundless, smooth 3-dimensional manifolds, the space forms are the only critical points of $\mathcal{F}_2$ if the critical value is nonnegative. In [10], Hu et al. extended this result to higher dimensions, and proved that, for n-dimensional $(n\geq 5)$, compact, boundless, smooth manifolds, the space forms are the only locally conformally flat critical points of $\mathcal{F}_2$ if the critical value of $\mathcal{F}_2$ is nonnegative. Gursky et al. developed a gluing procedure designed to obtain canonical metrics on connected sums of Einstein four-manifolds. By using certain quotients of $S^2\times S^2$ as one of the gluing factors, critical metrics on several non-simply-connected manifolds were obtained [11]. By using the curvature identity on 4-dimensional Riemannian manifolds, critical metrics for the squared L2-norm functionals of the curvature tensor, the Ricci tensor and the scalar curvature were studied [12].

In addition to the above functionals, we can get other examples of Riemannian functionals by integrating the quadratic polynomial for curvature tensor [1]. Among them, the most interesting Riemannian functional is

$$\mathcal{S}[g]:=\left(\int_{M^n}dvol_g\right)^{(4-n)/n}\int_{M^n}|A_g|^2\,dvol_g. \tag{1.4}$$

In this paper, we also call $\mathcal{S}$ the Schouten functional.

We know

$$|A_g|^2=|Ric_g|^2-\frac{3n-4}{4(n-1)^2}R_g^2=|E_g|^2+\frac{(n-2)^2}{4n(n-1)^2}R_g^2,$$

where $E_g:=Ric_g-(R_g/n)g$ is the trace-free Ricci tensor.

Obviously, $\mathcal{S}[g]=0$ if and only if (M^n,g) is Ricci-flat.

In [13], the authors proved that the infimum of the Schouten functional on $S^1\times S^3$ is zero. However, only the Ricci-flat metric is the metric with zero Schouten functional on the manifold, but there is no Ricci-flat metric on $S^1\times S^3$. A question naturally arises: ''Are there critical metrics of $\mathcal{S}$ on $S^1\times S^3$?'' This problem is the creative motivation of this paper.

Remark1.1. In general, if $M^n(n\geq 3)$ is an orientable, noncompact or compact n-dimensional smooth manifold with boundary, we can consider the critical point of the functional with respect to the variation with compact support. Then the Euler-Lagrange equation of the Schouten functional in [13] is still valid for noncompact or compact manifold with boundary (see proposition 3.5). Thus, all the conclusions derived from

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Euler-Lagrange equation in [13] are also valid for noncompact or compact manifold with boundary.

We firstly consider the manifold $M = I \times S^3$, where I is the interval $(0,l)$ or $(0,+\infty)$. We write the coordinate function on I as r, the tangent vector as $v_0 = \frac{\partial}{\partial r}$, and the cotangent vector as $\theta^0 = dr$ .Take the tangent frame field on S^3 according to the method in section 5 of [13], that is, let

$$S^3 = \{(x^1, x^2, x^3, x^4) \in R^4 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\}$$

be the unit 3-sphere in R^4.

Define vector fields v_1, v_2 and v_3 on S^3 by

$$v_1(x^1, x^2, x^3, x^4) = (x^2, -x^1, x^4, -x^3),$$
$$v_2(x^1, x^2, x^3, x^4) = (x^3, -x^4, -x^1, x^2),$$
$$v_3(x^1, x^2, x^3, x^4) = (x^4, x^3, -x^2, -x^1),$$

which constitute a basis for tangent vector fields on S^3. We denote the dual coframe field of $\{v_1, v_2, v_3\}$ as $\{\theta^1, \theta^2, \theta^3\}$.

Define a family of Riemannian metrics

$$g_t = dr^2 + f^2(r)[(\theta^1)^2 + (\theta^2)^2 + t^2(\theta^3)^2] \qquad (1.5)$$

on $M = I \times S^3$, where $f(r) > 0, t > 0$ is the parameter and satisfies $f(0) = 0, f'(0) = 1$. Then $\{v_1, v_2, \frac{1}{t}v_3\}$ constitutes the standard, orthogonal frame field on S^3, and its dual coframe field is $\{\xi^1, \xi^2, \xi^3\}$.

Remark1.2. Based on the standard metric, the Riemannian metric g_t in the form of (1.5) is constructed by introducing parameters in order to find the critical metrics of the Schouten functional on $I \times S^3$.

Now let's describe the main results of this paper.

Theorem1. For $\forall t > 0$, g_t is not the nontrivial, critical metric of the Schouten functional on $I \times S^3$.

Theorem2. There is no metric which likes g_t (where $f(r)$ is a periodic function) on $S^1 \times S^3$, making it a nontrivial, critical metric of the Schouten functional on $S^1 \times S^3$.

Here, the critical metric is called nontrivial, which means that it is neither locally conformally flat nor Einstein.

Remark1.3. According to the inference 3.1 in [13], every Einstein metric is the critical point of the Schouten functional $\mathcal{S}$ on a smooth n-dimensional manifold, so every Einstein metric is the critical metric of $\mathcal{S}$ on $S^1 \times S^3$.

The structure of this paper is as follows. In section 2, we firstly review the knowledge of Riemannian manifolds by using the moving frame method, and then give a fact. In section 3, we give some lemmas and propositions, as well as the proofs of theorem 1 and 2. In section 4, we summarize the research contents of this paper.

II. Preliminaries

Let M^n denote an n-dimensional smooth manifold and (M^n, g) be an n-dimensional Riemannian manifold with metric g. Let $\{e_1, \cdots, e_n\}$ be a local frame field of the tangent bundle of M^n, and $\{\omega^1, \cdots, \omega^n\}$ its dual coframe field. Throughout the paper we use the standard local notation and adopt the Einstein summation convention. For example, we write $g = g_{ij}\omega^i \otimes \omega^j$ for the local expression of g and set $(g^{ij}) = (g_{ij})^{-1}$. We use g^{ij} and g_{ij} to raise and lower indices as usual.

Recall that the connection forms $[\omega^i_j]$ of the Levi-Civita connection of (M^n, g) satisfy the structure equations:

$$d\omega^i = \omega^j \wedge \omega^i_j, \quad \omega^i_j + \omega^j_i = 0, \qquad (2.1)$$

$$d\omega^i_j = \omega^k_j \wedge \omega^i_k + \tfrac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l, \qquad (2.2)$$

where R^i_{jkl} are the components of the curvature tensor of (M^n, g). It is well-known that the Riemannian curvature tensor $R_{ijkl} = g_{im}R^m_{jkl}$ has an irreducible decomposition given by [1]

$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad - \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \qquad (2.3) \\ &= W_{ijkl} + \frac{1}{n-2}(A_{ik}g_{jl} - A_{il}g_{jk} + A_{jl}g_{ik} - A_{jk}g_{il}), \end{aligned}$$

where W_{ijkl}, R_{ij} and R are the components of the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of g, respectively, while A_{ij} are the components of the Schouten tensor A. It is well-known that the Weyl conformal curvature tensor in the decomposition (2.3) yields the conformally invariant part of the Riemannian curvature tensor, while the Schouten tensor describes the curvature property that is not conformally invariant.

The Ricci tensor R_{ij} and the scalar curvature R of (M^n, g) are defined by

$$R_{ij} = \sum_{k,l} g^{kl}R_{ikjl}, \quad R = \sum_{i,j} g^{ij}R_{ij}.$$

For an arbitrary tensor, such as T_{ijkl}, we define its norm as $|T|^2 = T_{ijkl}T_{mprs}g^{im}g^{jp}g^{kr}g^{ls}$.

Denote by ∇ and Δ_g the covariant differentiation operator of the Levi-Civita connection and the Laplacian on (M^n, g), respectively. For local expressions of the covariant derivatives we use the notation

$$R_{ij,k} = \nabla_k R_{ij}, \quad R_{ij,kl} = \nabla_l \nabla_k R_{ij}, \quad \Delta_g R_{ij} = g^{kl}R_{ij,kl},$$
$$A_{ij,k} = \nabla_k A_{ij}, \quad A_{ij,kl} = \nabla_l \nabla_k A_{ij}, \quad \Delta_g A_{ij} = g^{kl}A_{ij,kl}, \text{ etc.}$$

The first and the second order covariant derivatives of A_{ij} are given by

$$A_{ij,k}\omega^k = dA_{ij} - A_{kj}\omega^k_i - A_{ik}\omega^k_j,$$
$$A_{ij,kl}\omega^l = dA_{ij,k} - A_{lj,k}\omega^l_i - A_{il,k}\omega^l_j - A_{ij,l}\omega^l_k.$$

An n-dimensional Riemannian manifold (M^n, g) is called locally conformally flat if g is locally conformal equivalent to a flat Riemannian metric, that is, for any $p \in M$, there is a neighborhood U of point p and a smooth function ρ on U, so that $\tilde{g} = e^{2\rho}g$ is a flat Riemannian metric. The following proposition is well-known.

Proposition2.1. For a Riemannian manifold (M^n, g), if $n \geq 4$, then (M^n, g) is locally conformally flat if and only if $W_{ijkl} \equiv 0$.

III. Proof of Main Results

For manifold $I\times S^3$, we select

$$e_0=\frac{\partial}{\partial r}, e_1=f^{-1}v_1, e_2=f^{-1}v_2, e_3=f^{-1}\frac{1}{t}v_3.$$

Then $\{e_i\}_{i=0}^3$ is the integral, standard, orthogonal frame field on $(I\times S^3,g_t)$, whose dual coframe field $\{\omega^0,\omega^1,\omega^2,\omega^3\}$ is given by

$$\omega^0=dr, \tag{3.1}$$

$$\omega^1=f\xi^1, \tag{3.2}$$

$$\omega^2=f\xi^2, \tag{3.3}$$

$$\omega^3=f\xi^3, \tag{3.4}$$

Lemma3.1. If we define a family of Riemannian metrics

$$g_t=dr^2+f^2(r)[(\theta^1)^2+(\theta^2)^2+t^2(\theta^3)^2],\quad t>0$$

on $I\times S^3$, then the connection form of the Riemannian connection of $(I\times S^3,g_t)$ under the above standard, orthogonal frame field $\{e_i\}_{i=0}^3$ can be given by

$$\begin{aligned}&\omega_0^1=f^{-1}f'\omega^1, && \omega_0^2=f^{-1}f'\omega^2, \quad \omega_0^3=f^{-1}f'\omega^3,\\ &\omega_1^2=f^{-1}(2t^{-1}-t)\omega^3, && \omega_1^3=-f^{-1}t\omega^2, \quad \omega_2^3=f^{-1}t\omega^1.\end{aligned} \tag{3.5}$$

Proof. Under the standard, orthogonal frame $\{e_i\}_{i=0}^3$, the connection form $\{\omega_i^j\}$ of the Riemannian connection of $(I\times S^3,g_t)$ satisfies the following structural equation

$$d\omega^i=\omega^j\wedge\omega_j^i,\quad \omega_j^i+\omega_i^j=0, \tag{3.6}$$

and is uniquely determined by it. Next, we just need to verify that (3.5) satisfies (3.6).

It is known from the section 5 of [13]

$$d\xi^1=-2t^{-1}\xi^2\wedge\xi^3,\quad d\xi^2=2t^{-1}\xi^1\wedge\xi^3,\quad d\xi^3=-2t\xi^1\wedge\xi^2. \tag{3.7}$$

From (3.1)-(3.4) and (3.7), we obtain

$$\begin{aligned}&d\omega^0=0,\\ &d\omega^1=f'dr\wedge\xi^1+fd\xi^1=f'dr\wedge\xi^1-2ft^{-1}\xi^2\wedge\xi^3,\\ &d\omega^2=f'dr\wedge\xi^2+fd\xi^2=f'dr\wedge\xi^2+2ft^{-1}\xi^1\wedge\xi^3,\\ &d\omega^3=f'dr\wedge\xi^3+fd\xi^3=f'dr\wedge\xi^3-2ft\xi^1\wedge\xi^2.\end{aligned} \tag{3.8}$$

On the other hand, it is obtained from (3.5) and (3.6)

$$\begin{aligned}&d\omega^0=\omega^1\wedge\omega_1^0+\omega^2\wedge\omega_2^0+\omega^3\wedge\omega_3^0=0,\\ &d\omega^1=\omega^0\wedge\omega_0^1+\omega^2\wedge\omega_2^1+\omega^3\wedge\omega_3^1=f'dr\wedge\xi^1-2ft^{-1}\xi^2\wedge\xi^3,\\ &d\omega^2=\omega^0\wedge\omega_0^2+\omega^1\wedge\omega_1^2+\omega^3\wedge\omega_3^2=f'dr\wedge\xi^2+2ft^{-1}\xi^1\wedge\xi^3,\\ &d\omega^3=\omega^0\wedge\omega_0^3+\omega^1\wedge\omega_1^3+\omega^2\wedge\omega_2^3=f'dr\wedge\xi^3-2ft\xi^1\wedge\xi^2.\end{aligned} \tag{3.9}$$

Thus, from (3.8) and (3.9), it is known that (3.5) satisfies (3.6). □

Proposition3.1. If we define a family of Riemannian metrics

$$g_t=dr^2+f^2(r)[(\theta^1)^2+(\theta^2)^2+t^2(\theta^3)^2],\quad t>0$$

on $I\times S^3$, then the Riemannian curvature tensor R_{ijkl} of $(I\times S^3,g_t)$ under the above standard, orthogonal frame field $\{e_i\}_{i=0}^3$ can be given by

$$\begin{aligned}R_{01kl}&=-f^{-1}f''(\delta_{0k}\delta_{1l}-\delta_{0l}\delta_{1k}),\\ R_{02kl}&=-f^{-1}f''(\delta_{0k}\delta_{2l}-\delta_{0l}\delta_{2k}),\\ R_{03kl}&=-f^{-1}f''(\delta_{0k}\delta_{3l}-\delta_{0l}\delta_{3k}),\\ R_{12kl}&=[f^{-2}(4-3t^2)-f^{-2}ff'](\delta_{1k}\delta_{2l}-\delta_{1l}\delta_{2k}),\\ R_{13kl}&=(f^{-2}t^2-f^{-2}ff')(\delta_{1k}\delta_{3l}-\delta_{1l}\delta_{3k}),\\ R_{23kl}&=(f^{-2}t^2-f^{-2}ff')(\delta_{2k}\delta_{3l}-\delta_{2l}\delta_{3k}),\end{aligned} \tag{3.10}$$

where $0\le k,l\le 3$.

Proof. From (3.5), we know $\omega_0^1=f^{-1}f'\omega^1$, therefore

$$\begin{aligned}d\omega_0^1&=d(f^{-1}f')\wedge\omega^1+f^{-1}f'd\omega^1\\ &=(f^{-1}f')'\omega^0\wedge\omega^1+f^{-1}f'(f^{-1}f'\omega^0\wedge\omega^1\\ &\quad-2f^{-1}t^{-1}\omega^2\wedge\omega^3)\\ &=f^{-1}f''\omega^0\wedge\omega^1-2f^{-2}f't^{-1}\omega^2\wedge\omega^3.\end{aligned} \tag{3.11}$$

On the other hand, it is obtained from the structural equations (2.2) and (3.5)

$$\begin{aligned}d\omega_0^1&=\omega_0^k\wedge\omega_k^1+\tfrac{1}{2}R_{10kl}\omega^k\wedge\omega^l\\ &=\omega_0^2\wedge\omega_2^1+\omega_0^3\wedge\omega_3^1-\tfrac{1}{2}R_{01kl}\omega^k\wedge\omega^l\\ &=-2f^{-2}f't^{-1}\omega^2\wedge\omega^3-\tfrac{1}{2}R_{01kl}\omega^k\wedge\omega^l.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\begin{aligned}\tfrac{1}{2}R_{01kl}\omega^k\wedge\omega^l&=-f^{-1}f''\omega^0\wedge\omega^1\\ &=-\tfrac{1}{2}f^{-1}f''(\delta_{0k}\delta_{1l}-\delta_{0l}\delta_{1k})\omega^k\wedge\omega^l,\end{aligned}$$

therefore

$$R_{01kl}=-f^{-1}f''(\delta_{0k}\delta_{1l}-\delta_{0l}\delta_{1k}),\quad 0\le k,l\le 3.$$

Similarly, we can obtain other Riemannian curvature tensor R_{ijkl} under the standard, orthogonal frame field $\{e_i\}_{i=0}^3$. □

In particular, from (3.10), we can get

$$\begin{aligned}&R_{0101}=-f^{-1}f'', && R_{0202}=-f^{-1}f'',\\ &R_{0303}=-f^{-1}f'', && R_{1212}=f^{-2}(4-3t^2)-f^{-2}ff',\\ &R_{1313}=f^{-2}t^2-f^{-2}ff', && R_{2323}=f^{-2}t^2-f^{-2}ff',\end{aligned}$$

and $R_{ijkl}=0$ if any three of the indices are distinct. Therefore, we can get the following proposition.

Proposition3.2. If we define a family of Riemannian metrics

$$g_t=dr^2+f^2(r)[(\theta^1)^2+(\theta^2)^2+t^2(\theta^3)^2],\quad t>0$$

on $I\times S^3$, then the Ricci curvature tensor R_{ij} of $(I\times S^3,g_t)$ under the above standard, orthogonal frame field $\{e_i\}_{i=0}^3$ can be given by

$$\begin{aligned}&R_{00}=R_{0101}+R_{0202}+R_{0303}=-3f^{-1}f'',\\ &R_{11}=R_{1010}+R_{1212}+R_{1313}=-f^{-1}f''-2f^{-2}ff'+2f^{-2}(2-t^2),\\ &R_{22}=R_{2020}+R_{2121}+R_{2323}=-f^{-1}f''-2f^{-2}ff'+2f^{-2}(2-t^2),\\ &R_{33}=R_{3030}+R_{3131}+R_{3232}=-f^{-1}f''-2f^{-2}ff'+2f^{-2}t^2,\\ &R_{ij}=R_{ikjk}=R_{i0j0}+R_{i1j1}+R_{i2j2}+R_{i3j3}=0\quad if\ i\ne j,\end{aligned} \tag{3.13}$$

where $0\le i,j\le 3$.

From proposition3.2, we can easily get the scalar curvature R of $(I\times S^3,g_t)$ as follows

$$R=-6(f^{-1}f''+f^{-2}ff')+2f^{-2}(4-t^2). \tag{3.14}$$

Let's consider the Weyl curvature tensor of $(I\times S^3,g_t)$.

From(2.3), for $(I\times S^3,g_t)$, we have

$$\begin{aligned}W_{ijkl}=R_{ijkl}&-\tfrac{1}{2}(R_{ik}\delta_{jl}-R_{il}\delta_{jk}+R_{jl}\delta_{ik}-R_{jk}\delta_{il})\\ &+\tfrac{1}{6}R(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}).\end{aligned} \tag{3.15}$$

By simple calculation, we can get

$$\begin{aligned}W_{01kl}&=-\tfrac{2}{3}f^{-2}(1-t^2)(\delta_{0k}\delta_{1l}-\delta_{0l}\delta_{1k}),\\ W_{02kl}&=-\tfrac{2}{3}f^{-2}(1-t^2)(\delta_{0k}\delta_{2l}-\delta_{0l}\delta_{2k}),\end{aligned}$$

$$\begin{aligned} W_{03kl} &= \tfrac{4}{3} f^{-2}(1-t^2)(\delta_{0k}\delta_{3l}-\delta_{0l}\delta_{3k}),\\ W_{12kl} &= \tfrac{4}{3} f^{-2}(1-t^2)(\delta_{1k}\delta_{2l}-\delta_{1l}\delta_{2k}),\\ W_{13kl} &= -\tfrac{2}{3} f^{-2}(1-t^2)(\delta_{1k}\delta_{3l}-\delta_{1l}\delta_{3k}),\\ W_{23kl} &= -\tfrac{2}{3} f^{-2}(1-t^2)(\delta_{2k}\delta_{3l}-\delta_{2l}\delta_{3k}), \end{aligned} \tag{3.16}$$

where $(0 \le k, l \le 3)$.

From proposition2.1 and (3.16), we immediately obtain the following proposition.

Proposition3.3. $(I\times S^3, g_t)$ is locally conformally flat if and only if $t=1$.

Remark3.1. If $(I\times S^3, g_t)$ is locally conformally flat and $f(r)$ satisfies $ff''+(f')^2-1=0$, then the scalar curvature $R_{g_t}=0$. According to theorem4.1 in [13], at this time, the locally conformally flat metric g_t is the critical point of the Schouten functional $\mathcal{S}$.

Under the standard, orthogonal frame field, $E_{ij}=R_{ij}-\frac{1}{4}R\delta_{ij}$, so it's easy to get from propositions3.2 and (3.14)

$$\begin{aligned} E_{00} &= -\tfrac{3}{2}(f^{-1}f''-f^{-2}ff')-\tfrac{1}{2}f^{-2}(4-t^2),\\ E_{11} &= \tfrac{1}{2}(f^{-1}f''-f^{-2}ff')+\tfrac{1}{2}f^{-2}(4-3t^2),\\ E_{22} &= \tfrac{1}{2}(f^{-1}f''-f^{-2}ff')+\tfrac{1}{2}f^{-2}(4-3t^2),\\ E_{33} &= \tfrac{1}{2}(f^{-1}f''-f^{-2}ff')-\tfrac{1}{2}f^{-2}(4-5t^2),\\ E_{ij} &= 0 \quad if\ i\ne j, and \text{ for } 0\le i,j\le 3. \end{aligned} \tag{3.17}$$

where E_{ij} denotes the components of the trace-free Ricci tensor E_{g_t} of $(I\times S^3, g_t)$.

Therefore we can obtain the following Proposition.

Proposition3.4. $(I\times S^3, g_t)$ is Einstein manifold if and only if $t=1$, and $f(r)$ satisfies the equation $ff''-(f')^2+1=0$.

Proof. $(I\times S^3, g_t)$ is Einstein manifold if and only if the trace-free Ricci tensor E_{g_t} satisfies $E_{g_t}\equiv 0$, i.e. $E_{ij}\equiv 0$ $(0\le i,j\le 3)$.

Hence, we obtain

$$E_{00} = -\tfrac{3}{2}(f^{-1}f''-f^{-2}ff')-\tfrac{1}{2}f^{-2}(4-t^2)=0,$$

$$\begin{aligned} E_{11} &= \tfrac{1}{2}(f^{-1}f''-f^{-2}ff')+\tfrac{1}{2}f^{-2}(4-3t^2)=0,\\ E_{22} &= \tfrac{1}{2}(f^{-1}f''-f^{-2}ff')+\tfrac{1}{2}f^{-2}(4-3t^2)=0,\\ E_{33} &= \tfrac{1}{2}(f^{-1}f''-f^{-2}ff')-\tfrac{1}{2}f^{-2}(4-5t^2)=0. \end{aligned}$$

By solving the above equations, we can get $t=1$, and $f(r)$ satisfies the equation

$$ff''-(f')^2+1=0. \qquad \square$$

The first and the second order covariant derivatives of E_{g_t} are defined by

$$E_{ij,k}\omega^k = dE_{ij}-E_{kj}\omega_i^k-E_{ik}\omega_j^k, \tag{3.18}$$

$$E_{ij,kl}\omega^l = dE_{ij,k}-E_{lj,k}\omega_i^l-E_{il,k}\omega_j^l-E_{ij,l}\omega_k^l, \tag{3.19}$$

With(3.17)and(3.18), we obtain

$$\begin{aligned} E_{ij,k}\omega^k &= (E_{ii}-E_{jj})\omega_i^j \quad && if\ i\ne j,\\ E_{ii,k}\omega^k &= dE_{ii} && if\ i=j. \end{aligned}$$

By simple calculation, we get

Lemma3.2. The first covariant derivatives of E_{g_t} of $(I\times S^3, g_t)$ are given by

$$\begin{aligned} &E_{01,1}=E_{02,2}=-2f^{-1}f'(f^{-1}f''-f^{-2}ff')-2f^{-3}f'(2-t^2),\\ &E_{03,3}=-2f^{-1}f'(f^{-1}f''-f^{-2}ff')-2f^{-3}f't^2,\\ &E_{01,0}=E_{01,2}=E_{01,3}=E_{02,0}=E_{02,1}=E_{02,3}=E_{03,0}=E_{03,1}=E_{03,2}=0,\\ &E_{12,0}=E_{12,1}=E_{12,2}=E_{12,3}=0,\\ &E_{13,2}=-4f^{-3}t(1-t^2),\\ &E_{23,1}=4f^{-3}t(1-t^2),\\ &E_{13,0}=E_{13,1}=E_{13,3}=E_{23,0}=E_{23,2}=E_{23,3}=0,\\ &E_{00,0}=-\tfrac{3}{2}[-3f^{-2}ff''+f^{-1}f'''+2f^{-3}(f')^3]+f^{-3}f'(4-t^2),\\ &E_{11,0}=E_{22,0}=\tfrac{1}{2}[-3f^{-2}ff''+f^{-1}f'''\\ &\qquad +2f^{-3}(f')^3]-f^{-3}f'(4-3t^2),\\ &E_{33,0}=\tfrac{1}{2}[-3f^{-2}ff''+f^{-1}f'''+2f^{-3}(f')^3]+f^{-3}f'(4-5t^2),\\ &E_{00,1}=E_{00,2}=E_{00,3}=0,\\ &E_{11,1}=E_{11,2}=E_{11,3}=E_{22,1}=E_{22,2}=E_{22,3}=E_{33,1}=E_{33,2}=E_{33,3}=0. \end{aligned}$$

From lemma3.1, lemma3.2 and (3.19), with simple calculation, we obtain

Lemma3.3. The second covariant derivatives of E_{g_t} of $(I\times S^3, g_t)$ are given by

$$\begin{aligned} &E_{01,01}=2f^{-1}f'[4f^{-2}ff''-f^{-1}f'''-3f^{-3}(f')^3]\\ &\qquad +6f^{-4}ff'(2-t^2),\\ &E_{01,10}=-2[-5f^{-3}(f')^2f''+f^{-2}(f'')^2+f^{-2}ff'''+3f^{-4}(f')^4]\\ &\qquad +2(3f^{-4}ff'-f^{-3}f'')(2-t^2),\\ &E_{01,23}=E_{01,32}=4f^{-4}f't(1-t^2),\\ &E_{01,00}=E_{01,02}=E_{01,03}=E_{01,11}=E_{01,12}=E_{01,13}=0,\\ &E_{01,20}=E_{01,21}=E_{01,22}=E_{01,30}=E_{01,31}=E_{01,33}=0,\\ &E_{02,02}=2f^{-1}f'[4f^{-2}ff''-f^{-1}f'''-3f^{-3}(f')^3]\\ &\qquad +6f^{-4}ff'(2-t^2),\\ &E_{02,20}=-2[-5f^{-3}(f')^2f''+f^{-2}(f'')^2+f^{-2}ff'''\\ &\qquad +3f^{-4}(f')^4]+2(3f^{-4}ff'-f^{-3}f'')(2-t^2),\\ &E_{02,13}=E_{02,31}=-4f^{-4}f't(1-t^2),\\ &E_{02,00}=E_{02,01}=E_{02,03}=E_{02,21}=E_{02,22}=E_{02,23}=0,\\ &E_{02,10}=E_{02,11}=E_{02,12}=E_{02,30}=E_{02,32}=E_{02,33}=0,\\ &E_{03,03}=2f^{-1}f'[4f^{-2}ff''-f^{-1}f'''-3f^{-3}(f')^3]+6f^{-4}ff't^2,\\ &E_{03,30}=-2[-5f^{-3}(f')^2f''+f^{-2}(f'')^2+f^{-2}ff'''\\ &\qquad +3f^{-4}(f')^4]+2(3f^{-4}ff'-f^{-3}f'')t^2,\\ &E_{03,00}=E_{03,01}=E_{03,02}=E_{03,31}=E_{03,32}=E_{03,33}=0,\\ &E_{03,10}=E_{03,11}=E_{03,12}=E_{03,13}=E_{03,20}=E_{03,21}=E_{03,22}=E_{03,23}=0,\\ &E_{12,12}=E_{12,21}=-2f^{-2}(f')^2(f^{-1}f''-f^{-2}ff')\\ &\qquad +4f^{-4}t^2(1-t^2)-2f^{-4}ff'(2-t^2),\\ &E_{12,10}=E_{12,11}=E_{12,13}=E_{12,20}=E_{12,22}=E_{12,23}=0,\\ &E_{12,00}=E_{12,01}=E_{12,02}=E_{12,03}=E_{12,30}=E_{12,31}=E_{12,32}=E_{12,33}=0,\\ &E_{13,02}=E_{13,20}=12f^{-4}f't(1-t^2),\\ &E_{13,13}=-2f^{-2}(f')^2(f^{-1}f''-f^{-2}ff')-2f^{-4}ff'(2-t^2),\\ &E_{13,31}=-2f^{-2}(f')^2(f^{-1}f''-f^{-2}ff')\\ &\qquad -4f^{-4}t^2(1-t^2)-2f^{-4}ff't^2,\\ &E_{13,00}=E_{13,01}=E_{13,03}=E_{13,21}=E_{13,22}=E_{13,23}=0, \end{aligned}$$

$E_{13,10}=E_{13,11}=E_{13,12}=E_{13,30}=E_{13,32}=E_{13,33}=0,$
$E_{23,01}=E_{23,10}=-12f^{-4}f't(1-t^2),$
$E_{23,23}=-2f^{-2}(f')^2(f^{-1}f''-f^{-2}ff')-2f^{-4}ff'(2-t^2),$
$E_{23,32}=-2f^{-2}(f')^2(f^{-1}f''-f^{-2}ff')$
$\quad -4f^{-4}t^2(1-t^2)-2f^{-4}ff't^2,$
$E_{23,00}=E_{23,02}=E_{23,03}=E_{23,11}=E_{23,12}=E_{23,13}=0,$
$E_{23,20}=E_{23,21}=E_{23,22}=E_{23,30}=E_{23,31}=E_{23,33}=0,$
$E_{00,00}=-\frac{3}{2}[12f^{-3}(f')^2f''-3f^{-2}(f'')^2-4f^{-2}ff'''+f^{-1}f^{(4)}$
$\quad -6f^{-4}(f')^4]+(-3f^{-4}ff'+f^{-3}f'')(4-t^4),$
$E_{00,11}=E_{00,22}=\frac{1}{2}f^{-1}f'(17f^{-2}ff''-14f^{-3}(f')^3-3f^{-1}f''')$
$\quad +f^{-4}(f')^2(12-5t^2),$
$E_{00,33}=\frac{1}{2}f^{-1}f'(17f^{-2}ff''-14f^{-3}(f')^3-3f^{-1}f''')$
$\quad +f^{-4}(f')^2(4+3t^2),$
$E_{00,01}=E_{00,02}=E_{00,03}=E_{00,30}=E_{00,31}=E_{00,32}=0,$
$E_{00,10}=E_{00,12}=E_{00,13}=E_{00,20}=E_{00,21}=E_{00,23}=0,$
$E_{11,00}=\frac{1}{2}[12f^{-3}(f')^2f''-3f^{-2}(f'')^2-4f^{-2}ff'''+f^{-1}f^{(4)}$
$\quad -6f^{-4}(f')^4]+(3f^{-4}ff'-f^{-3}f'')(4-3t^2),$
$E_{11,11}=\frac{1}{2}f^{-1}f'(-11f^{-2}ff''+10f^{-3}(f')^3+f^{-1}f''')$
$\quad -f^{-4}(f')^2(12-7t^2),$
$E_{11,22}=\frac{1}{2}f^{-1}f'(-3f^{-2}ff''+2f^{-3}(f')^3+f^{-1}f''')$
$\quad -8f^{-4}t^2(1-t^2)-f^{-4}(f')^2(4-3t^2),$
$E_{11,33}=\frac{1}{2}f^{-1}f'(-3f^{-2}ff''+2f^{-3}(f')^3+f^{-1}f''')$
$\quad -f^{-4}(f')^2(4-3t^2),$
$E_{11,01}=E_{11,02}=E_{11,03}=E_{11,10}=E_{11,12}=E_{11,13}=0,$
$E_{11,20}=E_{11,21}=E_{11,23}=E_{11,30}=E_{11,31}=E_{11,32}=0,$
$E_{22,00}=\frac{1}{2}[12f^{-3}(f')^2f''-3f^{-2}(f'')^2-4f^{-2}ff'''+f^{-1}f^{(4)}$
$\quad -6f^{-4}(f')^4]+(3f^{-4}ff'-f^{-3}f'')(4-3t^2),$
$E_{22,11}=\frac{1}{2}f^{-1}f'(-3f^{-2}ff''+2f^{-3}(f')^3+f^{-1}f''')$
$\quad -8f^{-4}t^2(1-t^2)-f^{-4}(f')^2(4-3t^2),$
$E_{22,22}=\frac{1}{2}f^{-1}f'(-11f^{-2}ff''+10f^{-3}(f')^3+f^{-1}f''')$
$\quad -f^{-4}(f')^2(12-7t^2),$
$E_{22,33}=\frac{1}{2}f^{-1}f'(-3f^{-2}ff''+2f^{-3}(f')^3+f^{-1}f''')$
$\quad -f^{-4}(f')^2(4-3t^2),$
$E_{22,01}=E_{22,02}=E_{22,03}=E_{22,10}=E_{22,12}=E_{22,13}=0,$
$E_{22,20}=E_{22,21}=E_{22,23}=E_{22,30}=E_{22,31}=E_{22,32}=0,$
$E_{33,00}=\frac{1}{2}[12f^{-3}(f')^2f''-3f^{-2}(f'')^2-4f^{-2}ff'''+f^{-1}f^{(4)}$
$\quad -6f^{-4}(f')^4]+(-3f^{-4}ff'+f^{-3}f'')(4-5t^2),$
$E_{33,11}=E_{33,22}=\frac{1}{2}f^{-1}f'(-3f^{-2}ff''+2f^{-3}(f')^3+f^{-1}f''')$
$\quad +8f^{-4}t^2(1-t^2)+f^{-4}(f')^2(4-5t^2),$
$E_{33,33}=\frac{1}{2}f^{-1}f'(-11f^{-2}ff''+10f^{-3}(f')^3+f^{-1}f''')$
$\quad +f^{-4}(f')^2(4-9t^2),$
$E_{33,01}=E_{33,02}=E_{33,03}=E_{33,10}=E_{33,12}=E_{33,13}=0,$
$E_{33,20}=E_{33,21}=E_{33,23}=E_{33,30}=E_{33,31}=E_{33,32}=0.$

Lemma3.3 immediately implies

Lemma3.4. For $(I\times S^3, g_t)$, we have

$$\begin{aligned}
\Delta_{g_t}E_{00} &= \tfrac{3}{2}[5f^{-3}(f')^2f''-8f^{-4}(f')^4+f^{-2}ff'''+3f^{-2}(f'')^2\\
&\quad -f^{-1}f^{(4)}]+[4f^{-4}(f')^2+f^{-3}f''](4-t^2),\\
\Delta_{g_t}E_{11} &= \Delta_{g_t}E_{22} = -\tfrac{1}{2}[5f^{-3}(f')^2f''-8f^{-4}(f')^4+f^{-2}ff'''\\
&\quad +3f^{-2}(f'')^2-f^{-1}f^{(4)}]-8f^{-4}t^2(1-t^2)\\
&\quad -4f^{-4}(f')^2(2-t^2)-f^{-3}f''(4-3t^2),\\
\Delta_{g_t}E_{33} &= -\tfrac{1}{2}[5f^{-3}(f')^2f''-8f^{-4}(f')^4+f^{-2}ff'''\\
&\quad +3f^{-2}(f'')^2-f^{-1}f^{(4)}]+16f^{-4}t^2(1-t^2)\\
&\quad -4f^{-4}(f')^2t^2+f^{-3}f''(4-5t^2),\\
\Delta_{g_t}E_{01} &= \Delta_{g_t}E_{02}=\Delta_{g_t}E_{03}=\Delta_{g_t}E_{12}=\Delta_{g_t}E_{13}=\Delta_{g_t}E_{23}=0.
\end{aligned} \tag{3.20}$$

Proposition3.5. Let $M^n (n\ge 3)$ be a smooth n-manifold. Then a metric $g\in\mathcal{M}(M)$ is a critical point of the Schouten functional $\mathcal{S}$ if and only if it satisfies the following equations $(1\le i,j,k,l\le n)$

$$\Delta_g E_{ij}-\frac{(n-2)(2n-3)}{2(n-1)^2}R_{,ij}+\frac{(n-2)(2n-3)}{2n(n-1)^2}\Delta_g R g_{ij}$$
$$+2\sum_{k,l}E^{kl}W_{kilj}-\frac{4}{n-2}\sum_k E_i^k E_{kj}+\frac{n^2-8n+8}{2n(n-1)^2}RE_{ij}$$
$$+\frac{4}{n(n-2)}|E_g|^2 g_{ij}=0,$$

where $1\le i,j,k,l\le n$,

$$\frac{(n-2)^2}{n-1}\Delta_g R-(n-4)|E_g|^2-\frac{(n-2)^2(n-4)}{4n(n-1)^2}R_g^2$$
$$+(n-4)Vol(M^n,g)^{-\frac{4}{n}}\mathcal{S}[g]=0.$$

Proof. In [13], the authors give a detailed proof when the manifold is compact, boundless. If the manifold is noncompact or compact with boundary, the proof is similar to that of compact without boundary. At this time, we need to consider the critical point of the functional with respect to the variation with compact support set. Following the notation in [13], when the manifold is noncompact, we need to consider the case that the variation function h_{ij} has a compact support set. When the manifold is compact with boundary, we must require the variation function $h_{ij}|_{\partial M}=0$.

Here we will not give a proof. Please refer to proposition 3.1 and theorem3.1 in [13]. □

Thus, under the standard, orthogonal frame field, we get

Corollary3.1. Let M^4 be a smooth 4-manifold. Then a metric $g\in\mathcal{M}(M)$ is a critical point of $\mathcal{S}$ if and only if it satisfies the following equations

$$\begin{aligned}
&\Delta_g E_{ij}-\tfrac{5}{9}R_{,ij}+\tfrac{5}{36}\Delta_g R\delta_{ij}+2\sum_{k,l}E_{kl}W_{kilj}-2\sum_k E_{ik}E_{kj}\\
&-\tfrac{1}{9}RE_{ij}+\tfrac{1}{2}|E|^2\delta_{ij}=0,
\end{aligned} \tag{3.21}$$

where $0\le i,j,k,l\le 3$,

$$\Delta_g R=0. \tag{3.22}$$

The following corollary is immediately obtained from (3.22).

Corollary3.2. Let M^4 be a smooth 4-manifold. If $g\in\mathcal{M}(M)$ is a critical point of $\mathcal{S}$, then the scalar curvature R of g is constant.

Substituting (3.22) into (3.21), yields

$$\Delta_{g_t} E_{ij} + 2\sum_{k,l} E_{kl} W_{kilj} - 2\sum_k E_{ik} E_{kj} - \tfrac{1}{9} RE_{ij} + \tfrac{1}{2} |E|^2 \delta_{ij} = 0. \tag{3.23}$$

3.1 Proof of Theorem 1

Proof. For $(I \times S^3, g_t)$, under the above standard, orthogonal frame field $\{e_i\}_{i=0}^3$, when $i = j$, (3.23) can be written as

$$\Delta_{g_t} E_{ii} + 2\sum_k E_{kk} W_{kiki} - 2E_{ii}E_{ii} - \tfrac{1}{9} RE_{ii} + \tfrac{1}{2} |E|^2 = 0. \tag{3.24}$$

When $i \neq j$, (3.23) can be written as

$$\sum_k E_{kk} W_{kikj} = 0. \tag{3.25}$$

From (3.16) and (3.17), when $i \neq j$, (3.25) holds.

Now we only need to solve (3.24). From (3.16) and (3.17), we have

$$\begin{aligned}
&\sum_k E_{kk} W_{k0k0} = -\tfrac{16}{3} f^{-4}(1-t^2)^2, \\
&\sum_k E_{kk} W_{k1k1} = \tfrac{4}{3} f^{-2}(1-t^2)(f^{-1}f'' - f^{-2}ff') \\
&\qquad + \tfrac{4}{3} f^{-4}(1-t^2)(4-3t^2), \\
&\sum_k E_{kk} W_{k2k2} = \tfrac{4}{3} f^{-2}(1-t^2)(f^{-1}f'' - f^{-2}ff') \\
&\qquad + \tfrac{4}{3} f^{-4}(1-t^2)(4-3t^2), \\
&\sum_k E_{kk} W_{k3k3} = -\tfrac{8}{3} f^{-2}(1-t^2)(f^{-1}f'' - f^{-2}ff') \\
&\qquad - \tfrac{8}{3} f^{-4}(1-t^2)(2-t^2),
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
E_{00}^2 &= \tfrac{9}{4}(f^{-1}f'' - f^{-2}ff')^2 + \tfrac{3}{2} f^{-2}(f^{-1}f'' - f^{-2}ff')(4-t^2) \\
&\quad + \tfrac{1}{4} f^{-4}(4-t^2)^2, \\
E_{11}^2 &= \tfrac{1}{4}(f^{-1}f'' - f^{-2}ff')^2 + \tfrac{1}{2} f^{-2}(f^{-1}f'' - f^{-2}ff')(4-3t^2) \\
&\quad + \tfrac{1}{4} f^{-4}(4-3t^2)^2, \\
E_{22}^2 &= \tfrac{1}{4}(f^{-1}f'' - f^{-2}ff')^2 + \tfrac{1}{2} f^{-2}(f^{-1}f'' - f^{-2}ff')(4-3t^2) \\
&\quad + \tfrac{1}{4} f^{-4}(4-3t^2)^2, \\
E_{33}^2 &= \tfrac{1}{4}(f^{-1}f'' - f^{-2}ff')^2 - \tfrac{1}{2} f^{-2}(f^{-1}f'' - f^{-2}ff')(4-5t^2) \\
&\quad + \tfrac{1}{4} f^{-4}(4-5t^2)^2.
\end{aligned} \tag{3.27}$$

Therefore we obtain

$$\begin{aligned}
|E|^2 &= 3(f^{-1}f'' - f^{-2}ff')^2 + 2f^{-2}(f^{-1}f'' \\
&\quad - f^{-2}ff')(4-t^2) + f^{-4}(16 - 24t^2 + 11t^4).
\end{aligned} \tag{3.28}$$

Substituting (3.20), (3.26), (3.27), (3.28), (3.14) and (3.17) into (3.24), yields the following four equations

$$\begin{aligned}
&9f^{-1}f^{(4)} - 9f^{-2}ff''' - 81f^{-3}(f')^2 f'' + 84f^{-4}(f')^4 \\
&-3f^{-2}(f'')^2 + 6f^{-3}f''(4-t^2) - 32f^{-4}(f')^2(4-t^2) \\
&+\tfrac{4}{3} f^{-4}(40 - 56t^2 + 25t^4) = 0.
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&9f^{-1}f^{(4)} - 9f^{-2}ff''' - 81f^{-3}(f')^2 f'' + 84f^{-4}(f')^4 \\
&-3f^{-2}(f'')^2 - 2f^{-3}f''(4-13t^2) - 32f^{-4}(f')^2(5-2t^2) \\
&+4f^{-4}(40 - 112t^2 + 75t^4) = 0.
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&9f^{-1}f^{(4)} - 9f^{-2}ff''' - 81f^{-3}(f')^2 f'' + 84f^{-4}(f')^4 \\
&-3f^{-2}(f'')^2 - 2f^{-3}f''(4-13t^2) - 32f^{-4}(f')^2(5-2t^2) \\
&+4f^{-4}(40 - 112t^2 + 75t^4) = 0.
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
&9f^{-1}f^{(4)} - 9f^{-2}ff''' - 81f^{-3}(f')^2 f'' + 84f^{-4}(f')^4 \\
&-3f^{-2}(f'')^2 + 2f^{-3}f''(44-35t^2) - 32f^{-4}(f')^2(2+t^2) \\
&-4f^{-4}(40 - 168t^2 + 125t^4) = 0.
\end{aligned} \tag{3.32}$$

i.e.

$$\begin{cases}
9f^{-1}f^{(4)} - 9f^{-2}ff''' - 81f^{-3}(f')^2 f'' + 84f^{-4}(f')^4 \\
-3f^{-2}(f'')^2 + 6f^{-3}f''(4-t^2) - 32f^{-4}(f')^2(4-t^2) \\
+\tfrac{4}{3} f^{-4}(40 - 56t^2 + 25t^4) = 0, \\
9f^{-1}f^{(4)} - 9f^{-2}ff''' - 81f^{-3}(f')^2 f'' + 84f^{-4}(f')^4 \\
-3f^{-2}(f'')^2 - 2f^{-3}f''(4-13t^2) - 32f^{-4}(f')^2(5-2t^2) \\
+4f^{-4}(40 - 112t^2 + 75t^4) = 0, \\
9f^{-1}f^{(4)} - 9f^{-2}ff''' - 81f^{-3}(f')^2 f'' + 84f^{-4}(f')^4 \\
-3f^{-2}(f'')^2 + 2f^{-3}f''(44-35t^2) - 32f^{-4}(f')^2(2+t^2) \\
-4f^{-4}(40 - 168t^2 + 125t^4) = 0.
\end{cases} \tag{3.33}$$

We first prove this conclusion: for $\forall t > 0$, there is no metric g_t with $f(r)$ as a constant, so that it is a nontrivial, critical metric of $\mathcal{S}[g_t]$ on $I \times S^3$. We prove it by means of reduction to absurdity. Assuming that $f(r)$ is a constant, the equations are obtained from (3.33)

$$\begin{cases}
40 - 56t^2 + 25t^4 = 0, \\
40 - 112t^2 + 75t^4 = 0, \\
40 - 168t^2 + 125t^4 = 0.
\end{cases}$$

By solving the above equations, we know that for $\forall t > 0$, the equations have no solution. The proof is completed.

Next, we solve the equations (3.33) with assuming that $f(r)$ is not a constant.

In the equations (3.33), the first equation minus the second equation, and the third equation minus the second equation, we get

$$\begin{aligned}
&32f^{-3}f''(1-t^2) + 32f^{-4}(f')^2(1-t^2) \\
&-\tfrac{160}{3} f^{-4}(2-5t^2)(1-t^2) = 0, \\
&96f^{-3}f''(1-t^2) + 96f^{-4}(f')^2(1-t^2) \\
&-160f^{-4}(2-5t^2)(1-t^2) = 0.
\end{aligned} \tag{3.34}$$

Obviously, the two equations in (3.34) are actually the same equation. After simplification, we get

$$3f^{-3}f''(1-t^2) + 3f^{-4}(f')^2(1-t^2) - 5f^{-4}(2-5t^2)(1-t^2) = 0.$$

Now we only consider the case of $t \neq 1$. So the above equation can be written as

$$ff'' + (f')^2 - \tfrac{5}{3}(2-5t^2) = 0. \tag{3.35}$$

Next, we use the reduction method to solve this equation. Let

$$y = \frac{df}{dr} = f',$$

then

$$f'' = \frac{d^2 f}{dr^2} = \frac{dy}{dr} = \frac{dy}{df}\frac{df}{dr} = y\frac{dy}{df}.$$

Substituting the above equation into (3.35), we obtain a variable separable equation

$$fy\,dy = \left[\tfrac{5}{3}(2-5t^2) - y^2\right] df. \tag{3.36}$$

Due to $f(r) > 0$, when $\tfrac{5}{3}(2-5t^2) - y^2 \neq 0$, the equation (3.36) can be written as the following equivalent equation

$$\frac{y}{\tfrac{5}{3}(2-5t^2) - y^2} dy = \frac{df}{f}.$$

Integrating the above equation to get

$$\ln f + \tfrac{1}{2}\ln|\tfrac{5}{3}(2-5t^2) - y^2| = C_1,$$

where C_1 is an arbitrary constant.

Therefore

$$y^2 = \tfrac{5}{3}(2-5t^2) + C_2 f^{-2},$$

where $C_2 = \pm e^{2C_1} \neq 0$. In addition, the special solution $y^2 = \tfrac{5}{3}(2-5t^2)$ is obtained from $\tfrac{5}{3}(2-5t^2) - y^2 = 0$.

Therefore, the general solution of equation (3.36) is

$$y^2 = \tfrac{5}{3}(2-5t^2) + Cf^{-2}, \tag{3.37}$$

where C is an arbitrary constant.

The equation (3.37) is actually a first-order differential equation for a fixed C, i.e.

$$\frac{df}{dr} = \pm\sqrt{\tfrac{5}{3}(2-5t^2) + Cf^{-2}}. \tag{3.38}$$

The equation (3.38) is also a variable separable equation.

If $\tfrac{5}{3}(2-5t^2) = 0$, i.e. $t = \sqrt{\tfrac{2}{5}}$, then the equation (3.38) can be written as

$$fdf = \pm\sqrt{C}dr.$$

By solving this equation, we obtain

$$f = \sqrt{\pm 2\sqrt{C}r + 2C_3}, \tag{3.39}$$

where C_3 is an arbitrary constant.

When $t = \sqrt{\tfrac{2}{5}}$, substituting (3.39) into equations (3.33), yields

$$4f^2 - 19C = 0,$$

i.e.

$$4(\pm 2\sqrt{C}r + 2C_3) - 19C = 0.$$

Obviously, this equation does not hold. Thus, we can get that when $t = \sqrt{\tfrac{2}{5}}$, g_t is not a nontrivial, critical metric of $\mathcal{S}$.

If $\tfrac{5}{3}(2-5t^2) \neq 0$, the general solution of the equation (3.38) is

$$f = \sqrt{\tfrac{5}{3}(2-5t^2)(r+C_4)^2 - \frac{C}{\tfrac{5}{3}(2-5t^2)}}, \tag{3.40}$$

where C_4 is an arbitrary constant.

Substituting (3.40) into (3.33), yields

$$4f^{-2}[\tfrac{20}{3}(2-5t^2)(r+C_4)^2(35-223t^2+350t^4)$$
$$+\frac{C}{\tfrac{5}{3}(2-5t^2)}(50-343t^2+500t^4)] = 0.$$

Owing to the arbitrariness of C and C_4, we can get

$$\begin{cases} 35-223t^2+350t^4 = 0, \\ 50-343t^2+500t^4 = 0. \end{cases}$$

Obviously, this equation also does not hold. Thus, we can get that for $\forall t > 0$ and $t \neq \sqrt{\tfrac{2}{5}}$, g_t is not a nontrivial, critical metric of $\mathcal{S}$ on $I \times S^3$.

To sum up, for $\forall t > 0$, g_t is not a nontrivial, critical metric of $\mathcal{S}$ on $I \times S^3$. □

Remark3.2. In theorem1, the existence problem of nontrivial, critical metric g_t of the Schouten functional on $I \times S^3$ is transformed into the existence problem of solutions of differential equations (3.33). Now we will apply the conclusion of theorem1 to $S^1 \times S^3$.

3.2 Proof of Theorem2

Proof. In the proof of theorem1, we can easily see that there is no periodic function $f(r)$ and parameter t_0, so that g_{t_0} is a nontrivial, critical metric of $\mathcal{S}[g_t]$ on $I \times S^3$. So we have the same conclusion on $S^1 \times S^3$.

Remark3.3. In theorem2, unfortunately, the Riemannian metric we defined on the manifold, although there is a critical metric of the Schouten functional, there is no nontrivial critical metric. In the subsequent research process, we can further explore the nontrivial critical metric of the Schouten functional by modifying the parameters of g_t.

IV. Conclusion

In this paper, we define the Riemannian metrics of the form g_t on $I \times S^3$, and then transform the existence problem of the critical metric of the Schouten functional into the existence problem of solutions of the differential equation. This paper provides an idea and method to define and find the critical metric of the Schouten functional on a 4-manifold. Moreover, the method for finding critical metric given in this paper can be extended to other 4- manifolds, which is also our future research work.

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