# Characterizations of Fuzzy Functions on Time Scales

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***Abstract*—This paper handles with the characterization theorem for $\Delta_{gH}$ fuzzy functions on $\mathbb{T}$ (time scales) through the $\Delta$-differentiability of their end point functions. We proposed a relationship between the $\Delta_{gH}$-derivative of level-wise function $F_\beta$ and the delta differentiability of the endpoint functions $\underline{F}_\beta$ and $\overline{F}_\beta$. We extended the results to fuzzy integro dynamic equations (FIDEs) on $\mathbb{T}$ (FIDETs) which translates FIDET into an equivalent system of crisp FIDETs. This laid the foundation for the methods of finding the analytic and approximate solutions of FIDETs.**

***Index Terms*—time scales, Fuzzy-valued function, Hukuhara difference, gH- difference.**

## I. Introduction

THE first path to study FDEs depends on the H-difference. Bede [3] introduced gH-derivative of fuzzy functions. Hilger studied $\mathbb{T}$ to combine the continuous and discrete systems [4]. Using the H-difference, Hong [6] established the results on set valued functions on $\mathbb{T}$. Lupulescu [12] studied the interval functions on $\mathbb{T}$. In [5], te author established few results of fuzzy functions on $\mathbb{T}$. For literature on fuzzy differential and difference equations, [9], [13]-[14], [20], [21]. For detailed study on $\mathbb{T}$ we refer to [1]-[2], [7], [8], [10]-[11], [15]. In [16]- [19], Vasavi et.al., established the existence as well as uniqueness criteria for FDEs on $\mathbb{T}$ under $\Delta_H$-derivative, $\Delta_{SH}$-derivative(second type), generalized delta derivative ($\Delta_g$-derivative).

The basic concepts concerned to fuzzy as well as $\mathbb{T}$ are given in section 2. The next section presents definitions and proposed a relationship between the $\Delta_{gH}$-derivative of level-wise function $F_\beta$ and the delta differentiability of the endpoint functions $\underline{F}_\beta$ and $\overline{F}_\beta$.

## II. Preliminaries

$\mathbb{R}_F$ denotes the set of fuzzy functions whose values are compactly supported, upper semi-continuous, normal as well as fuzzy convex on $\mathbb{R}$ whose $\beta$-cuts are defined as usual and the metric is the supremum metric [9].

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## III. $\Delta_{gH}$-Derivative for Fuzzy Functions on Time Scales

We present few results and then propose a relationship between the $\Delta_{gH}$-derivative of level-wise function $G_\beta$ and $\Delta$-derivative of endpoint functions $\underline{\mathcal{G}_\beta}$ and $\overline{\mathcal{G}_\beta}$. Throughout $\mathcal{G}:\mathbb{T}\to\mathbb{R}_F$ be a fuzzy function.

*Definition 3.1:* [5] Let $\Delta_{gH}\mathcal{G}(\ell)\in\mathbb{R}_F$ and for given $\delta>0$, there exists $\xi>0$, such that

$$d[(\mathcal{G}(\ell+h)\ominus_{gH}\mathcal{G}(\sigma(\ell)),\Delta_{gH}\mathcal{G}(\ell)(h-\mu(\ell))]\leq\delta(h-\mu(\ell)),$$

$$d[(\mathcal{G}(\sigma(\ell))\ominus_{gH}G(\ell-h),\Delta_{gH}\mathcal{G}(\ell)(h+\mu(\ell))]\leq\delta(h+\mu(\ell)),$$

$\forall\ \ell+h,\ell-h\in U_{\mathbb{T}}$ with $0<h<\xi$. If $\Delta_{gH}\mathcal{G}(\ell)$ exists, then $\mathcal{G}$ is $\Delta_{gH}$-differentiable on $\mathbb{T}^k$.

*Theorem 3.1:* [5] Denote $[\mathcal{G}(\ell)]_\beta=[\underline{\mathcal{G}}_\beta(\ell),\overline{\mathcal{G}}_\beta(\ell)]$, for each $\beta\in[0,1]$. Then if $\Delta_{gH}\mathcal{G}$ exists, then the real functions $\underline{\mathcal{G}}_\beta$ and $\overline{\mathcal{G}}_\beta$ are $\Delta$-derivable and

$$[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]_\beta=\begin{cases}(i)\ \left[\underline{\mathcal{G}}_\beta^{\Delta}(\ell),\overline{\mathcal{G}}_\beta^{\Delta}(\ell)\right], \text{ (or)}\\(ii)\ \left[\overline{\mathcal{G}}_\beta^{\Delta}(\ell),\underline{\mathcal{G}}_\beta^{\Delta}(\ell)\right].\end{cases}$$

If $\mathcal{G}$ is $\Delta_{gH}$-differentiable as in (i), then $\mathcal{G}$ is called to be $\Delta_{1,gH}$-differentiable, otherwise $\Delta_{2,gH}$-differentiable.

*Remark 3.1:* The above theorem gives the form of the $\beta$-level sets of $\mathcal{G}_\beta^{\Delta_{gH}}(\ell)$, whenever the endpoint functions are differentiable. In view of this lemma, two cases arise:

(i) It may happen that $\mathcal{G}^{\Delta_{gH}}(\ell)$ exists and the endpoint functions may not be $\Delta$-differentiable.

(ii) In such a case, what is $\mathcal{G}^{\Delta_{gH}}(\ell)$ in terms of its endpoint functions.

From the following example, it is clear that the answer to the above remark is positive i.e. whenever $I$ is $\Delta_{gH}$-differentiable, the endpoint functions may not be $\Delta$-differentiable.

*Example 3.1:* Let $\mathcal{G}$ be as follows

$$[\mathcal{G}(\ell)]_\beta=\left[\frac{-1}{(1+|\sigma(\ell)|)(1+\beta)},\frac{1}{(1+|\sigma(\ell)|)(1+\beta)}\right],$$

where $\underline{\mathcal{G}}_\beta(\ell)=\dfrac{-1}{(1+|\sigma(\ell)|)(1+\beta)}$,

$\overline{\mathcal{G}}_\beta(\ell)=\dfrac{1}{(1+|\sigma(\ell)|)(1+\beta)}$

For $\mathbb{T}=\mathbb{Z}$, $\sigma(\ell)=\ell+1$. Then for all $\ell\neq-1$, $\overline{\mathcal{G}}_\beta(\ell),\underline{\mathcal{G}}_\beta(\ell)$ are delta derivable. At $\ell=-1$, the left and right delta derivatives are not same.

$$(\underline{\mathcal{G}}_\beta)^{\Delta}(\ell)=\begin{cases}\dfrac{-1}{\ell(\ell+1)(1+\beta)},\ \ell<-1\\\dfrac{1}{(\ell+2)(\ell+3)(1+\beta)},\ \ell>-1\end{cases},$$

$$(\overline{\mathcal{G}}_\beta)^{\Delta}(\ell)=\begin{cases}\dfrac{1}{\ell(\ell+1)(1+\beta)},\ \ell<-1\\\dfrac{-1}{(\ell+2)(\ell+3)(1+\beta)},\ \ell>-1\end{cases}$$

Now for the gH-difference and $\xi \neq -1$, we have

$$\begin{aligned}&\frac{[\mathcal{G}(\xi-1)\ominus_{gH}\mathcal{G}(-1)]_\beta}{\xi}\\&=\frac{1}{\xi}\left[\frac{-1}{(1+|\xi|)(1+\beta)},\frac{1}{(1+|\xi|)(1+\beta)}\right]\ominus_{gH}\\&\qquad\left[\frac{-1}{(1+\beta)},\frac{1}{(1+\beta)}\right]\\&=\frac{1}{\xi(1+\beta)}\left[\min\left\{\frac{-|\xi|}{(1+|\xi|)},\frac{|\xi|}{(1+|\xi|)}\right\},\right.\\&\qquad\left.\max\left\{\frac{-|\xi|}{(1+|\xi|)},\frac{|\xi|}{(1+|\xi|)}\right\}\right]\\&=\frac{1}{(1+\beta)}\left[\frac{-1}{(1+|\xi|)},\frac{1}{(1+|\xi|)}\right].\end{aligned}$$

Thus, the limit $\mathcal{G}^{\Delta_{gH}}(-1) = \lim_{\xi\to 0}\frac{[\mathcal{G}(\xi-1)\ominus_{gH}\mathcal{G}(-1)]_\beta}{\xi} = \left[\frac{-1}{(1+\beta)},\frac{1}{(1+\beta)}\right]$, As the gH-difference exists, $\mathcal{G}$ is $\Delta_{gH}$-differentiable but the end point functions $\underline{\mathcal{G}}_\beta(\xi), \overline{\mathcal{G}}_\beta(\xi)$ are not delta differentiable at $\xi = -1$.

*Proposition 3.1:* For $\ell \in \mathbb{T}^k$.

(i) Let $\ell$ be right-scattered. $\mathcal{G}(\sigma(\ell)) \ominus_{gH} \mathcal{G}(\ell)$, gH-difference exists for $\ell \in \mathbb{T}^k$,

$$\begin{cases} len([\mathcal{G}(\sigma(\ell))]_\beta) \geq len([\mathcal{G}(\ell)]_\beta),\\ \underline{\mathcal{G}}_\beta(\sigma(\ell)) - \underline{\mathcal{G}}_\beta(\ell) \text{ is nondecreasing w.r. to } \beta,\\ \overline{\mathcal{G}}_\beta(\sigma(\ell)) - \overline{\mathcal{G}}_\beta(\ell), \text{ is nonincreasing w.r. to } \beta.\end{cases}$$

$$\begin{cases} len([\mathcal{G}(\sigma(\ell))]_\beta) \leq len([\mathcal{G}(\ell)]_\beta),\\ \underline{\mathcal{G}}_\beta(\sigma(\ell)) - \underline{\mathcal{G}}_\beta(\ell) \text{ is nonincreasing w.r. to } \beta,\\ \overline{\mathcal{G}}_\beta(\sigma(\ell)) - \overline{\mathcal{G}}_\beta(\ell), \text{ is nondecreasing w.r. to } \beta.\end{cases}$$

(ii) Let $\ell \in \mathbb{T}^k$ be right-dense. The differences $\mathcal{G}(\ell+h)\ominus_{gH}\mathcal{G}(\ell)$, $\mathcal{G}(\ell)\ominus_{gH}\mathcal{G}(\ell-h)$ exists for $\ell-h, \ell, \ell+h \in \mathbb{T}^k$, $0<|h|<\delta$. Then:

$$\begin{cases} len([\mathcal{G}(\ell+h)]_\beta) \geq len([\mathcal{G}(\ell)]_\beta),\\ \underline{\mathcal{G}}_\beta(\ell+h) - \underline{\mathcal{G}}_\beta(\ell) \text{ is nondecreasing w.r. to } \beta,\\ \overline{\mathcal{G}}_\beta(\ell+h) - \overline{\mathcal{G}}_\beta(\ell), \text{ is nonincreasing w.r. to } \beta.\end{cases}$$

or

$$\begin{cases} len([\mathcal{G}(\ell+h)]_\beta) \leq len([\mathcal{G}(\ell)]_\beta),\\ \underline{\mathcal{G}}_\beta(\ell+h) - \underline{\mathcal{G}}_\beta(\ell) \text{ is nonincreasing w.r. to } \beta,\\ \overline{\mathcal{G}}_\beta(\ell+h) - \overline{\mathcal{G}}_\beta(\ell), \text{ is nondecreasing w.r. to } \beta.\end{cases}$$

The following theorem explains the form of the $\beta$-level sets of $\Delta_{gH}\mathcal{G}$ on $\mathbb{T}$.

*Theorem 3.2:* For $\ell \in \mathbb{T}^k$, $\Delta_{gH}\mathcal{G}(\ell)$ exists $\Leftrightarrow$ one among the following holds:

(a) The $\Delta$- derivative of $\underline{\mathcal{G}}_\beta, \overline{\mathcal{G}}_\beta$ exists at $\ell$, and either

$$[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)] = [(\underline{\mathcal{G}}_\beta)^\Delta(\ell), (\overline{\mathcal{G}}_\beta)^\Delta(\ell)],$$

or

$$[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)] = [(\overline{\mathcal{G}}_\beta)^\Delta(\ell), (\underline{\mathcal{G}}_\beta)^\Delta(\ell)].$$

(b) $(\underline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ and $(\overline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ exist and

$$(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell) = (\overline{\mathcal{G}}_\beta)^\Delta_-(\ell), \quad \text{and}$$
$$(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell) = (\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$$

and either

$$\begin{aligned}[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)] &= [(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell), (\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)]\\ &= [(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell), (\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)].\end{aligned}$$

or

$$\begin{aligned}[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)] &= [(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell), (\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)]\\ &= [(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell), (\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)].\end{aligned}$$

*Proof:* Let $\mathcal{G}$ be $\Delta_{gH}$-differentiable at $\ell \in \mathbb{T}^k$, then (a) holds from Theorem 5 in [5]. Conversely suppose that $\underline{\mathcal{G}}_\beta, \overline{\mathcal{G}}_\beta$ are $\Delta$-derivable at $\ell \in \mathbb{T}^k$. Let $\ell$ be right-scattered.

$$len\left(\frac{1}{\mu(\ell)}[\mathcal{G}_\beta(\sigma(\ell))]\right) \geq len\left(\frac{1}{\mu(\ell)}[\mathcal{G}_\beta(\ell)]\right), \quad (1)$$

or

$$len\left(\frac{1}{\mu(\ell)}[\mathcal{G}_\beta(\sigma(\ell))]\right) \leq len\left(\frac{1}{\mu(\ell)}[\mathcal{G}_\beta(\ell)]\right). \quad (2)$$

From (1), $\underline{\mathcal{G}}_\beta(\sigma(\ell)) - \underline{\mathcal{G}}_\beta(\ell)$ is nondecreasing, $\overline{\mathcal{G}}_\beta(\sigma(\ell)) - \overline{\mathcal{G}}_\beta(\ell)$ is nonincreasing. Therefore $(\underline{\mathcal{G}}_\beta)^\Delta(\ell)$ is nondecreasing and $(\overline{\mathcal{G}}_\beta)^\Delta(\ell)$ is nonincreasing. Clearly, $(\underline{\mathcal{G}})^\Delta(\ell) \leq (\overline{\mathcal{G}})^\Delta(\ell)$, which implies $(\underline{\mathcal{G}}_\beta)^\Delta(\ell) \leq (\overline{\mathcal{G}}_\beta)^\Delta(\ell)$. Hence $(\underline{\mathcal{G}}_\beta)^\Delta(\ell)$, $(\overline{\mathcal{G}}_\beta)^\Delta(\ell)$. Hence,

$$\begin{aligned}&\left[\frac{\mathcal{G}(\sigma(\ell))\ominus_{gH}\mathcal{G}(\ell)}{\mu(\ell)}\right]_\beta\\&=\left[\min\left\{\frac{\underline{\mathcal{G}}_\beta(\sigma(\ell))-\underline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)},\frac{\overline{\mathcal{G}}_\beta(\sigma(\ell))-\overline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)}\right\},\right.\\&\qquad\left.\max\left\{\frac{\underline{\mathcal{G}}_\beta(\sigma(\ell))-\underline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)},\frac{\overline{\mathcal{G}}_\beta(\sigma(\ell))-\overline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)}\right\}\right]\\&=[(\underline{\mathcal{G}}_\beta)^\Delta(\ell),(\overline{\mathcal{G}}_\beta)^\Delta(\ell)],\end{aligned}$$

which is a fuzzy interval. Hence $\mathcal{G}$ is $\Delta_{gH}$-differentiable.

From (2), the function $(\underline{\mathcal{G}}_\beta)^\Delta(\ell)$ is nonincreasing, $(\overline{\mathcal{G}}_\beta)^\Delta(\ell)$ is nondecreasing. Hence,

$$\left[\frac{\mathcal{G}(\sigma(\ell))\ominus_{gH}\mathcal{G}(\ell)}{\mu(\ell)}\right]_\beta = [(\overline{\mathcal{G}}_\beta)^\Delta(\ell), (\underline{\mathcal{G}}_\beta)^\Delta(\ell)], \ ,$$

which is a fuzzy interval. Hence, $\mathcal{G}$ is $\Delta_{gH}$-differentiable.

If $\ell$ is right-dense, $\sigma(\ell) = \ell$, $\mu(\ell) = 0$.

$$len\left(\frac{1}{h}[\mathcal{G}_\beta(\ell+h)]\right) \geq len\left(\frac{1}{h}[\mathcal{G}_\beta(\ell)]\right), \quad (3)$$

or

$$len\left(\frac{1}{h}[\mathcal{G}_\beta(\ell+h)]\right) \leq len\left(\frac{1}{h}[\mathcal{G}_\beta(\ell)]\right), \quad (4)$$

From (3), $\underline{\mathcal{G}}_\beta(\ell+h) - \underline{\mathcal{G}}_\beta(\ell)$ is nondecreasing and $\overline{\mathcal{G}}_\beta(\ell+h) - \overline{I}_\beta(\ell)$ is nonincreasing. Hence,

$$\begin{aligned}&\left[\lim_{h\to 0}\frac{\mathcal{G}(\ell+h)\ominus_{gH}\mathcal{G}(\ell)}{h}\right]_\beta =\\&\left[\min\left\{\lim_{h\to 0}\frac{\underline{\mathcal{G}}_\beta(\ell+h)-\underline{\mathcal{G}}_\beta(\ell)}{h},\lim_{h\to 0}\frac{\overline{\mathcal{G}}_\beta(\ell+h)-\overline{\mathcal{G}}_\beta(\ell)}{h}\right\},\right.\\&\left.\max\left\{\lim_{h\to 0}\frac{\underline{\mathcal{G}}_\beta(\ell+h)-\underline{\mathcal{G}}_\beta(\ell)}{h},\lim_{h\to 0}\frac{\overline{\mathcal{G}}_\beta(\ell+h)-\overline{\mathcal{G}}_\beta(\ell)}{h}\right\}\right]\\&=[(\underline{\mathcal{G}}_\beta)^\Delta(\ell),(\overline{\mathcal{G}}_\beta)^\Delta(\ell)],\end{aligned}$$

is a fuzzy interval. Hence $\mathcal{G}$ is $\Delta_{gH}$-differentiable. From (4),

$$\left[\lim_{h\to 0}\frac{\mathcal{G}(\ell+h)\ominus_{gH}\mathcal{G}(\ell)}{h}\right]_\beta=[(\overline{\mathcal{G}}_\beta)^\Delta(\ell),(\underline{\mathcal{G}}_\beta)^\Delta(\ell)],$$

is a fuzzy interval and hence $\mathcal{G}$ is $\Delta_{gH}$-differentiable.

Now suppose that (b) is valid i.e. $(\underline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$, $(\overline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ exist and $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)$ and $(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$ and either

$$[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]=[(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)]=[(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell),(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)],$$

or

$$[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]=[(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)]=[(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)].$$

Here three cases will arise: (i) $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)<(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$,
(ii) $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$,
(iii) $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)>(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$.

(i) Let $\ell\in\mathbb{T}^k$ be right-scattered, $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)<(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$,

$$\begin{aligned}&\left[\frac{\mathcal{G}(\sigma(\ell))\ominus_{gH}\mathcal{G}(\ell)}{\mu(\ell)}\right]_\beta\\&=\left[\min\left\{\frac{\overline{\mathcal{G}}_\beta(\sigma(\ell))-\overline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)},\frac{\underline{\mathcal{G}}_\beta(\sigma(\ell))-\underline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)}\right\},\right.\\&\qquad\left.\max\left\{\frac{\overline{\mathcal{G}}_\beta(\sigma(\ell))-\overline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)},\frac{\underline{\mathcal{G}}_\beta(\sigma(\ell))-\underline{\mathcal{G}}_\beta(\ell)}{\mu(\ell)}\right\}\right]\\&=[(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)],\end{aligned}$$

is a fuzzy interval. Hence, $\mathcal{G}$ is $\Delta_{gH}$-differentiable. When $\ell$ is right-dense, the result is similar.

(ii) Assume $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$. Since $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)$ and $(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$, we have $\underline{\mathcal{G}}_\beta$, $\overline{\mathcal{G}}_\beta$ are $\Delta$-differentiable. Hence from (a), $\mathcal{G}$ is $\Delta_{gH}$-differentiable.

(iii) In a similar way as in (i), if $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)>(\overline{\mathcal{G}}_\beta)^\Delta_+(\mathcal{G})$, then $\mathcal{G}$ is $\Delta_{gH}$-differentiable and

$$\begin{aligned}[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]&=[(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)]\\&=[(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)].\end{aligned}$$

Analogously, if $\Delta_{gH}\mathcal{G}(\ell)$ exists $\in\mathbb{T}^k$, $(\underline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$, $(\overline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ exist and $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)$, $(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$, then

$$\begin{aligned}[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]&=[(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)]\\&=[(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell),(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)],\end{aligned}$$

or

$$\begin{aligned}[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]&=[(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)]\\&=[(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)].\end{aligned}$$

■

*Theorem 3.3:* If $\mathcal{G}(\ell)$ is $\Delta_{gH}$-differentiable at $\ell\in\mathbb{T}^k$, $\Leftrightarrow$ one among the four holds:

(i) $\underline{\mathcal{G}}_\beta,\overline{\mathcal{G}}_\beta$ are $\Delta$-differentiable at $\ell$, $(\underline{\mathcal{G}}_\beta)^\Delta$ is nondecreasing, $(\overline{\mathcal{G}}_\beta)^\Delta(\ell)$ is nonincreasing functions of $\beta$, and $(\underline{\mathcal{G}}_1)^\Delta(s)\le(\overline{\mathcal{G}}_1)^\Delta(s)$. In this case,

$$[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]=[(\underline{\mathcal{G}}_\beta)^\Delta(\ell),(\overline{\mathcal{G}}_\beta)^\Delta(\ell)],$$

(ii) $\underline{\mathcal{G}}_\beta,\overline{\mathcal{G}}_\beta$ are $\Delta$-differentiable at $\ell$, $(\underline{\mathcal{G}}_\beta)^\Delta$ is nonincreasing, $(\overline{\mathcal{G}}_\beta)^\Delta(\ell)$ is nondecreasing functions of $\beta$, and $(\overline{\mathcal{G}}_1)^\Delta(\ell)\le(\underline{\mathcal{G}}_1)^\Delta(\ell)$. In this case,

$$[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]=[(\overline{\mathcal{G}}_\beta)^\Delta(\ell),(\underline{\mathcal{G}}_\beta)^\Delta(\ell)].$$

(iii) $(\underline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ and $(\overline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ exist and $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)$ is nondecreasing and $(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$ is nonincreasing functions of $\beta$, and $(\underline{\mathcal{G}}_1)^\Delta_+(\ell)\le(\overline{\mathcal{G}}_1)^\Delta_+(\ell)$. In this case,

$$\begin{aligned}[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]&=[(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)]\\&=[(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell),(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)],\end{aligned}$$

(iv) $(\underline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ and $(\overline{\mathcal{G}}_\beta)^\Delta_{+/-}(\ell)$ exist and $(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)$ is nonincreasing and $(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell)=(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell)$ is nondecreasing as functions of $\beta$, and $(\overline{\mathcal{G}}_1)^\Delta_+(\ell)\le(\underline{\mathcal{G}}_1)^\Delta_+(\ell)$. In this case,

$$\begin{aligned}[\mathcal{G}_\beta^{\Delta_{gH}}(\ell)]&=[(\overline{\mathcal{G}}_\beta)^\Delta_+(\ell),(\underline{\mathcal{G}}_\beta)^\Delta_+(\ell)]\\&=[(\underline{\mathcal{G}}_\beta)^\Delta_-(\ell),(\overline{\mathcal{G}}_\beta)^\Delta_-(\ell)].\end{aligned}$$

The below example illustrates the form of the $\beta$-level sets for the $\Delta_{gH}$-differentiability of $I$ on time scales when the end-point functions are not $\Delta$-differentiable.

*Example 3.2:* For $p\in\mathbb{T}^k$, $\mathcal{G}$ defined by the $\beta$-cuts

$$[\mathcal{G}(p)]_\beta=[-1(1-\beta)(p^3+p),2(1-\beta)(p^3+p)].$$

i.e. $\underline{\mathcal{G}}_\beta(p)=2(1-\beta)(p^3+p)$, $\overline{\mathcal{G}}_\beta(p)=-1(1-\beta)(p^3+p)$, $p<0$,
$\underline{\mathcal{G}}_\beta(p)=-1(1-\beta)(p^3+p)$, $\overline{\mathcal{G}}_\beta(p)=2(1-\beta)(p^3+p)$, $p>0$.

When $\mathbb{T}=\mathbb{Z}$, for all $p\neq 0$, $\underline{\mathcal{G}}_\beta$, $\overline{\mathcal{G}}_\beta$ are $\Delta$-differentiable.

At $p=0$, the functions $\underline{\mathcal{G}}_\beta$, $\overline{\mathcal{G}}_\beta$ are not $\Delta$-differentiable, they are differentiable but they are not same.

$$\begin{aligned}(\underline{\mathcal{G}}_\beta)^\Delta(p)&=2(1-\beta)(3p^2+3p+2),\ p<0\\&\quad-1(1-\beta)(3p^2+3p+2),\ p>0\end{aligned}$$

$$(\overline{\mathcal{G}}_\beta)^\Delta(p)=\begin{cases}-1(1-\beta)(3p^2+3p+2),\ p<0\\2(1-\beta)(3p^2+3p+2),\ p>0.\end{cases}$$

Here $(\underline{\mathcal{G}}_\beta)^\Delta_+(p)=(\overline{\mathcal{G}}_\beta)^\Delta_-(p)$, $(\underline{\mathcal{G}}_\beta)^\Delta_-(p)=(\overline{\mathcal{G}}_\beta)^\Delta_+(p)$ are respectively nondecreasing, nonincreasing functions of $\beta$, and $(\underline{\mathcal{G}}_1)^\Delta_+(p)\le(\overline{\mathcal{G}}_1)^\Delta_+(p)$. Hence from Theorem 3.3 (iii),

$$[\mathcal{G}_\beta^{\Delta_{gH}}(p)]=[-1(1-\beta)(3p^2+3p+2),2(1-\beta)(3p^2+3p+2)].$$

*Example 3.3:* For $p\in\mathbb{T}^k$, $\mathcal{G}$ defined by the $\beta$-cuts

$$[\mathcal{G}(p)]_\beta=\left[\frac{-1}{(1+|\sigma(p)|)(1+\beta)},\frac{1}{(1+|\sigma(p)|)(1+\beta)}\right],$$

where $\underline{\mathcal{G}}_\beta(p)=\dfrac{-1}{(1+|\sigma(p)|)(1+\beta)}$,
$\overline{\mathcal{G}}_\beta(p)=\dfrac{1}{(1+|\sigma(p)|)(1+\beta)}$.

If $\mathbb{T}=\mathbb{Z}$, $\sigma(p)=p+1$. Then for all $p\neq -1$, $\underline{\mathcal{G}}_\beta(p),\overline{\mathcal{G}}_\beta(p)$ are delta derivable. At $p=-1$, they are right differentiable but not left delta differentiable.

$$(\underline{\mathcal{G}}_\beta)^\Delta(p)=\begin{cases}\dfrac{-1}{p(p+1)(1+\beta)},\ p<-1\\\dfrac{1}{(p+2)(p+3)(1+\beta)},\ p>-1,\end{cases}$$

$$(\overline{\mathcal{G}}_\beta)^\Delta(p) = \begin{cases} \frac{1}{p(p+1)(1+\beta)}, & p < -1 \\ \frac{-1}{(p+2)(p+3)(1+\beta)}, & p > -1. \end{cases}$$

Then from Theorem 2 (i) in [5] at $p = -1$, we have

$$\begin{aligned} &\frac{[\mathcal{G}(\sigma(-1)) \ominus_{gH} \mathcal{G}(-1)]}{\mu(-1)} \\ &= [\mathcal{G}(\sigma(-1)) \ominus_{gH} \mathcal{G}(-1)] \\ &= \frac{1}{2}\left[\frac{-1}{(1+\beta)}, \frac{1}{(1+\beta)}\right] \ominus_{gH} \left[\frac{-1}{(1+\beta)}, \frac{1}{(1+\beta)}\right] \\ &= \frac{1}{(1+\beta)}\left[\min\left\{\frac{-1}{2}, \frac{1}{2}\right\}, \max\left\{\frac{-1}{2}, \frac{1}{2}\right\}\right] \\ &= \frac{1}{(1+\beta)}\left[\frac{-1}{2}, \frac{1}{2}\right]. \end{aligned}$$

Hence the limit $I^{\Delta_{gH}}(-1) = \frac{1}{(1+\beta)}\left[\frac{-1}{2}, \frac{1}{2}\right]$. As the gH-difference exists, $\mathcal{G}$ is $\Delta_{gH}$-differentiable but the endpoint functions $\underline{\mathcal{G}}_\beta(s)$, $\overline{\mathcal{G}}_\beta(s)$ are not delta derivable at $s = -1$.

## IV. Fuzzy Integro Dynamic Equations on time scales (FIDETs)

The main aim of this study is to present the characterization theorem for fuzzy integro dynamic equation on time scales (FIDET) which translates FIDET into an equivalent system of crisp FIDETs is of the form:

$$z^{\Delta_{gH}}(p) = \mathcal{F}(p, y(p)) + \int_{p_0}^{p} \mathcal{G}(p, q, z(q))\Delta q, p_0, q \in [p_0, p_0+a]_{\mathbb{T}}, \quad (1)$$

$$z(p_0) = z_0,$$

where $\mathcal{F} : [p_0, p_0 + a]_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $\mathcal{G} : [p_0, p_0 + a]^2_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ are rd-continuous fuzzy functions, $z^\Delta$ denotes $\Delta_{gH}$-derivative of $z$, $p \in \mathbb{T}$, $z_0 \in \mathbb{R}_{\mathcal{F}}$.

To solve this, express $zp$ in $\beta$-level representation $[z_p]_\beta = [\underline{z}_\beta(p), \overline{z}_\beta(p)]$ and $[z(p_0)]_\beta = [\underline{z_0}_\beta(p), \overline{z_0}_\beta(p)]$. From Zadeh's extension principle if $z(p)$ is fuzzy, then

$$\begin{aligned} &[\mathcal{F}(p, y(p))]_\beta \\ &= [\underline{\mathcal{F}}_\beta(p, z(p)), \overline{\mathcal{F}}_\beta(p, z(p))] \\ &= [\underline{\mathcal{F}}_{1,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \overline{\mathcal{F}}_{2,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p))] \end{aligned}$$

$\mathcal{G}$ can be expressed as

$$\begin{aligned} &[\mathcal{G}(p, q, z(q))]_\beta \\ &= [\underline{\mathcal{G}}_\beta(p, q, z(q)), \overline{\mathcal{G}}_\beta(p, q, z(q))] \\ &= [\underline{\mathcal{G}}_{1,\beta}(p, q, \underline{z}_\beta(q), \overline{z}_\beta(q)), \overline{\mathcal{G}}_{2,\beta}(p, q, \underline{z}_\beta(q), \overline{z}_\beta(q))] \end{aligned}$$

*Definition 4.1:* Let $z : [p_0, p_0 + a]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$ and $z^{\Delta_{1,gH}}$, $z^{\Delta_{2,gH}}$ exists. If $z$ and $z^{\Delta_{1,gH}}$ satisfy (1), it is called (i)-solution, otherwise (ii)-solution.

Now, we represent FIDETs (1) in terms of its $\beta$-cuts, where the new system consists of two crisp IDEs for each type of differentiability. For convenience, we are considering (i) and (ii) cases of Theorem 3.3.

(i) If $z(p)$ is $\Delta_{1,gH}$ differentiable, then $[z^{\Delta_{gH}}(p)]_\beta = [\underline{z}^\Delta_\beta(p), \overline{z}^\Delta_\beta(p)]$ and FIDETs (1) is translated into

$$\begin{aligned} &\underline{z}^{\Delta_{gH}}(p) \\ &= \underline{\mathcal{F}}_\beta(p, \underline{z}(p), \overline{z}(p)) + \int_{p_0}^{p} \underline{\mathcal{G}}_\beta(p, q, \underline{y}(q), \overline{y}(q))\Delta q, \\ &\overline{z}^{\Delta_{gH}}(p) \\ &= \overline{\mathcal{F}}_\beta(p, \underline{z}(p), \overline{z}(p)) + \int_{p_0}^{p} \overline{\mathcal{G}}_\beta(p, q, \underline{y}(q), \overline{y}(q))\Delta q, \end{aligned} \quad (2)$$

subject to $\underline{z}_\beta(p_0) = \underline{z_0}_\beta, \overline{z}_\beta(p_0) = \overline{z_0}_\beta$.

(ii) If $z(p)$ is $\Delta_{2,gH}$ differentiable, then $[z^{\Delta_{gH}}(p)]_\beta = [\overline{z}^\Delta_\beta(p), \underline{z}^\Delta_\beta(p)]$ and FIDETs (1) are translated into

$$\begin{aligned} &\underline{z}^{\Delta_{gH}}(p) \\ &= \overline{\mathcal{F}}_\beta(p, \underline{z}(p), \overline{z}(p)) + \int_{p_0}^{p} \overline{\mathcal{G}}_\beta(p, q, \underline{z}(q), \overline{z}(q))\Delta q, \\ &\overline{z}^{\Delta_{gH}}(p) \\ &= \underline{\mathcal{F}}_\beta(p, \underline{z}(p), \overline{z}(p)) + \int_{p_0}^{p} \underline{\mathcal{G}}_\beta(p, q, \underline{z}(q), \overline{z}(q))\Delta s, \end{aligned} \quad (3)$$

subject to $\underline{z}_\beta(p_0) = \underline{z_0}_\beta, \overline{z}_\beta(p_0) = \overline{z_0}_\beta$.

Obviously, $[\underline{z}_\beta(p), \overline{z}_\beta(p)$ and its $\Delta_{gH}$-derivative $[\underline{z}^{\Delta_{gH}}(p), \overline{z}^{\Delta_{gH}}(p)]$ are valid level sets for each $\beta \in [0, 1]$.

To obtain the approximate solution of (1), without loss of generality assume $\mathcal{G}(p, q, y(q)) = K(p, q)\mathcal{G}(z(p))$, where the $\beta$-level representation of $\mathcal{G}(y(p))$ is

$$\begin{aligned} &\mathcal{G}(z(p)) \\ &= [\underline{\mathcal{G}}(z(p)), \overline{\mathcal{G}}(z(p))] \\ &= [\underline{\mathcal{G}}_{1,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \overline{\mathcal{G}}_{2,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p))] \end{aligned}$$

Hence (1) can be expressed as

$z^{\Delta_{gH}}(p) = \mathcal{F}(p, z(p)) + \int_{p_0}^{p} K(p, q)\mathcal{G}(z(p)\Delta q, p_0, q \in [p_0, p_0 + a]_{\mathbb{T}}$.

To solve (1), express it by equivalent crisp IDEs

$$\begin{aligned} &\underline{z}^{\Delta_{gH}}(p) \\ &= \underline{\mathcal{F}}_{1,\beta}(p, \underline{z}(p), \overline{z}(p)) + \int_{p_0}^{p} K_{1,\beta}(p, q, \underline{z}_\beta(q), \overline{z}_\beta(q))\Delta q, \end{aligned}$$

$$\begin{aligned} &\overline{z}^{\Delta_{gH}}(p) \\ &= \underline{\mathcal{F}}_{2,\beta}(p, \underline{z}(p), \overline{z}(p)) + \int_{p_0}^{p} K_{2,\beta}(p, q, \underline{y}_\beta(q), \overline{y}_\beta(s))\Delta s, \end{aligned}$$

subject to $\underline{z}_\beta(p_0) = \underline{z_0}_\beta, \overline{z}_\beta(p_0) = \overline{z_0}_\beta$, where

$$K_{1,\beta}(p, q, \underline{z}_\beta(q), \overline{z}_\beta(q)) = \begin{cases} K(p, q)\underline{\mathcal{G}}_{1,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \\ \quad K(p, q) \geq 0, \\ K(p, q)\underline{\mathcal{G}}_{2,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \\ \quad K(p, q) < 0, \end{cases}$$

$$K_{2,\beta}(p, q, \underline{z}_\beta(q), \overline{z}_\beta(q)) = \begin{cases} K(p, q)\underline{\mathcal{G}}_{2,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \\ \quad K(p, q) \geq 0, \\ K(p, q)\underline{\mathcal{G}}_{1,\beta}(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \\ \quad K(p, q) < 0, \end{cases}$$

*Definition 4.2:* A function $\mathcal{F} : [p_0, p_0 + a]_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $\mathcal{G} : [p_0, p_0 + a]^2_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ are

(i) rd-continuous, if $\mathcal{F}(p, u(p)), \mathcal{G}(p, u(p), v(p))$ are rd-continuous and $u, v : [p_0, p_0 + a]_{\mathbb{T}} \to \mathbb{R}_{\mathcal{F}}$.

(ii) Lipschitz continuous w.r.to the last argument in fuzzy sense if there exists $\ell_1, \ell_2 > 0$ such that

$$\begin{aligned} &D(\mathcal{F}(p, u(p)), \mathcal{F}(p, v(p))) \leq \ell_1 D(u(p), v(p)) \\ &D(\mathcal{G}(p, s, u(p)), F(p, s, v(p))) \leq \ell_2 D(u(p), v(p)) \end{aligned} \tag{4}$$

for each $p, s \in [p_0, p_0 + a]_{\mathbb{T}}$. Then FIDETs (1) have two unique solutions, one is $\Delta_{1,gH}$-differentiable and the other is $\Delta_{2,gH}$-differentiable on $[p_0, p_0 + a]_{\mathbb{T}}$.

*Theorem 4.1:* Let $\mathcal{F}, \mathcal{G}$ are bounded and $p_0 \in \mathbb{T}$ with $\inf \mathbb{T} \leq p_0 - a, \sup \mathbb{T} \geq p_0 + a$ such that

$$\begin{aligned} &(i)[\mathcal{F}(p, z(p))]_\beta \\ &\quad = [\underline{\mathcal{F}}_\beta(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \overline{\mathcal{F}}_\beta(p, \underline{z}_\beta(p), \overline{z}_\beta(p)), \\ &\quad [\mathcal{G}(p, s, z(s))]_\beta \\ &\quad = [\underline{\mathcal{G}}_\beta(p, s, \underline{z}_\beta(s), \overline{z}_\beta(s)), \overline{\mathcal{G}}_\beta(p, s, \underline{z}_\beta(s), \overline{z}_\beta(s))], \end{aligned}$$

(ii) $\underline{\mathcal{F}}_\beta, \overline{\mathcal{F}}_\beta, \underline{\mathcal{G}}_\beta, \overline{\mathcal{G}}_\beta$ are rd-equicontinuous uniformly in $\beta \in [0, 1]$, uniformly bounded on any bounded set, and uniformly Lipschitz in the second, third argument, i.e., $\ell_1, \ell_2 > 0$ and

$$\begin{aligned} &|\underline{\mathcal{F}}_\beta(p, \underline{u}_\beta(p), \overline{u}_\beta(p)) - \underline{\mathcal{F}}_\beta(p, \underline{v}_\beta(p), \overline{v}_\beta(p))| \\ &\qquad \leq \ell_1 \max\{|\underline{u}_\beta(p) - \overline{u}_\beta(p))|, |\underline{v}_\beta(p) - \overline{v}_\beta(p))|\}, \end{aligned}$$

and

$$\begin{aligned} &|\overline{\mathcal{F}}_\beta(p, \underline{u}_\beta(p), \overline{u}_\beta(p)) - \overline{\mathcal{F}}_\beta(p, \underline{v}_\beta(p), \overline{v}_\beta(p))| \\ &\qquad \leq \ell_1 \max\{|\underline{u}_\beta(p) - \overline{u}_\beta(p))|, |\underline{v}_\beta(p) - \overline{v}_\beta(p))|\}, \\ &|\underline{\mathcal{G}}_\beta(p, \underline{u}_\beta(p), \overline{u}_\beta(p)) - \underline{\mathcal{G}}_\beta(p, \underline{v}_\beta(p), \overline{v}_\beta(p)) \\ &\qquad \leq \ell_2 \max\{|\underline{u}_\beta(p) - \overline{u}_\beta(p))|, |\underline{v}_\beta(p) - \overline{v}_\beta(p))|\} \end{aligned}$$

and

$$\begin{aligned} &|\overline{\mathcal{G}}_\beta(p, \underline{u}_\beta(p), \overline{u}_\beta(p)) - \overline{\mathcal{G}}_\beta(p, \underline{v}_\beta(p), \overline{v}_\beta(p)) \\ &\qquad \leq \ell_2 \max\{|\underline{u}_\beta(p) - \overline{u}_\beta(p))|, |\underline{v}_\beta(p) - \overline{v}_\beta(p))|\} \end{aligned}$$

for all $p, s \in [p_0, p_0 + a]_{\mathbb{T}}$, $\beta \in [0, 1]$, then for $\Delta_{1,gH}$-differentiability,the FIDET (1) on $\mathbb{T}$ is equivalent to (2) and for $\Delta_{2,gH}$-differentiability,the FIDET (1) on $\mathbb{T}$ is equivalent to (3).

*Proof:* Let $z$ be $\Delta_{1,gH}$-differentiable. The rd-equicontinuity of $\underline{\mathcal{F}}_\beta, \overline{\mathcal{F}}_\beta, \underline{\mathcal{G}}_\beta$ and $\overline{\mathcal{G}}_\beta$ implies the rd-continuity of $\mathcal{F}, \mathcal{G}$. Further, the Lipschitz property ensures that the fuzzy functions $\mathcal{F}, \mathcal{G}$ satisfy Lipschitz property in the metric space $(\mathbb{R}_F, D)$ as follows:

$$\begin{aligned} &D(\mathcal{F}(p, u(p)), \mathcal{F}(p, v(p))) \\ &= \sup_{\beta \in [0,1]} \max\{|\underline{\mathcal{F}}_\beta(p, \underline{u}_\beta(p), \overline{u}_\beta(p)) - \underline{\mathcal{F}}_\beta(p, \underline{v}_\beta(p), \overline{v}_\beta(p))|, \\ &\qquad |\overline{\mathcal{F}}_\beta(p, \underline{u}_\beta(p), \overline{u}_\beta(p)) - \overline{\mathcal{F}}_\beta(p, \underline{v}_\beta(p), \overline{v}_\beta(p))|\} \\ &\leq \ell_1 \sup_{\beta \in [0,1]} \max\{|\underline{u}_\beta(p) - \overline{u}_\beta(p))|, |\underline{v}_\beta(p) - \overline{v}_\beta(p))|\} \\ &= \ell_1 D(u, v). \end{aligned}$$

In a similar way, $D(\mathcal{G}(p, s, u(s)), G(p, s, v(s))) \leq \ell_2 D(u, v)$.

From the Lipschitz as well as boundedness condition, it follows that FIDET (1) has unique $\Delta_{1,gH}$-differentiable solution and a unique $\Delta_{2,gH}$-differentiable solution.

Conversely, suppose that (1) has a solution $(\underline{z}_\beta(p)), \overline{z}_\beta(p))$ with $\beta \in [0, 1]$, whereas the Lipschitz condition implies the uniqueness along with existence of fuzzy solution $z(p)$. As $z$ is $\Delta_{1,gH}$-differentiable, then $\underline{z}_\beta$ and $\overline{z}_\beta$ are the endpoints of $[z(p)_\beta]$, solution of (2). Since the solution of (2) is unique, we have

$[z(p)_\beta] = [\underline{z}(p)_\beta, \overline{z}(p)_\beta] = [z(p)]_\beta$, which means, FIDET (1) is equivalent to (2). ∎

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